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Published in:
IEEE Transactions on Cybernetics

DOI:
10.1109/TCYB.2020.3022024

Publication date:
2020

Practical Exponential Stability of Impulsive Stochastic Reaction–Diffusion Systems With Delays

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***Abstract*—This article studies the practical exponential stability of impulsive stochastic reaction–diffusion systems (ISRDSs) with delays. First, a direct approach and the Lyapunov method are developed to investigate the *p*th moment practical exponential stability and estimate the convergence rate. Note that these two methods can also be used to discuss the exponential stability of systems in certain conditions. Then, the practical stability results are successfully applied to the impulsive reaction–diffusion stochastic Hopfield neural networks (IRDSHNNs) with delays. By the illustration of four numerical examples and their simulations, our results in this article are proven to be effective in dealing with the problem of practical exponential stability of ISRDSs with delays, and may be regarded as stabilization results.**

***Index Terms*—Hopfield neural networks, impulses, Lyapunov method, practical exponential stability, stochastic reaction–diffusion systems with delays.**

I. Introduction

THE THEORY of stochastic systems has been extensively studied for many years because of the investigation of numerous physical and engineering problems [1]–[5]. We notice that time delays could not be ignored in many practical systems, such as neural networks, ecological systems, and electric circuits, and may lead to oscillation, instability, or other degradation of system performance [6]–[8]. Besides, diffusion effects usually inevitably occur in man-made neural networks when electrons transport in a nonuniform electromagnetic field. And it is also common to consider them in other real-world processes, like a chemical reaction and biological immigration, since the effects always influence the stability of systems [9]–[11]. What is more, impulses always exist in basic models to describe the dynamical processes that are subject to sudden changes in their states, and they have been widely used to stabilize and synchronize nonlinear unstable dynamical systems and chaotic systems [12]–[14]. Therefore, it is of prime importance to consider the delay effects, reaction–diffusion effects, and impulsive effects on the dynamical behavior of systems, and these effects also have attracted considerable interest [15]–[20].

In the past few years, researchers have paid a lot of attention to exponential stability or stabilization of systems. (See [20]–[29] and the references therein.) For example, Yang and Xu [25] analyzed the global exponential stability of impulsive delayed systems by establishing an impulsive delayed differential inequality. Wu *et al.* [28] discussed the stability and stabilization of stochastic neural networks with neutral type by combining a Lyapunov–Krasovskii functional with the linear matrix inequalities. Furthermore, Wei *et al.* [20] considered the global exponential stability in the mean-square sense of stochastic impulsive reaction–diffusion system with stabilizing impulses.

On the other hand, we notice that the desired state of a system may be mathematically unstable, but the system may oscillate sufficiently in a small neighborhood of this state. In this case, it is still important to discuss the performance since it is considered acceptable. This case yields the concept of practical stability, which aims to obtain the ultimate boundedness of state trajectory, and this concept is more useful for many problems, like the traveling of a space vehicle between two points and keeping the temperature of a chemical process within certain bounds [30]. Besides, practical stability is also suitable in those situations, such as the delayed logistic systems, the switched delayed systems, and so on [31], [32]. Many interesting results on the practical stability of different systems have been reported [32]–[37]. For instance, Xu and Zhai [34] used a direct method to study the practical stability and stabilization problems for hybrid and switched systems. Caraballo *et al.* [37] investigated the *p*th moment practical exponential stability and almost sure practical exponential stability of impulsive stochastic delayed systems with the Lyapunov–Razumikhin method. However, to the best of our knowledge, those works have not been done for impulsive stochastic reaction–diffusion systems (ISRDSs) with delays yet.

Manuscript received December 19, 2019; revised May 13, 2020 and August 10, 2020; accepted August 31, 2020. This work was supported in part by the National Natural Science Foundation of China under Grant 32072976, Grant 11861131004, Grant 11771014, Grant 11771040, and Grant 31772844; in part by the Major Basic Research Projects of the Shandong Natural Science Foundation under Grant 2018A07; in part by the China Scholarship Council under Grant 201906330009; and in part by the National Key Research and Development Program of China under Grant 2018YFD0901601. This article was recommended by Associate Editor G.-P. Liu. *(Corresponding authors: Linshan Wang; Yangfan Wang.)*

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Digital Object Identifier 10.1109/TCYB.2020.3022024

In this article, we study the practical exponential stability of ISRDSs with delays, and present sufficient conditions which may imply the exponential stability under certain circumstances, and extend some results in [37]–[40]. A precise description of the systems will be given in the next section. The main contributions of this article are listed as follows.

1) The pth moment practical exponential stability and the convergence rate of ISRDSs with delays are studied for the first time in two ways: a) a direct approach and b) the Lyapunov method. With the direct approach, practical stability theorems of the systems with stabilizing impulses and destabilizing impulses are established. And using the Lyapunov method, the systems with stabilizing and destabilizing impulses are investigated simultaneously, but there is a threshold for the product of all impulsive strengths.
2) The practical stabilization results of the systems can be derived from our proposed results, which is verified by examples. Also, the exponential stability of the systems can be obtained by the practical exponential stability if the origin is an equilibrium point.
3) By applying the theoretical results to the impulsive reaction–diffusion stochastic Hopfield neural networks (IRDSHNNs) with delays, some easy-to-test algebraic criteria for the practical stability of the networks are proposed.
4) Four numerical examples are given to demonstrate the applicability of our results. The effects of diffusion terms and time delays on the practical exponential stability of systems are also illustrated by these four examples.

II. Preliminaries

Notations: Let $\mathbb{R}$ be the set of real numbers, $\mathbb{Z}_+$ be the set of positive integer numbers, and $\mathbb{R}^l$ be the l-dimensional real space equipped with the Euclidean norm $|\cdot|$. $(\Omega, \mathscr{F}, \mathbb{P})$ is a complete probability space with filtration $\{\mathscr{F}_t\}_{t\geq 0}$. $L^2(\mathcal{O})^n$ denotes a Hilbert space with the norm $\|\boldsymbol{u}\| = (\int_{\mathcal{O}}|\boldsymbol{u}(t,\boldsymbol{x})|^2 d\boldsymbol{x})^{1/2}$, and $(\cdot,\cdot)$ is the inner product. $H_0^1(\mathcal{O})$ is a Hilbert space with the norm $|||\boldsymbol{u}||| = \|\nabla \boldsymbol{u}\|$. $C^b([-\tau,0]\times \mathcal{O}, L^2(\mathcal{O})^n)$ represents the Banach space of all continuous functions from $[-\tau,0]\times\mathcal{O}$ to $L^2(\mathcal{O})^n$, with the norm $\|\boldsymbol{\phi}\|_C = \sup_{\theta\in[-\tau,0]}\|\boldsymbol{\phi}(\theta)\|$. $C^b_{\mathscr{F}_0}$ denotes the family of $\mathscr{F}_0$-measurable bounded $C^b([-\tau,0]\times\mathcal{O}, L^2(\mathcal{O})^n)$-valued stochastic variables $\boldsymbol{\phi}$ with $E\|\boldsymbol{\phi}\|_C < \infty$. $\|\boldsymbol{B}\|_F = [\mathrm{tr}(\boldsymbol{B}\boldsymbol{B}^T)]^{1/2}$ is the Frobenius norm, and $\|\boldsymbol{B}\|_{\max} = \max_{ij}\{|b_{ij}|\}$ is the max norm, where $\boldsymbol{B} = (b_{ij})_{n\times m}$, and tr is the trace operator. Let $\boldsymbol{W}(t,\boldsymbol{x}) = \sum_{n=1}^{\infty}\sqrt{\lambda_n}\beta_n(t)\boldsymbol{e}_n(\boldsymbol{x})$, where $\lambda_n \geq 0$ $(n = 1,2,\ldots,)$ are nonnegative real numbers, $\{\beta_n(t)\}_{n=1}^{\infty}$ is a sequence of standard Brownian motions mutually independent over $(\Omega,\mathscr{F},\mathbb{P})$, and $\{\boldsymbol{e}_n(\boldsymbol{x})\}_{n=1}^{\infty}$ is a complete orthonormal basis in $L^2(\mathcal{O})^m$. Let Q be a positive definite, self-adjoint, and Hilbert–Schmidt operator defined by $Qe_n = \lambda_n e_n$ with a finite trace $\mathrm{tr}Q = \sum_{n=1}^{\infty}\lambda_n < \infty$. $\mathfrak{L}_2^0(\Re, L^2(\mathcal{O})^n)$ is the space of all Hilbert–Schmidt operators from $\Re \triangleq Q^{(1/2)}(L^2(\mathcal{O})^n)$ into $L^2(\mathcal{O})^n$ with norm $\|\Phi\|_* \triangleq \sqrt{\mathrm{tr}(\Phi Q\Phi^*)}$, where Φ^* is the adjoint of Φ. $M_2^{n,m}[t_0,t]$ is the set of those nonanticipating functions for which the $n\times m$-matrix-valued functions $G(t,\omega)$ are with probability 1 satisfied $\int_{t_0}^{t}|G(s,\omega)|^2 ds<\infty$, and $M_2^{n,m} = \bigcap_{t>t_0}M_2^{n,m}[t_0,t]$. $\boldsymbol{u}(t_k^+)$ and $\boldsymbol{u}(t_k^-)$ represent the right-hand and left-hand limit of $\boldsymbol{u}(t_k)$, respectively. $PC([-\tau,0]\times\mathcal{O}, L^2(\mathcal{O})^n)$ is a Banach space of functions from $[-\tau,0]\times\mathcal{O}$ to $L^2(\mathcal{O})^n$, which are continuous everywhere except for some t_k at which $\boldsymbol{u}(t_k^-)$ and $\boldsymbol{u}(t_k^+)$ exist and $\boldsymbol{u}(t_k) = \boldsymbol{u}(t_k^+)$. Other notations are the same as those in [41].

In this article, we consider the following ISRDSs with delays:

$$\begin{cases} d\boldsymbol{u} = (\nabla\cdot(\boldsymbol{D}(\boldsymbol{x})\circ\nabla\boldsymbol{u}) + \boldsymbol{f}(t,\boldsymbol{u}_t))dt \\ \qquad + \boldsymbol{G}(t,\boldsymbol{u}_t)d\boldsymbol{W}(t,\boldsymbol{x}),\ t\neq t_k \\ \boldsymbol{u}(t_k) - \boldsymbol{u}(t_k^-) = \boldsymbol{P}_k\boldsymbol{u}(t_k^-),\ k\in\mathbb{Z}_+ \\ \boldsymbol{u}\ |_{\boldsymbol{x}\in\partial\mathcal{O}} = 0,\ t\geq t_0\geq 0 \\ \boldsymbol{u}(t_0+\theta,\boldsymbol{x},\omega) = \boldsymbol{\phi}(\theta,\boldsymbol{x},\omega)\in C^b_{\mathscr{F}_0} \\ \theta\in[-\tau,0],\ \boldsymbol{x}\in\mathcal{O},\ \omega\in\Omega \end{cases} \tag{1}$$

where $\boldsymbol{x} = (x_1,x_2,\ldots,x_l)^T \in \mathbb{R}^l$, $\boldsymbol{u} = (u_1(t,\boldsymbol{x},\omega), u_2(t,\boldsymbol{x},\omega),\ldots,u_n(t,\boldsymbol{x},\omega))^T$, $\boldsymbol{u}_t = \boldsymbol{u}(t+\theta,\boldsymbol{x},\omega) = (u_1(t+\theta,\boldsymbol{x},\omega), u_2(t+\theta,\boldsymbol{x},\omega),\ldots,u_n(t+\theta,\boldsymbol{x},\omega))^T$, $\theta\in[-\tau,0]$. $\boldsymbol{D}(\boldsymbol{x}) = (D_{ik}(\boldsymbol{x}))_{n\times l}$, $\nabla\boldsymbol{u} = (\nabla u_1,\nabla u_2,\ldots,\nabla u_n)^T$, $\nabla u_i = ((\partial u_i/\partial x_1),(\partial u_i/\partial x_2),\ldots,(\partial u_i/\partial x_l))$, $i = 1,2,\ldots,n$. $\nabla\cdot(\boldsymbol{D}(\boldsymbol{x})\circ\nabla\boldsymbol{u}) = (\sum_{j=1}^{l}[(\partial(D_{1j}(\boldsymbol{x})(\partial u_1/\partial x_j)))/\partial x_j], \sum_{j=1}^{l}[(\partial(D_{2j}(\boldsymbol{x})(\partial u_2/\partial x_j)))/\partial x_j],\ldots,\sum_{j=1}^{l}[(\partial(D_{nj}(\boldsymbol{x})(\partial u_n/\partial x_j)))/\partial x_j])^T$. $\boldsymbol{f}(t,\boldsymbol{u}_t) = (f_1(t,\boldsymbol{u}_t), f_2(t,\boldsymbol{u}_t),\ldots,f_n(t,\boldsymbol{u}_t))^T$, and $\boldsymbol{G} = (G_{ij})_{n\times m}\in M_2^{n,m}$ are the Borel measurable drift function and diffusion matrix, respectively. $\boldsymbol{\phi}(\theta,\boldsymbol{x},\omega) = (\phi_1(\theta,\boldsymbol{x},\omega),\phi_2(\theta,\boldsymbol{x},\omega),\ldots,\phi_n(\theta,\boldsymbol{x},\omega))^T$ is the initial data, and $\mathcal{O}$ is an open connected and bounded subset of $\mathbb{R}^l$ with a sufficiently regular boundary $\partial\mathcal{O}$. Moreover, the impulsive times t_k satisfy $t_0 < t_1 < \cdots < t_k < \cdots$, and $\lim_{k\to\infty}t_k = \infty$. $\boldsymbol{P}_k = \mathrm{diag}(p_{1k},p_{2k},\ldots,p_{nk})$ is the impulsive matrix at time t_k (see [20], [26], [37]). We assume that $\boldsymbol{u}$ is right continuous at $t = t_k$, that is, $\boldsymbol{u}(t_k) = \boldsymbol{u}(t_k^+)$. Hence, the solutions to (1) are piecewise right-hand continuous functions with discontinuities at $t = t_k$ for $k\in\mathbb{Z}_+$.

Throughout this article, we make the following assumptions.

(H₁): There exists $\alpha > 0$ such that $D_{ij}(\boldsymbol{x}) \geq \alpha$, $i = 1,2,\ldots,n$, $j = 1,2,\ldots,l$.

(H₂): There exists $\rho > 0$ such that $\|\boldsymbol{f}(t,\boldsymbol{u}) - \boldsymbol{f}(t,\boldsymbol{v})\| \vee \|\boldsymbol{G}(t,\boldsymbol{u}) - \boldsymbol{G}(t,\boldsymbol{v})\|_* \leq \rho\|\boldsymbol{u}-\boldsymbol{v}\|$, where $\boldsymbol{u},\boldsymbol{v}\in L^2(\mathcal{O})^n$.

(H₃): There exists $K > 0$ such that $\|\boldsymbol{f}(t,\boldsymbol{u})\|^2\vee\|\boldsymbol{G}(t,\boldsymbol{u})\|_*^2 \leq K^2(1+\|\boldsymbol{u}\|^2)$, where $\boldsymbol{u}\in L^2(\mathcal{O})^n$.

Here, we define a linear operator as follows:

$$\mathscr{A} : \mathcal{D}(\mathscr{A}) \to L^2(\mathcal{O})^n,\ \ \mathscr{A}\boldsymbol{u} = \nabla\cdot(\boldsymbol{D}(\boldsymbol{x})\circ\nabla\boldsymbol{u}) \tag{2}$$

where $\boldsymbol{u}\in\mathcal{D}(\mathscr{A})$, and $\mathcal{D}(\mathscr{A}) = H^2(\mathcal{O})^n\cap H_0^1(\mathcal{O})^n \subset L^2(\mathcal{O})^n$.

Definition 1 [37]: For $p > 0$, system (1) is said to be the pth moment practically exponentially stable if there exist positive constants λ, M_1, and M_2 such that for all $\boldsymbol{\phi}\in C^b_{\mathscr{F}_0}$

$$E\|\boldsymbol{u}(t,\boldsymbol{x},\omega)\|^p \leq M_1 E\|\boldsymbol{\phi}\|_C^p e^{-\lambda(t-t_0)} + M_2,\quad t\geq t_0. \tag{3}$$

Remark 1: The inequality (3) shows that $\boldsymbol{u}(t)$ is ultimately bounded by a small bound M_2, that is, $E\|\boldsymbol{u}(t,\boldsymbol{x},\omega)\|^p$ is small for sufficiently large t. As can be seen later, M_2 depends on $\boldsymbol{f}(t,\boldsymbol{0})$ and $\boldsymbol{G}(t,\boldsymbol{0})$ in this article, and in particular, $M_2 = 0$ if

This article has been accepted for inclusion in a future issue of this journal. Content is final as presented, with the exception of pagination.

$\boldsymbol{f}(t, \mathbf{0}) = \mathbf{0}$ and $\boldsymbol{G}(t, \mathbf{0}) = \mathbf{0}$. So the practical exponential stability we discuss in this article implies the exponential stability of the origin.

Lemma 1 (Poincaré Inequality [42], [43]): Let $\mathcal{O}$ be an open bounded domain in $\mathbb{R}^l$ with a smooth boundary, then $\|\boldsymbol{u}\| \leq \beta^{-1}|||\boldsymbol{u}|||$, $\boldsymbol{u} \in H_0^1(\mathcal{O})$, where β depends on the domain $\mathcal{O}$.

III. Practical Stability of Mild Solutions: Direct Approach

In this section, we discuss the practical stability of (1) using a direct approach. First, let $\delta = \sup_{k\in\mathbb{Z}_+} \|\boldsymbol{I} + \boldsymbol{P_k}\|^2_{\max}$.

Theorem 1: Let (H_1)–(H_3) hold. Suppose that $0 < \delta \leq 1$, and $2\alpha\beta^2 - 1 - 4\rho^2 > 0$. Then, system (1) is practically exponentially stable in the mean-square sense, and the convergence rate is greater than or equal to λ, where $\lambda - 2\alpha\beta^2 + 1 + 4\rho^2 e^{\lambda\tau} < 0$.

Proof: Like the proof in [44] and [45], we can obtain the existence–uniqueness of mild solution to system (1). Let $\boldsymbol{u}(t)$ be a mild solution to (1) and $V(t) = \|\boldsymbol{u}(t)\|^2$, then for $t \in (t_{k-1}, t_k)$

$$dV(t) = 2(\boldsymbol{u}, \mathscr{A}\boldsymbol{u})dt + 2(\boldsymbol{u}, \boldsymbol{f}(t, \boldsymbol{u}_t))dt + 2(\boldsymbol{u}, \boldsymbol{G}(t, \boldsymbol{u}_t)) \times dW(t, \boldsymbol{x}) + tr\big(\boldsymbol{G}(t, \boldsymbol{u}_t)Q\boldsymbol{G}^*(t, \boldsymbol{u}_t)\big)dt. \tag{4}$$

Integrating both sides of (4) from t_{k-1} to t, and then taking the expectation and the derivative may lead to

$$D^+EV(t) = 2E(\boldsymbol{u}, \mathscr{A}\boldsymbol{u}) + 2E(\boldsymbol{u}, \boldsymbol{f}(t, \boldsymbol{u}_t)) + E\|\boldsymbol{G}(t, \boldsymbol{u}_t)\|^2_*. \tag{5}$$

From (H_1), Lemma 1, and the Gauss formula, one obtains

$$\begin{aligned} 2E(\boldsymbol{u}, \mathscr{A}\boldsymbol{u}) &= -2E\int_{\mathcal{O}} \sum_{i=1}^{n}\sum_{j=1}^{l} D_{ij}\left(\frac{\partial u_i}{\partial x_j}\right)^2 d\boldsymbol{x} \\ &\leq -2\alpha E|||\boldsymbol{u}|||^2 \\ &\leq -2\alpha\beta^2 EV(t). \end{aligned} \tag{6}$$

It then follows from (H_2) and the Young inequality that:

$$2E(\boldsymbol{u}, \boldsymbol{f}(t, \boldsymbol{u}_t)) \leq EV(t) + 2\rho^2 E \sup_{s\in[t-\tau,t]} V(s) + 2\|\boldsymbol{f}(t, \mathbf{0})\|^2 \tag{7}$$

$$E\|\boldsymbol{G}(t, \boldsymbol{u}_t)\|^2 \leq 2\rho^2 E \sup_{s\in[t-\tau,t]} V(s) + 2\|\boldsymbol{G}(t, \mathbf{0})\|^2_*. \tag{8}$$

Therefore, according to (5)–(8) and (H_3), one can deduce that for $t \in (t_{k-1}, t_k)$

$$\begin{aligned} D^+EV(t) &\leq -\Big(2\alpha\beta^2 - 1\Big)EV(t) + 4\rho^2 E\overline{V(t)} \\ &\quad + 2\|\boldsymbol{f}(t, \mathbf{0})\|^2 + 2\|\boldsymbol{G}(t, \mathbf{0})\|^2_* \\ &\leq -\Big(2\alpha\beta^2 - 1\Big)EV(t) + 4\rho^2 E\overline{V(t)} + 4K^2 \end{aligned} \tag{9}$$

where $\overline{V(t)} = \sup_{s\in[t-\tau,t]} V(s)$. Since $2\alpha\beta^2 - 1 - 4\rho^2 > 0$, then there exists $\lambda > 0$ such that $\lambda - 2\alpha\beta^2 + 1 + 4\rho^2 e^{\lambda\tau} < 0$. Then, we choose $M > 1$, and let $\mu(t) = ME\|\boldsymbol{\phi}\|^2_C e^{-\lambda(t-t_0)} + [4K^2/(2\alpha\beta^2 - 1 - 4\rho^2)]$. Next, we claim that $EV(t) < \mu(t)$, $t \geq t_0$. We will first show that for $t \in [t_0, t_1)$, $EV(t) < \mu(t)$. In fact, it is clear that

$$\begin{aligned} EV(t_0) &\leq E\|\boldsymbol{\phi}\|^2_C < ME\|\boldsymbol{\phi}\|^2_C + \tfrac{4K^2}{2\alpha\beta^2-1-4\rho^2} \\ &= \mu(t_0). \end{aligned} \tag{10}$$

Then, suppose that there exists $t' \in (t_0, t_1)$ such that

$$EV(t') = \mu(t') \tag{11}$$

$$EV(t) < \mu(t), \quad t \in \big[t_0 - \tau, t'\big) \tag{12}$$

$$D^+EV(t') \geq D^+\mu(t'). \tag{13}$$

It then follows that:

$$\begin{aligned} D^+\mu(t') &> -\Big(2\alpha\beta^2 - 1\Big)EV(t') + 4\rho^2 E\overline{V(t')} + 4K^2 \\ &\geq D^+EV(t') \end{aligned} \tag{14}$$

which is a contradiction with (13). Thus, $EV(t) < \mu(t)$, $t \in [t_0, t_1)$. Note that $0 < \delta \leq 1$, then we have

$$\begin{aligned} EV(t_1) &\leq EV(t_1^-) < ME\|\boldsymbol{\phi}\|^2_C e^{-\lambda(t_1-t_0)} \\ &\quad + \tfrac{4K^2}{2\alpha\beta^2-1-4\rho^2}. \end{aligned} \tag{15}$$

Then, similar to the proof on $[t_0, t_1)$, we have $EV(t) < \mu(t)$, $t \in [t_1, t_2)$. By simple induction, it can be deduced that

$$E\|\boldsymbol{u}(t)\|^2 \leq ME\|\boldsymbol{\phi}\|^2_C e^{-\lambda(t-t_0)} + \tfrac{4K^2}{2\alpha\beta^2-1-4\rho^2}. \tag{16}$$

Therefore, system (1) is practically exponentially stable in the mean-square sense. ■

Theorem 2: Suppose that (H_1)–(H_3) hold, then we have the following.

1) If $0 < \delta \leq 1$, and
$$2\alpha\beta^2 - 1 - \tfrac{\ln\delta}{\overline{h}} - \tfrac{4\rho^2}{\delta} > 0 \tag{17}$$
where $\overline{h} = \sup_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} < \infty$, then (1) is practically exponentially stable in the mean-square sense, and the convergence rate is greater than or equal to λ, where λ satisfies $\lambda - 2\alpha\beta^2 + 1 + (\ln\delta/\overline{h}) + (4\rho^2/\delta)e^{\lambda\tau} = 0$.

2) If $\delta > 1$, and
$$2\alpha\beta^2 - 1 - \tfrac{\ln\delta}{\underline{h}} - 4\rho^2\delta > 0 \tag{18}$$
where $\underline{h} = \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} < \infty$, then (1) is practically exponentially stable in the mean-square sense, and the convergence rate is greater than or equal to λ, where λ satisfies $\lambda - 2\alpha\beta^2 + 1 + (\ln\delta/\underline{h}) + 4\rho^2\delta e^{\lambda\tau} = 0$.

Proof: Let $V(t) = \|\boldsymbol{u}(t)\|^2$. According to the proof of Theorem 1, one can obtain that for $t \in (t_{k-1}, t_k)$

$$\begin{aligned} D^+EV(t) &\leq -(2\alpha\beta^2 - 1)EV(t) + 4\rho^2 E\overline{V(t)} \\ &\quad + 4K^2. \end{aligned} \tag{19}$$

Also

$$EV(t_k) = E\|\boldsymbol{u}(t_k)\|^2 \leq \|\boldsymbol{I} + \boldsymbol{P_k}\|^2_{\max} EV(t_k^-). \tag{20}$$

For any $\varepsilon > 0$, let $\nu(t)$ be a solution to the following systems:

$$\begin{cases} D^+\nu(t) = -\big(2\alpha\beta^2 - 1\big)\nu(t) + 4\rho^2\overline{\nu(t)} \\ \qquad\quad + 4K^2 + \varepsilon, \quad t \neq t_k \\ \nu(t_k) = \|\boldsymbol{I} + \boldsymbol{P_k}\|^2_{\max}\nu(t_k^-), \quad k \in \mathbb{Z}_+ \\ \nu(t_0 + \theta) = E\|\boldsymbol{\phi}(\theta, \boldsymbol{x}, \omega)\|^2, \quad \theta \in [-\tau, 0]. \end{cases} \tag{21}$$

Then, in terms of [20, Lemma 3], we can obtain that

$$EV(t) \le \nu(t), \quad t \ge t_0. \tag{22}$$

From the formula for the variation of parameters in [25], we obtain

$$\nu(t) = W(t,t_0)\nu(t_0) + \int_{t_0}^{t} W(t,s) \times \left[4\rho^2\overline{\nu(s)} + 4K^2 + \varepsilon\right]ds \tag{23}$$

where $W(t,s)$, $t, s \ge 0$ is the Cauchy matrix of linear system

$$\begin{cases} D^+ w(t) = -\left(2\alpha\beta^2 - 1\right)w(t), & t \ne t_k \\ w(t_k) = \|\boldsymbol{I} + \boldsymbol{P_k}\|_{\max}^2 w\left(t_k^-\right), & k \in \mathbb{Z}_+. \end{cases} \tag{24}$$

Case (I): If $0 < \delta \le 1$, then we define $a = 2\alpha\beta^2 - 1 - (\ln\delta/\overline{h})$. By the representation of the Cauchy matrix, we obtain

$$W(t,s) = e^{-(2\alpha\beta^2-1)(t-s)} \prod_{s<t_k\le t} \|\boldsymbol{I} + \boldsymbol{P_k}\|_{\max}^2 \le e^{-\left(a+\frac{\ln\delta}{\overline{h}}\right)(t-s)} \delta^{\frac{t-s}{\overline{h}}-1} = \frac{1}{\delta}e^{-a(t-s)} \tag{25}$$

where $t \ge s \ge t_0$. Accordingly, for $t \ge t_0$

$$\nu(t) \le \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-a(t-t_0)} + \int_{t_0}^{t} \frac{1}{\delta}e^{-a(t-s)} \times \left[4\rho^2\overline{\nu(s)} + 4K^2 + \varepsilon\right]ds. \tag{26}$$

Notice that $2\alpha\beta^2 - 1 - (\ln\delta/\overline{h}) - (4\rho^2/\delta) > 0$, then there is a constant $\lambda > 0$ such that $\lambda - 2\alpha\beta^2 + 1 + (\ln\delta/\overline{h}) + (4\rho^2/\delta)e^{\lambda\tau} = 0$. It is clear that for $t \in [t_0 - \tau, t_0]$

$$\nu(t) \le E\|\boldsymbol{\phi}\|_C^2 \le \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-\lambda(t-t_0)} + \frac{4K^2+\varepsilon}{\left(2\alpha\beta^2-1-\frac{\ln\delta}{\overline{h}}-\frac{4\rho^2}{\delta}\right)\delta}. \tag{27}$$

Next, we claim that for $t \ge t_0$

$$\nu(t) \le \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-\lambda(t-t_0)} + \frac{4K^2+\varepsilon}{\left(2\alpha\beta^2-1-\frac{\ln\delta}{\overline{h}}-\frac{4\rho^2}{\delta}\right)\delta}. \tag{28}$$

If this is not true, then there exists $t^* > t_0$ such that

$$\nu\left(t^*\right) > \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-\lambda(t^*-t_0)} + \frac{4K^2+\varepsilon}{\left(2\alpha\beta^2-1-\frac{\ln\delta}{\overline{h}}-\frac{4\rho^2}{\delta}\right)\delta} \tag{29}$$

$$\nu(t) \le \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-\lambda(t-t_0)} + \frac{4K^2+\varepsilon}{\left(2\alpha\beta^2-1-\frac{\ln\delta}{\overline{h}}-\frac{4\rho^2}{\delta}\right)\delta} \tag{30}$$

where $t \in [t_0 - \tau, t^*)$. Thus

$$\begin{aligned} \nu\left(t^*\right) \le{}& \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-a(t^*-t_0)} + \int_{t_0}^{t^*} \frac{1}{\delta}e^{-a(t^*-s)} \\ &\times \left[4\rho^2 \cdot \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-\lambda(s-\tau-t_0)} + 4K^2 + \varepsilon \right. \\ &\left. + 4\rho^2\frac{4K^2+\varepsilon}{\left(2\alpha\beta^2-1-\frac{\ln\delta}{\overline{h}}-\frac{4\rho^2}{\delta}\right)\delta}\right]ds \\ ={}& \int_{t_0}^{t^*} \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{\lambda t_0 - at^*}(a-\lambda)e^{(a-\lambda)s}ds \\ &+ \int_{t_0}^{t^*} \frac{4K^2+\varepsilon}{\left(2\alpha\beta^2-1-\frac{\ln\delta}{\overline{h}}-\frac{4\rho^2}{\delta}\right)\delta}ae^{-a(t^*-s)}ds \\ &+ \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-a(t^*-t_0)} \\ <{}& \frac{E\|\boldsymbol{\phi}\|_C^2}{\delta}e^{-\lambda(t^*-t_0)} + \frac{4K^2+\varepsilon}{\left(2\alpha\beta^2-1-\frac{\ln\delta}{\overline{h}}-\frac{4\rho^2}{\delta}\right)\delta} \end{aligned} \tag{31}$$

which contradicts (29). Therefore, (28) holds, and the proof is completed.

Case (II): If $\delta > 1$, then let $a = 2\alpha\beta^2 - 1 - (\ln\delta/\underline{h})$. By the representation of the Cauchy matrix, we obtain

$$W(t,s) = e^{-(2\alpha\beta^2-1)(t-s)} \prod_{s<t_k\le t} \|\boldsymbol{I} + \boldsymbol{P_k}\|_{\max}^2 \le e^{-\left(a+\frac{\ln\delta}{\underline{h}}\right)(t-s)} \delta^{\frac{t-s}{\underline{h}}+1} = \delta e^{-a(t-s)} \tag{32}$$

where $t \ge s \ge t_0$. Similar to the proof of case (I), we have

$$\nu(t) \le \delta E\|\boldsymbol{\phi}\|_C^2 e^{-\lambda(t-t_0)} + \delta\frac{4K^2+\varepsilon}{2\alpha\beta^2-1-\frac{\ln\delta}{\underline{h}}-\frac{4\rho^2}{\delta}} \tag{33}$$

which completes the proof. ■

Remark 2: When $\boldsymbol{f}(t,\boldsymbol{0}) = \boldsymbol{0}$ and $\boldsymbol{G}(t,\boldsymbol{0}) = \boldsymbol{0}$, one can infer the exponential stability of the trivial solution to (1) from the proof of Theorem 2.

Remark 3: We notice that (17) may imply $2\alpha\beta^2 - 1 - 4\rho^2 > 0$ if $4\rho^2\overline{h} \ge 1$. In this case, Theorem 1 includes some results of Theorem 2-1).

IV. Practical Stability of Mild Solutions: The Lyapunov Method

In this section, we develop the Lyapunov method to study the pth moment practical exponential stability.

Theorem 3: System (1) is the pth moment practically exponentially stable if there exist constants $\omega_1 > 0$, $\omega_2 > 0$, $a > 0$, $b > 0$, $c \ge 0$, $\sigma_k > 0$, $\gamma > 1$, $\overline{h} > 0$, and $\mathcal{N} \in \mathbb{Z}_+$ and a function $V \in C^{1,2}([t_0 - \tau, \infty) \times L^2(\mathcal{O})^n; \mathbb{R}_+)$ such that:

1) $\omega_1\|\boldsymbol{u}\|^p \le V(t,\boldsymbol{u}(t)) \le \omega_2\|\boldsymbol{u}\|^p$;
2) $LV(t,\boldsymbol{u}(t)) \le aV(t,\boldsymbol{u}(t)) + b\overline{V(t,\boldsymbol{u}(t))} + c$, where $\overline{V(t,\boldsymbol{u}(t))} = \sup_{s\in[t-\tau,t]} V(s,\boldsymbol{u}(s))$, $t \ge t_0$, $t \ne t_k$, $k \in \mathbb{Z}_+$;
3) $EV(t_k, \boldsymbol{u}(t_k^-) + \boldsymbol{P_k}\boldsymbol{u}(t_k^-)) \le (1/\sigma_k)EV(t_k^-, \boldsymbol{u}(t_k^-))$, where $\sigma_{\mathcal{N}+k} = \sigma_k$, $k \in \mathbb{Z}_+$;
4) $\overline{h} = \sup_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} < \infty$ and $a\overline{h} + b\sigma_{\mathcal{N}}\overline{h} < \ln\gamma$;
5)

$$\begin{cases} \prod_{1\le j\le\mathcal{N}-1}\left(\frac{\gamma}{\sigma_j} \vee 1\right) \le \frac{\sigma_{\mathcal{N}}}{\gamma}, & \mathcal{N} \ge 2 \\ \sigma_k \equiv \gamma, & \mathcal{N} = 1. \end{cases}$$

Moreover, the convergence rate is greater than or equal to λ, where λ satisfies $a\overline{h} + b\sigma_{\mathcal{N}}\overline{h}e^{\lambda\tau} < \ln\gamma - \lambda\overline{h}$.

Proof: On the basis of Condition 4), one can obtain that there exist $\lambda > 0$ and $\varepsilon_0 > 0$ such that

$$a\overline{h} + \frac{\gamma+\varepsilon_0}{\gamma}b\sigma_{\mathcal{N}}\overline{h}e^{\lambda\tau} < \ln\gamma - \lambda\overline{h}. \tag{34}$$

Let $V(t) = V(t,\boldsymbol{u}(t))$, $V_0 = \sup_{s\in[t_0-\tau,t_0]} V(s)$, and $\Psi(t) = V(t)e^{\lambda(t-t_0)}$, $t \ge t_0$.

First, we claim that for any $\varepsilon \in (0, \varepsilon_0]$

$$E\Psi(t) < (\gamma+\varepsilon)\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t-t_0)}\right), \quad t \in [t_0, t_1). \tag{35}$$

It is easy to find that $E\Psi(t_0) = EV(t_0) \le EV_0 + (c/\lambda)$. If (35) is not true for $t \in (t_0, t_1)$, then there exist $t_0 \le \underline{t} < \bar{t} < t_1$ such that

$$E\Psi(\bar{t}) = (\gamma+\varepsilon)\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(\bar{t}-t_0)}\Big) \tag{36}$$

$$E\Psi(\underline{t}) = EV_0 + \tfrac{c}{\lambda}e^{\lambda(\underline{t}-t_0)} \tag{37}$$

$$E\Psi(t) \ge EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}, \quad t \in [\underline{t}, \bar{t}] \tag{38}$$

$$E\Psi(t) \le (\gamma+\varepsilon)\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big), \quad t \in [\underline{t}, \bar{t}]. \tag{39}$$

Also, for $t \in [t_0 - \tau, \bar{t}]$

$$EV(t)\Big[e^{\lambda(t-t_0)} \vee 1\Big] \le (\gamma+\varepsilon)\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big). \tag{40}$$

Then, one can obtain that for $t \in [\bar{t}, \underline{t}]$

$$\begin{aligned} D^+E\Psi(t) &= e^{\lambda(t-t_0)}\big[aEV(t) + \lambda EV(t) + bE\overline{V(t)} + c\big] \\ &\le (a+\lambda)e^{\lambda(t-t_0)}EV(t) + be^{\lambda\tau}(\gamma+\varepsilon) \\ &\quad \times \Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big) + ce^{\lambda(t-t_0)} \\ &\le E\Psi(t)[a + \lambda + be^{\lambda\tau}(\gamma+\varepsilon)] + ce^{\lambda(t-t_0)} \end{aligned}$$

which implies that

$$\begin{aligned} E\Psi(\bar{t}) &\le E\Psi(\underline{t})e^{\int_{\underline{t}}^{\bar{t}}(a+\lambda+be^{\lambda\tau}(\gamma+\varepsilon))dt} \\ &\quad + \int_{\underline{t}}^{\bar{t}} ce^{\lambda(s-t_0)}e^{\int_s^{\bar{t}}(a+\lambda+be^{\lambda\tau}(\gamma+\varepsilon))dt}ds. \end{aligned}$$

Note that $(\sigma_{\mathcal{N}}/\gamma) \ge 1$. Together with (34), it then follows that:

$$\begin{aligned} &(\gamma+\varepsilon)\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(\bar{t}-t_0)}\Big) \\ &\le \gamma^{\frac{\bar{t}-\underline{t}}{h}}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(\underline{t}-t_0)}\Big) + \gamma^{\frac{\bar{t}-\underline{t}}{h}} c\int_{\underline{t}}^{\bar{t}} e^{\lambda(s-t_0)}ds \\ &< \gamma\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(\bar{t}-t_0)}\Big) \end{aligned} \tag{41}$$

which is a contradiction, so (35) holds. From the arbitrary of ε, we have $E\Psi(t) \le \gamma(EV_0 + (c/\lambda)e^{\lambda(t-t_0)})$, $t \in [t_0, t_1)$. Notice that

$$\begin{aligned} E\Psi(t_1) &= EV(t_1)e^{\lambda(t_1-t_0)} \le \tfrac{1}{\sigma_1}EV(t_1^-)e^{\lambda(t_1-t_0)} \\ &\le \tfrac{\gamma}{\sigma_1}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t_1-t_0)}\Big). \end{aligned} \tag{42}$$

There are two cases. If $(\gamma/\sigma_1) \le 1$, then $E\Psi(t_1) \le EV_0 + (c/\lambda)e^{\lambda(t_1-t_0)}$. Similar to the above discussion on $[t_0, t_1)$, we can deduce that $E\Psi(t) \le \gamma(EV_0+(c/\lambda)e^{\lambda(t-t_0)})$, $t \in [t_1, t_2)$. If $(\gamma/\sigma_1) > 1$, then we can derive that $E\Psi(t) \le (\gamma^2/\sigma_1)(EV_0 + (c/\lambda)e^{\lambda(t-t_0)})$, $t \in [t_1, t_2)$. In fact, we only need to prove that, for any $\varepsilon \in (0, \varepsilon_0]$, $E\Psi(t) < [(\gamma(\gamma+\varepsilon))/\sigma_1](EV_0 + (c/\lambda)e^{\lambda(t-t_0)})$, $t \in [t_1, t_2)$. Suppose that this is not true, then one may choose $t_1 \le t_* < t^* < t_2$ such that

$$E\Psi(t^*) = \tfrac{\gamma(\gamma+\varepsilon)}{\sigma_1}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t^*-t_0)}\Big) \tag{43}$$

$$E\Psi(t_*) = \tfrac{\gamma}{\sigma_1}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t_*-t_0)}\Big) \tag{44}$$

$$E\Psi(t) \ge \tfrac{\gamma}{\sigma_1}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big), \ t \in [t_*, t^*] \tag{45}$$

$$E\Psi(t) \le \tfrac{\gamma(\gamma+\varepsilon)}{\sigma_1}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big), \ t \in [t_*, t^*] \tag{46}$$

and considering $E\Psi(t) \le \gamma(EV_0 + (c/\lambda)e^{\lambda(t-t_0)})$, $t \in [t_0, t_1)$, we have

$$EV(t)\Big[e^{\lambda(t-t_0)} \vee 1\Big] \le \tfrac{\gamma(\gamma+\varepsilon)}{\sigma_1}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big) \tag{47}$$

where $t \in [t_0 - \tau, t^*]$. Then

$$D^+E\Psi(t) \le E\Psi(t)\big[a + \lambda + be^{\lambda\tau}(\gamma+\varepsilon)\big] + ce^{\lambda(t-t_0)} \tag{48}$$

where $t \in [t_*, t^*]$. Similar to (41), we have

$$\begin{aligned} &\tfrac{\gamma(\gamma+\varepsilon)}{\sigma_1}\Big(EV_0 + \frac{c}{\lambda}e^{\lambda(t^*-t_0)}\Big) \\ &\le \gamma^{\frac{t^*-t_*}{h}}\tfrac{\gamma}{\sigma_1}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t_*-t_0)}\Big) + \gamma^{\frac{t^*-t_*}{h}}\tfrac{c}{\lambda} \\ &\times \tfrac{\gamma}{\sigma_1}\Big(e^{\lambda(t^*-t_0)} - e^{\lambda(t_*-t_0)}\Big) \\ &\le \tfrac{\gamma^2}{\sigma_1}\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t^*-t_0)}\Big) \end{aligned} \tag{49}$$

which leads to a contradiction. So, if $(\gamma/\sigma_1) > 1$, we have $E\Psi(t) \le (\gamma^2/\sigma_1)(EV_0 + (c/\lambda)e^{\lambda(t-t_0)})$, $t \in [t_1, t_2)$. Then, it can be derived that

$$E\Psi(t) \le \Big(\tfrac{\gamma}{\sigma_1} \vee 1\Big)\gamma\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big), \quad t \in [t_0, t_2). \tag{50}$$

If $\mathcal{N} = 1$, then $\sigma_k \equiv \gamma$. From (50), one may obtain

$$E\Psi(t) \le \gamma\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big), \quad t \in [t_0, t_2). \tag{51}$$

Notice that $E\Psi(t_2) \le (1/\sigma_2)E\Psi(t_2^-) \le (\gamma/\sigma_2)(EV_0 + (c/\lambda)e^{\lambda(t_2-t_0)}) = EV_0 + (c/\lambda)e^{\lambda(t_2-t_0)}$. Similar to the proof on $[t_0, t_1)$, we can deduce that

$$E\Psi(t) \le \gamma\Big(EV_0 + \tfrac{c}{\lambda}e^{\lambda(t-t_0)}\Big), \quad t \ge t_0, \quad \mathcal{N} = 1. \tag{52}$$

If $\mathcal{N} > 1$, suppose that

$$E\Psi(t) \le \prod_{1\le j\le l-1}\Big(\frac{\gamma}{\sigma_j} \vee 1\Big)\gamma\Big(EV_0 + \frac{c}{\lambda}e^{\lambda(t-t_0)}\Big) \tag{53}$$

where $t \in [t_0, t_l)$, $2 \le l < \mathcal{N}$, and $l \in \mathbb{Z}_+$. Next, we will prove that for any $\varepsilon \in (0, \varepsilon_0]$

$$E\Psi(t) < \prod_{1\le j\le l}\Big(\frac{\gamma}{\sigma_j} \vee 1\Big)(\gamma+\varepsilon)\Big(EV_0 + \frac{c}{\lambda}e^{\lambda(t-t_0)}\Big) \tag{54}$$

where $t \in [t_l, t_{l+1})$. This may lead to

$$E\Psi(t) \le \prod_{1\le j\le l}\Big(\frac{\gamma}{\sigma_j} \vee 1\Big)\gamma\Big(EV_0 + \frac{c}{\lambda}e^{\lambda(t-t_0)}\Big) \tag{55}$$

where $t \in [t_0, t_{l+1})$. It follows from (53) that:

$$\begin{aligned} E\Psi(t_l) &\le \frac{1}{\sigma_l}EV(t_l^-)e^{\lambda(t_l-t_0)} \\ &\le \prod_{1\le j\le l}\Big(\frac{\gamma}{\sigma_j} \vee 1\Big)\Big(EV_0 + \frac{c}{\lambda}e^{\lambda(t_l-t_0)}\Big). \end{aligned} \tag{56}$$

If (54) does not hold, then there exist $t_l \le t_\alpha < t^\alpha < t_{l+1}$ such that

$$\begin{aligned} E\Psi(t^\alpha) &= \prod_{1\le j\le l}\Big(\frac{\gamma}{\sigma_j} \vee 1\Big)(\gamma+\varepsilon) \\ &\quad \times \Big(EV_0 + \frac{c}{\lambda}e^{\lambda(t^\alpha-t_0)}\Big) \end{aligned}$$

$$E\Psi(t_\alpha) = \prod_{1\le j\le l}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t_\alpha - t_0)}\right)$$
$$E\Psi(t) \ge \prod_{1\le j\le l}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right)$$
$$t\in\left[t_\alpha, t^\alpha\right]$$
$$E\Psi(t) \le \prod_{1\le j\le l}\left(\frac{\gamma}{\sigma_j}\vee 1\right)(\gamma+\varepsilon)$$
$$\times\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right),\ t\in\left[t_\alpha, t^\alpha\right]$$
$$EV(t)\left[e^{\lambda(t-t_0)}\vee 1\right] \le \prod_{1\le j\le l}\left(\frac{\gamma}{\sigma_j}\vee 1\right)(\gamma+\varepsilon)$$
$$\times\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right)$$
$$t\in\left[t_0-\tau, t^\alpha\right].$$

From Condition 2), we can then observe that for $t \in [t_\alpha, t^\alpha]$

$$D^+E\Psi(t) \le E\Psi(t)\left[a+\lambda+be^{\lambda\tau}(\gamma+\varepsilon)\right] + ce^{\lambda(t-t_0)}.$$

Like (49), we can obtain that this contradicts with (34). Thus, (55) holds, which implies that

$$E\Psi(t) \le \prod_{1\le j\le \mathcal{N}-1}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\gamma\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right) \quad (57)$$

where $t \in [t_0, t_{\mathcal{N}})$. By Condition 5), one can obtain that

$$E\Psi(t_{\mathcal{N}}) \le \frac{\gamma}{\sigma_{\mathcal{N}}}\prod_{1\le j\le \mathcal{N}-1}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t_{\mathcal{N}} - t_0)}\right)$$
$$\le EV_0 + \frac{c}{\lambda}e^{\lambda(t_{\mathcal{N}} - t_0)}. \quad (58)$$

Next, we claim that $E\Psi(t) \le \gamma(EV_0 + (c/\lambda)e^{\lambda(t-t_0)})$, $t \in [t_{\mathcal{N}}, t_{\mathcal{N}+1})$, which is equal to prove, for any $\varepsilon \in (0, \varepsilon_0]$

$$E\Psi(t) < (\gamma+\varepsilon)\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right),\quad t\in\left[t_{\mathcal{N}}, t_{\mathcal{N}+1}\right). \quad (59)$$

Similarly, we assume that this is not true, which implies that we can choose $t_{\mathcal{N}} \le t_\beta < t^\beta < t_{\mathcal{N}+1}$ such that

$$E\Psi\left(t^\beta\right) = (\gamma+\varepsilon)\left(EV_0 + \frac{c}{\lambda}e^{\lambda\left(t^\beta - t_0\right)}\right) \quad (60)$$
$$E\Psi\left(t_\beta\right) = EV_0 + \frac{c}{\lambda}e^{\lambda\left(t_\beta - t_0\right)} \quad (61)$$
$$E\Psi(t) \ge EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)},\ t\in\left[t_\beta, t^\beta\right] \quad (62)$$
$$E\Psi(t) \le (\gamma+\varepsilon)\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right),\ t\in\left[t_\beta, t^\beta\right]. \quad (63)$$

Also

$$EV(t)\left[e^{\lambda(t-t_0)}\vee 1\right] \le \prod_{1\le j\le \mathcal{N}-1}\left(\frac{\gamma}{\sigma_j}\vee 1\right)(\gamma+\varepsilon)$$
$$\times\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right)$$
$$\le \frac{\sigma_{\mathcal{N}}}{\gamma}(\gamma+\varepsilon)\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right)$$
$$t\in\left[t_0-\tau, t^\beta\right] \quad (64)$$

which leads to

$$D^+E\Psi(t) \le (a+\lambda)E\Psi(t) + b\frac{\sigma_{\mathcal{N}}}{\gamma}(\gamma+\varepsilon)$$
$$\times\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right)e^{\lambda\tau} + ce^{\lambda(t-t_0)}$$
$$\le E\Psi(t)\left[a+\lambda+be^{\lambda\tau}\frac{\sigma_{\mathcal{N}}}{\gamma}(\gamma+\varepsilon)\right]$$
$$+ ce^{\lambda(t-t_0)},\quad t\in\left[t_\beta, t^\beta\right]. \quad (65)$$

Then, combining (34) and (60) with (61), one can observe that

$$(\gamma+\varepsilon)\left(EV_0 + \frac{c}{\lambda}e^{\lambda\left(t^\beta - t_0\right)}\right) \le \gamma\left(EV_0 + \frac{c}{\lambda}e^{\lambda\left(t^\beta - t_0\right)}\right) \quad (66)$$

which is a contradiction. Thus, (59) holds. In this way, we have

$$\begin{cases} E\Psi(t) \le \gamma\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t-t_0)}\right), \quad t\in\left[t_{\mathcal{N}}, t_{\mathcal{N}+1}\right) \\ E\Psi(t) \le \left(\frac{\gamma}{\sigma_1}\vee 1\right)\gamma\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t-t_0)}\right), \quad t\in\left[t_{\mathcal{N}+1}, t_{\mathcal{N}+2}\right) \\ \cdots \\ E\Psi(t) \le \prod_{1\le j\le \mathcal{N}-1}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\gamma\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t-t_0)}\right) \\ \qquad t\in\left[t_{2\mathcal{N}-1}, t_{2\mathcal{N}}\right) \\ E\Psi(t) \le \gamma\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t-t_0)}\right), \quad t\in\left[t_{2\mathcal{N}}, t_{2\mathcal{N}+1}\right) \\ \cdots \end{cases}$$

Therefore, it can be derived that

$$E\Psi(t) \le \max_{1\le i\le \mathcal{N}}\prod_{1\le j\le i}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\gamma\left(EV_0 + \frac{c}{\lambda}e^{\lambda(t - t_0)}\right),\quad t\ge t_0.$$

That is, for $t \ge t_0$

$$EV(t) \le \max_{1\le i\le \mathcal{N}}\prod_{1\le j\le i}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\gamma EV_0 e^{-\lambda(t-t_0)}$$
$$+ \max_{1\le i\le \mathcal{N}}\prod_{1\le j\le i}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\gamma\frac{c}{\lambda}$$

that is

$$E\|\boldsymbol{u}(t)\|^p \le \max_{1\le i\le \mathcal{N}}\prod_{1\le j\le i}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\frac{\gamma\omega_2}{\omega_1}E\|\boldsymbol{\phi}\|_C^p e^{-\lambda(t-t_0)}$$
$$+ \max_{1\le i\le \mathcal{N}}\prod_{1\le j\le i}\left(\frac{\gamma}{\sigma_j}\vee 1\right)\frac{\gamma c}{\lambda\omega_1}$$

and the proof is completed. ■

Remark 4: We mention that Condition 3) in Theorem 3 means the impulses are periodic. Especially, if $\mathcal{N} = 1$, that is, $\sigma_k = \sigma$, then it follows from the definition of σ_k in Condition 3) that the system is subject to stabilizing impulses.

If $\boldsymbol{P}_k = 0$, $k = 1, 2, \ldots$, in (1), then similar to the proof of Theorem 3, we have the following pth moment practical exponential stability for the system (1) without impulses.

Theorem 4: System (1) without impulses is the pth moment practically exponentially stable if there exist constants $\omega_1 > 0$, $\omega_2 > 0$, $a > 0$, $b > 0$, and $c \ge 0$, and a function $V \in C^{1,2}([t_0-\tau, \infty) \times L^2(\mathcal{O})^n; \mathbb{R}_+)$ such that:

1) $\omega_1\|\boldsymbol{u}\|^p \le V(t, \boldsymbol{u}(t)) \le \omega_2\|\boldsymbol{u}\|^p$;
2) $LV(t, \boldsymbol{u}(t)) \le aV(t, \boldsymbol{u}(t)) + b\overline{V(t, \boldsymbol{u}(t))} + c$, where $\overline{V(t, \boldsymbol{u}(t))} = \sup_{s\in[t-\tau,t]} V(s, \boldsymbol{u}(s))$, $t \ge t_0$, $t \ne t_k$, $k \in \mathbb{Z}_+$;
3) $a + b < 0$.

Define $V(t, \boldsymbol{u}(t)) = \|\boldsymbol{u}(t)\|^2$ in Theorem 3, then we have the following theorem.

Theorem 5: Assume that (H_1)–(H_3) hold. If there exist $\gamma > 1$ and $\mathcal{N} \in \mathbb{Z}_+$ such that $2\alpha\beta^2 - 1 - [4\rho^2/(\delta_{\mathcal{N}}) + (\ln \gamma/\overline{h})] > 0$, $\delta_{\mathcal{N}+k} = \delta_k$, and

$$\begin{cases} \gamma^{\mathcal{N}} \prod_{1\le j\le \mathcal{N}-1}\left(\delta_j \vee \frac{1}{\gamma}\right) \le \frac{1}{\delta_{\mathcal{N}}}, & \mathcal{N} \ge 2 \\ \delta_k \equiv \frac{1}{\gamma}, & \mathcal{N} = 1 \end{cases}$$

where $\delta_k = \|\boldsymbol{I}+\boldsymbol{P_k}\|^2_{\max} > 0$ and $\overline{h} = \sup_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} < \infty$, then (1) is practically exponentially stable in the mean-square sense.

Remark 5: Notice that if $\mathcal{N} = 1$, then Theorem 5 is consistent with Theorem 2 case (I), which means Theorem 5 contains some of the results in Theorem 2 case (I). It is also worthwhile pointing out that, when the product of all δ_k is greater than 1, Theorem 2 case (II) may work, but Theorem 5 cannot. However, if the product is less than 1, and there are much greater impulses, then we can apply Theorem 5 to discuss the practical stability of systems. Therefore, Theorems 2 and 5 can be used for different systems.

Remark 6: According to Theorems 3 and 4, one may deduce the exponential stability of the trivial solution to (1) if $c = 0$ in Condition 2). Similarly, if $\boldsymbol{f}(t, \boldsymbol{u})$ and $\boldsymbol{G}(t, \boldsymbol{u})$ satisfy $\boldsymbol{f}(t, \boldsymbol{0}) = \boldsymbol{0}$ and $\boldsymbol{G}(t, \boldsymbol{0}) = \boldsymbol{0}$, it then follows from Theorem 5 that the trivial solution is exponentially stable.

Remark 7: Note that if $\boldsymbol{D} = \boldsymbol{0}$, then (1) becomes the impulsive stochastic system with delays. Caraballo *et al.* [37] discussed the practical stability of the system with stabilizing impulses by the Lyapunov method, and Wang *et al.* [38] studied the stabilization problem. Letting $\boldsymbol{P}_k = \boldsymbol{0}$, then (1) is the stochastic reaction–diffusion systems with delays, and the exponential stability has been investigated in [39] and [40]. So Theorems 1–4 include some results in [37]–[40] as special cases.

Remark 8: Theorems 2 and 5 provide some sufficient conditions for practical exponential stability. These can be viewed as stabilization results because systems without impulsive effects may be unstable, while the ones with impulses may become practically stable, which will be verified in Example 3.

V. Applications

In this section, we consider the following IRDSHNNs with delays:

$$\begin{cases} d\boldsymbol{u} = (\mathscr{A}\boldsymbol{u} - \boldsymbol{A}\boldsymbol{u} + \boldsymbol{C}\boldsymbol{f}(\boldsymbol{u}(t-\tau, \boldsymbol{x})) + \boldsymbol{J})dt \\ \qquad + \boldsymbol{G}(\boldsymbol{u}(t-\tau, \boldsymbol{x}))d\boldsymbol{W}(t, \boldsymbol{x}), \ \ t \ne t_k \\ \boldsymbol{u}(t_k, \boldsymbol{x}) - \boldsymbol{u}\left(t_k^-, \boldsymbol{x}\right) = \boldsymbol{P_k}\boldsymbol{u}\left(t_k^-, \boldsymbol{x}\right), \ \ k \in \mathbb{Z}_+ \\ \boldsymbol{u}(t, \boldsymbol{x}) \mid_{x\in\partial\mathcal{O}} = 0, \ \ t \ge 0 \\ \boldsymbol{u}(\theta, \boldsymbol{x}) = \boldsymbol{\phi}(\theta, \boldsymbol{x}) \in C^b_{\mathscr{F}_0}, \ \ -\tau \le \theta \le 0, \ \ \boldsymbol{x} \in \mathcal{O} \end{cases} \tag{67}$$

where $\boldsymbol{x} \in \mathbb{R}^l$, $\omega \in \Omega$, $\boldsymbol{u} = (u_1(t, \boldsymbol{x}, \omega), u_2(t, \boldsymbol{x}, \omega), \ldots, u_n(t, \boldsymbol{x}, \omega))^T$. $\boldsymbol{A} = \operatorname{diag}(a_1, a_2, \ldots, a_n)$, $a_{\min} = \min\{a_1, a_2, \ldots, a_n\}$, $a_i > 0$, $i = 1, 2, \ldots, n$. $\boldsymbol{C} = (c_{ij})_{n\times n}$, $\boldsymbol{J} = (J_1, J_2, \ldots, J_n)^T$, $\boldsymbol{P_k} = \operatorname{diag}(p_{1k}, p_{2k}, \ldots, p_{nk})$. $\boldsymbol{f}(\boldsymbol{u}) = (f_1(u_1), f_2(u_2), \ldots, f_n(u_n))^T$, $\boldsymbol{G} = (G_{ij})_{n\times m} \in M_2^{n,m}$. The physical meanings of parameters of (67) are similar to those in [8].

We make the following assumptions for the neural networks.

(A_1): There exists $\alpha > 0$ such that $D_{ij}(\boldsymbol{x}) \ge \alpha$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, l$.

(A_2): There exists $\rho \ge 0$ such that $\|\boldsymbol{f}(\boldsymbol{u}) - \boldsymbol{f}(\boldsymbol{v})\| \vee \|\boldsymbol{G}(\boldsymbol{u}) - \boldsymbol{G}(\boldsymbol{v})\|_* \le \rho\|\boldsymbol{u} - \boldsymbol{v}\|$.

Corollary 1: Suppose (A_1) and (A_2) hold. If $0 < \delta \le 1$, and $2\alpha\beta^2 + 2a_{\min} - 2 - [(2\rho^2(\|\boldsymbol{C}\|^2_F + 1))/\delta] - (\ln \delta/\overline{h}) > 0$, where $\delta = \sup_{k\in\mathbb{Z}_+}\|\boldsymbol{I} + \boldsymbol{P_k}\|^2_{\max}$, and $\overline{h} = \sup_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} < \infty$, then (67) is practically exponentially stable in the mean-square sense.

Proof: From [46], one can derive the existence–uniqueness of mild solution $\boldsymbol{u}(t)$ to (67). Choose $V(t) = \|\boldsymbol{u}(t)\|^2$. For $t \in (t_{k-1}, t_k)$, from the Itô formula [1], we can deduce that

$$\begin{aligned} \frac{dEV(t)}{dt} &= 2E(\boldsymbol{u}, \mathscr{A}\boldsymbol{u}) - 2E(\boldsymbol{u}, \boldsymbol{A}\boldsymbol{u}) + 2E(\boldsymbol{u}, \boldsymbol{C}\boldsymbol{f}(\boldsymbol{u}(t-\tau, \boldsymbol{x}))) \\ &\quad + 2E(\boldsymbol{u}, \boldsymbol{J}) + E\|\boldsymbol{G}(\boldsymbol{u}(t-\tau, \boldsymbol{x}))\|^2_* \\ &\triangleq I_1 + I_2 + I_3 + I_4 + I_5. \end{aligned} \tag{68}$$

Similar to the proof of Theorem 1, we have

$$I_1 \le -2\alpha E\||\boldsymbol{u}\||^2 \le -2\alpha\beta^2 EV(t). \tag{69}$$

Applying the positiveness of a_i and the Young inequality leads to

$$I_2 \le -2a_{\min}EV(t) \tag{70}$$

$$I_4 \le EV(t) + \|\boldsymbol{J}\|^2. \tag{71}$$

Combining (H_2) with the Young inequality, one may derive that

$$\begin{aligned} I_3 &\le EV(t) + \|\boldsymbol{C}\|^2_F\|\boldsymbol{f}(\boldsymbol{u}(t-\tau, \boldsymbol{x}))\|^2 \\ &\le EV(t) + 2\|\boldsymbol{C}\|^2_F\rho^2 EV(t-\tau) \\ &\quad + 2\|\boldsymbol{C}\|^2_F\|\boldsymbol{f}(\boldsymbol{0})\|^2. \end{aligned} \tag{72}$$

Similarly

$$I_5 \le 2\rho^2 EV(t-\tau) + 2\|\boldsymbol{G}(\boldsymbol{0})\|^2_*. \tag{73}$$

Thus, we can conclude that

$$\begin{aligned} \frac{dEV(t)}{dt} &\le -\left(2\alpha\beta^2 + 2a_{\min} - 2\right)EV(t) \\ &\quad + 2\rho^2\left(\|\boldsymbol{C}\|^2_F + 1\right)E\overline{V(t)} + 2\|\boldsymbol{C}\|^2_F\|\boldsymbol{f}(\boldsymbol{0})\|^2 \\ &\quad + \|\boldsymbol{J}\|^2 + 2\|\boldsymbol{G}(\boldsymbol{0})\|^2_*. \end{aligned} \tag{74}$$

Then, the practical exponential stability in the mean-square sense can be obtained by imitating the proof of Theorem 2. ■

Similarly, we have the following results.

Corollary 2: Suppose (A_1) and (A_2) hold. If $\delta > 1$, and $2\alpha\beta^2 + 2a_{\min} - 2 - 2\rho^2\delta(\|\boldsymbol{C}\|^2_F + 1) - (\ln \delta/\underline{h}) > 0$, where $\delta = \sup_{k\in\mathbb{Z}_+}\|\boldsymbol{I} + \boldsymbol{P_k}\|^2_{\max}$, and $\underline{h} = \inf_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} < \infty$, then (67) is practically exponentially stable in the mean-square sense.

Now, we shall apply Theorem 5 to (67). The following results can be deduced by (74).

Corollary 3: Assume that (A_1) and (A_2) hold. If there exist constant $\gamma > 1$, and $\mathcal{N} \in \mathbb{Z}_+$ such that $\delta_{\mathcal{N}+k} = \delta_k$

$$2\alpha\beta^2 + 2a_{\min} - 2 - \frac{2\rho^2\left(\|\boldsymbol{C}\|^2_F + 1\right)}{\delta_{\mathcal{N}}} + \frac{\ln \gamma}{\overline{h}} > 0$$

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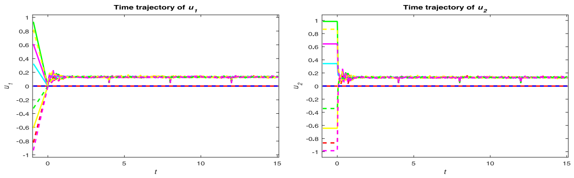

Fig. 1. Trajectory of u_1 and u_2 to (75) in Example 1.1.

and

$$\begin{cases} \gamma^{\mathcal{N}} \prod_{1\le j\le \mathcal{N}-1}\left(\delta_j \vee \frac{1}{\gamma}\right) \le \frac{1}{\delta_{\mathcal{N}}}, & \mathcal{N} \ge 2 \\ \delta_k \equiv \frac{1}{\gamma}, & \mathcal{N} = 1 \end{cases}$$

where $\delta_k = \|\boldsymbol{I}+\boldsymbol{P_k}\|_{\max}^2 > 0$, and $\overline{h} = \sup_{k\in\mathbb{Z}_+}\{t_k - t_{k-1}\} < \infty$, then (67) is practically exponentially stable in the mean-square sense.

Remark 9: We mention that, if $\boldsymbol{P_k} = \boldsymbol{0}$, then Corollaries 1–3 become the practical exponential stability of stochastic delayed reaction–diffusion Hopfield neural networks without impulses, which has been discussed in [47]. So our results include some of the results in [47].

VI. Examples

Our results in this article provide some sufficient conditions for practical exponential stability of (1), and they can be used in many different systems. In this section, four examples are given to illustrate the effectiveness of our proposed results.

Example 1: Consider the following IRDSHNNs with delays:

$$\begin{cases} du_1 = (\triangle u_1 - 16.1u_1 + 0.5\tanh(u_1(t-1,x)) \\ \qquad + 0.5\tanh(u_2(t-1,x)) + 2)dt \\ \qquad + \tanh(u_1(t-1,x))dW, \ \ t \ne t_k \\ u_1(t_k) - u_1\left(t_k^-\right) = pu_1\left(t_k^-\right) \\ du_2 = (\triangle u_2 - 16.1u_2 + 0.5\tanh(u_1(t-1,x)) \\ \qquad + 0.5\tanh(u_2(t-1,x)) + 2)dt \\ \qquad + \tanh(u_2(t-1,x))dW, \ \ t \ne t_k \\ u_2(t_k) - u_2\left(t_k^-\right) = pu_2\left(t_k^-\right) \\ u_i|_{x\in\partial\mathcal{O}} = 0, \ \ t \ge 0, \ \ i = 1, 2 \\ (u_1(\theta), u_2(\theta))^T = (\sin(0.2\pi x)\theta, \sin(0.2\pi x))^T \\ \theta \in [-1, 0], \ \ x \in \mathcal{O} \end{cases} \tag{75}$$

where $\mathcal{O} = (0, 20)$ and $t_k = 4k$, $k \in \mathbb{Z}_+$. $W = \sum_{n=1}^{\infty}(1/n)B_n(t)e_n(x)$, where $\{B_n(t)\}_{n=1}^{\infty}$ are independent standard Brownian motions, and $e_n(x) = \sqrt{(1/20)}\sin(n\pi x/20)$.

Example 1.1: Let $p = (1/e) - 1$. Then, all assumptions in Corollary 1 are fulfilled with $\alpha = 1$, $\beta \ge 0.05$, $a_{\min} = 16.1$, $\|\boldsymbol{C}\|_F^2 = 1$, $\rho = 1$, $\overline{h} = 4$, and $\delta = (1/e^2) < 1$. Therefore, based on Corollary 1, (75) is practically exponentially stable in the mean-square sense. This can be verified by Figs. 1 and 2. In order to give a clear description, the trajectory of $\|\boldsymbol{u}\|^2$ has been given in Fig. 3.

Example 1.2: Let $p = e - 1$, then $\delta = e^2 > 1$. Based on Corollary 2, one can derive that (75) is practically exponentially stable. Figs. 4 and 5 show the trajectories of u_1, u_2, and $\|\boldsymbol{u}\|^2$, which is consistent with our results.

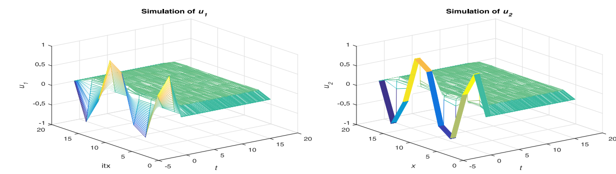

Fig. 2. Simulation in $\mathbb{R}^3$ of u_1 and u_2 to (75) in Example 1.1.

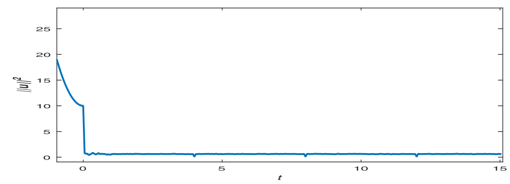

Fig. 3. $\|u\|^2$ of (75) in Example 1.1.

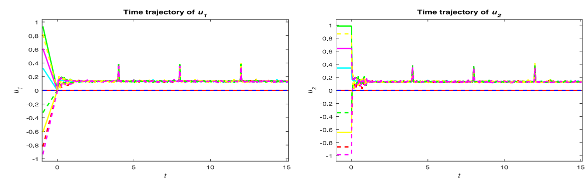

Fig. 4. Trajectory of u_1 and u_2 to (75) in Example 1.2.

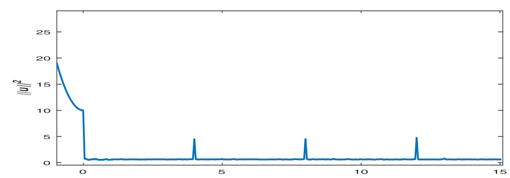

Fig. 5. $\|u\|^2$ of (75) in Example 1.2.

Example 2: Consider the following 1-D ISRDSs with delays:

$$\begin{cases} du = (\triangle u - 5u + 0.5\sin(u(t-1,x)) + J)dt \\ \qquad + 0.5\tanh(u(t-1,x))dW, \ \ t \ne t_k \\ u(t_k) - u\left(t_k^-\right) = p_k u\left(t_k^-\right) \\ u|_{x\in\partial\mathcal{O}} = 0, \ \ t \ge 0 \\ u(\theta) = \sin(0.4\pi x)\left(2\theta - \theta^2\right), \ \ \theta \in [-1, 0], \ \ x \in \mathcal{O} \end{cases} \tag{76}$$

where $\mathcal{O}$ and W are the same as those in Example 1.

Example 2.1: Let $t_k = k$, $k \in \mathbb{Z}_+$, $p_{3k-1} = (1/\sqrt{0.35}) - 1$, $p_{3k-2} = (1/\sqrt{0.83}) - 1$, and $p_{3k} = (1/\sqrt{6}) - 1$. Then, we choose $V = \|\boldsymbol{u}(t)\|^2$, $\mathcal{N} = 3$, $\gamma = 1.2$, $\sigma_1 = 0.35$, $\sigma_2 = 0.83$, and $\sigma_3 = 6$. We notice that the assumptions in Corollary 3 can be perfectly satisfied. So if $J = 1$, one may obtain the practical exponential stability of (76), which is shown in Fig. 6(a). Also, it follows from Corollary 3 that (76) with $J = 0$ is exponentially stable, as shown in Fig. 6(b). Moreover, Fig. 7 (red and black) shows the trajectories of $\|u(t)\|^2$.

Example 2.2: Let $t_k = 2k$, $k \in \mathbb{Z}_+$, and $p_k = (1/\sqrt{8}) - 1$. If $V = \|\boldsymbol{u}(t)\|^2$, $\mathcal{N} = 1$, and $\sigma_k = \gamma = 8$, then it can be deduced from Corollary 3 that (76) is practically exponentially stable when $J = 1$, and exponentially stable when $J = 0$. To verify our results, the simulation results are shown in Fig. 6(c) and (d), respectively. One can also observe the

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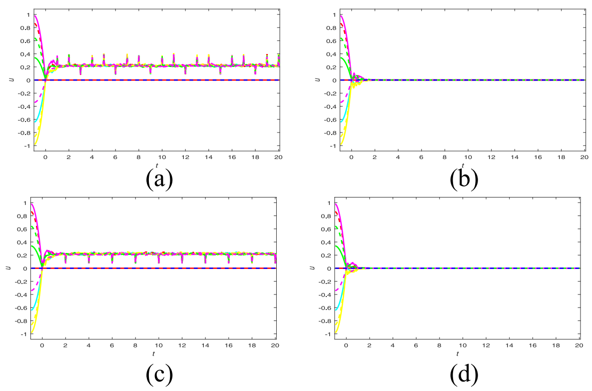

Fig. 6. (a) Trajectory of system (76) in Example 2.1 with $J = 1$. (b) Trajectory of system (76) in Example 2.1 with $J = 0$. (c) Trajectory of system (76) in Example 2.2 with $J = 1$. (d) Trajectory of system (76) in Example 2.2 with $J = 0$.

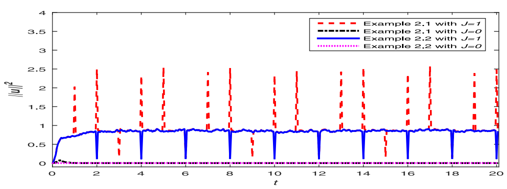

Fig. 7. $\|u\|^2$ of (76) in Example 2.

practical stability of (76) from the simulation of $\|u(t)\|^2$ in Fig. 7 (blue and magenta).

Example 3: Consider the following ISRDSs with delays:

$$\begin{cases} du_1 = (\Delta u_1 + (0.05 + 0.7\cos(0.5t))u_1 - 0.6u_2 \\ \qquad + 0.01\cos(0.5t)u_1(t-1,x))dt \\ \qquad + \tanh(u_1(t-1,x))dW, \ t \neq t_k \\ u_1(t_k) - u_1(t_k^-) = p_k u_1(t_k^-) \\ du_2 = (\Delta u_2 + 0.6u_1 + (0.05 + 0.7\cos(0.5t))u_2 \\ \qquad - 0.01u_1(t-1,x) - 0.01u_2(t-1,x))dt \\ \qquad + \tanh(u_2(t-1,x))dW, \ t \neq t_k \\ u_2(t_k) - u_2(t_k^-) = p_k u_2(t_k^-) \\ u_i|_{x\in\partial\mathcal{O}} = 0, \ t \geq 0, \ i = 1,2 \\ (u_1(\theta), u_2(\theta))^T = (\sin(0.2\pi x)\theta, \sin(0.2\pi x))^T \\ \theta \in [-1,0], \ x \in \mathcal{O} \end{cases} \tag{77}$$

where $t_k = 0.01k$, $k \in \mathbb{Z}_+$. $\mathcal{O}$ and W are the same as those in Example 1.

Example 3.1: Let $p_k = 0$, then (77) becomes the stochastic reaction–diffusion systems without impulses. We cannot derive the practical stability of (77) from Theorem 1 or Theorem 4. But from Fig. 8, we can infer that the solution to (77) would be divergent with the increasing of time and thus it is not practically stable.

Example 3.2: If $p_k = e^{-1} - 1$, then according to Theorem 2, one may observe that (77) can become exponentially stable with the impulses. The state trajectory is portrayed in Fig. 9, from which we can also obtain the stability.

Example 3.3: Let $p_{2k} = (1/\sqrt{14}) - 1$ and $p_{2k-1} = \sqrt{2} - 1$. Then, one may choose $V = \|\boldsymbol{u}(t)\|^2$, $\mathcal{N} = 2$, $\gamma = 1.2$, $\sigma_1 = 0.5$, and $\sigma_2 = 14$. Using Theorem 5, it can be derived that the impulsive control can exponentially stabilize the system (77), which is shown in Fig. 10.

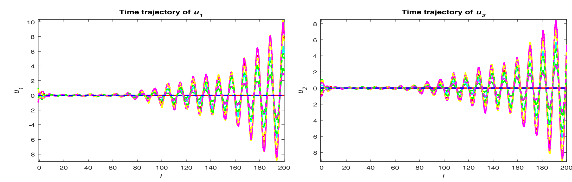

Fig. 8. Trajectory of u_1 and u_2 to (77) in Example 3.1.

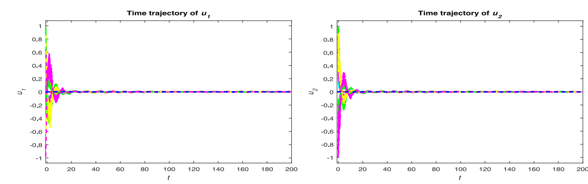

Fig. 9. Trajectory of u_1 and u_2 to (77) in Example 3.2.

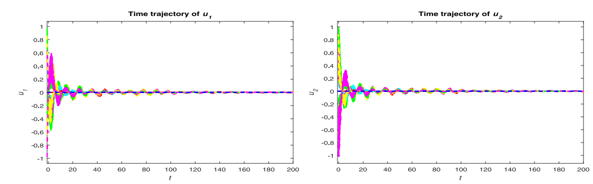

Fig. 10. Trajectory of u_1 and u_2 to (77) in Example 3.3.

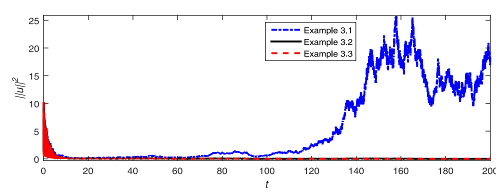

Fig. 11. $\|\boldsymbol{u}\|^2$ of (77) in Example 3.

Remark 10: Fig. 11 illustrates the trajectories of $\|\boldsymbol{u}\|^2$ in Examples 3.1–3.3. It then can be obtained that the impulses given by Examples 3.2 and 3.3 can stabilize the system.

Example 4: Consider the following 1-D systems:

$$\begin{cases} du = [D\Delta u - u(t-\tau,x)]dt + \tanh(u(t-\tau,x))dW \\ u|_{x\in\partial\mathcal{O}} = 0, \ t \geq 0 \\ u(\theta) = \sin(0.4\pi x)\big(2\theta - \theta^2\big), \ \theta \in [-\tau, 0], \ x \in \mathcal{O} \end{cases} \tag{78}$$

where $\mathcal{O}$ and W are the same as those in Example 1.

Example 4.1: Let $D = 35$, then Fig. 12 shows the trajectories of $\|u(t)\|^2$ with $\tau = 0$, $\tau = 3$, and $\tau = 20$. From Fig. 12, we obtain that the convergence rate of (78) tends to decrease with the increase of time delay. It then follows that time delays may affect the convergence rate of systems.

Example 4.2: Let $\tau = 3$, then the simulation results of $\|u(t)\|^2$ with $D = 10$, $D = 20$, and $D = 35$ are illustrated in Fig. 13, respectively, which demonstrate the effect of diffusion terms. It is easily seen that (78) with $D = 10$ is not practically stable, but the one with $D = 35$ is stable.

Remark 11: One can observe from Example 4 that the time delays and diffusion terms may influence the stability of systems. Therefore, we cannot ignore the effect of them when discussing the dynamical behavior of systems.

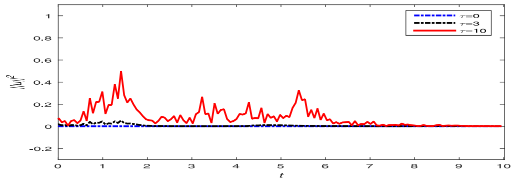

Fig. 12. $\|u\|^2$ of (78) in Example 4.1.

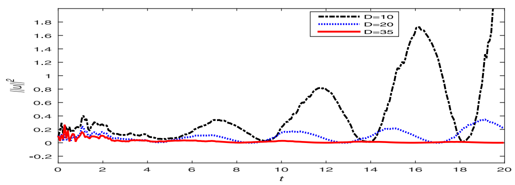

Fig. 13. $\|u\|^2$ of (78) in Example 4.2.

VII. Conclusion

In this article, a direct approach and the Lyapunov method are developed to study the practical exponential stability of ISRDSs with delays. Those two ways can be used for the systems with different impulses, and are also applicable when discussing the exponential stability under certain conditions. The proposed results are applied to the IRDSHNNs with delays to obtain some algebraic criteria. Numerical examples are given to demonstrate the effectiveness of our theoretical results, which also illustrate the effects of diffusion terms and time delays. Notice that the concept of practical stability is more suitable for many systems, such as the delayed logistic equations and the switched delayed systems. It is interesting to investigate the practical stability of these systems in the future. Another topic is to extend our results to systems with more complex impulses, such as state-dependent impulses and delayed impulses.

Acknowledgment

The authors would like to thank the editors and reviewers for their constructive comments and suggestions.

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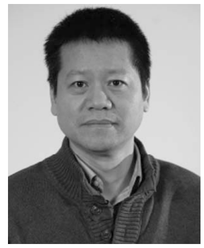

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