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Generalised mathematical models for 3D magnetic reconnection at null points

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Ali Al-Hachami

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Generalised Mathematical Models for 3D Magnetic Reconnection at Null Points

By

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“One who does not thank people does not thank God”

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Declaration

I declare that the following thesis is my own composition and that it has not been submitted before in application for a higher degree.

Ali K. H. Al-Hachami
Certification

This is to certify that Ali K. H. Al-Hachami has complied with all the requirements for the submission of this Doctor of Philosophy thesis to the University of Dundee.

Dr D. I. Pontin
Abstract

Plasmas occur in many technical, laboratory and space environments, and often behave in a highly ideal manner. This means that advection of the plasma can store large amounts of energy in the magnetic field. This energy is released when a sudden change in the magnetic topology of the field occurs—facilitated by the process of ‘magnetic reconnection’. A great deal of research has been focussed on understanding the reconnection process and we now appreciate that the 3D process is critically different from early 2D models.

The magnetic field in many astrophysical plasmas, for example in the solar corona, is known to have a highly complex—and clearly three-dimensional—structure. Turbulent plasma motions in high-$\beta$ regions where field lines are anchored, such as the solar interior, can store large amounts of energy in the magnetic field. This energy can only be released when magnetic reconnection occurs. Reconnection may only occur in locations where huge gradients of the magnetic field develop, and one candidate for such locations are magnetic null points, known to be abundant for example in the solar atmosphere. Reconnection leads to changes in the topology of the magnetic field, and energy being released as heat, kinetic energy and acceleration of particles. Thus reconnection is responsible for many dynamic processes, for instance solar flares and jets in the solar atmosphere.

The aim of this thesis is to investigate the properties of magnetic reconnection
at a 3D null point. One key focus will be to understand the dependence of the process on the symmetry of the magnetic field around the null. In particular we examine the rate of reconnection of magnetic flux at the null point, as well as how the current sheet forms and its properties.

According to our present understanding, there are three main modes of magnetic reconnection that may occur at 3D nulls, spine-fan reconnection, torsional spine reconnection and torsional fan reconnection. We first consider the spine-fan reconnection mode. It is found that the basic structure of the mode of magnetic reconnection considered is unaffected by varying the magnetic field symmetry, that is, the plasma flow is found to cross both the spine and fan of the null. However the peak intensity and dimensions of the current sheet are dependent on the symmetry/asymmetry of the field lines. As a result, the reconnection rate is also found to be strongly dependent on the field asymmetry.

In addition, the properties of the torsional spine and torsional fan modes of magnetic reconnection at 3D nulls are investigated. New analytical models are developed which for the first time include a current layer that is fully spatially localised around the spine or fan of the null. The principal aim is to investigate the effect of varying the degree of asymmetry of the null point magnetic field on the resulting reconnection process – where previous studies always considered a non-generic radially symmetric null. Analytical solutions are derived for the steady kinematic equations at a three dimensional null point. In these models the electric current lies parallel to either the fan or spine. In order to confirm the results of kinematic models, numerical simulations are performed in which the full set of resistive MHD equations are solved. It is found that the geometry of the current layers within which torsional spine and torsional fan reconnection occur is strongly dependent on the symmetry of the magnetic field. Torsional spine reconnection still occurs in a narrow tube around the spine, but with elliptical
cross-section when the fan eigenvalues are different. The eccentricity of the ellipse increases as the degree of asymmetry increases, with the short axis of the ellipse being along the strong field direction. The spatiotemporal peak current, and the peak reconnection rate attained, are found not to depend strongly on the degree of asymmetry. For torsional fan reconnection, the reconnection occurs in a planar disk in the fan surface, which is again elliptical when the symmetry of the magnetic field is broken. The short axis of the ellipse is along the weak field direction, with the current being peaked in these weak field regions. The peak current and peak reconnection rate in this case are clearly dependent on the asymmetry, with the peak current increasing but the reconnection rate decreasing as the degree of asymmetry is increased.
Chapter 1

Introduction

1.1 Background

We can describe most of the matter in the universe as plasma, such as stars, the interstellar medium and inter planetary medium, since they are all made of ionized gases. Although plasma generally does not exist on the surface of the earth, an understanding of plasma is essential. This is because our planet Earth is part of this universe, and it interacts with its surroundings (see Figure 1.1). In addition, the ionosphere and upper atmosphere of Earth, the Earth radiation belts (Van Allen belts) are also known to be ionized. Plasma occurs in many technical, laboratory and space environments. In these different plasma environments, it is found that the magnetic field plays a crucial role in the dynamic process that appear (solar flares, Coronal Mass Ejections and so on).

In most of the universe, the magnetic field is frozen into the plasma, and moves around with it. However, in some localised regions, the magnetic field lines are able to slip through the plasma, and may break and rejoin. This process of
Figure 1.1: Solar magnetism directly affects the Earth and the rest of the solar system. The solar wind shapes the Earth’s magnetosphere and magnetic storms are illustrated here as approaching Earth. Image from the NASA website (http://www.nasa.gov/connect/apps.html).

Magnetic reconnection is essential in many areas of plasma physics. Magnetic reconnection is the topological change of a magnetic configuration through breaking and rejoining of magnetic field lines. In this thesis by ‘magnetic topology’ we are referring to features such as magnetic nulls, spine, fans and separators as well as, for example, the amount of flux in distinct topological domains. These changes allow the release of magnetic energy, during which magnetic energy is converted into fast-particle energy, heat and kinetic flow energy in a relatively short time.

1.1.1 Equations of MHD

In this thesis, we will assume the magnetohydrodynamic (MHD) approximation. The equations of MHD are as follows:
Mass conservation

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}), \quad (1.1) \]

The equation of motion

or, momentum conservation

\[ \rho \frac{D\mathbf{v}}{Dt} = -\nabla P + \mathbf{J} \times \mathbf{B} + \mathbf{F}, \quad (1.2) \]

Gas law

\[ P = \rho \mathcal{R} T, \quad (1.3) \]

Ampere’s law

\[ \nabla \times \mathbf{B} = \mu \mathbf{J}, \quad (1.4) \]

Solenoidal constraint

\[ \nabla \cdot \mathbf{B} = 0, \quad (1.5) \]

Faraday’s law

\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E}, \quad (1.6) \]
Ohm’s law

\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \mathbf{R}. \quad (1.7) \]

These equations must also be supplemented by an appropriate energy equation. The form of energy equation we have used in the numerical simulations described in this thesis is

\[ \frac{\partial e}{\partial t} = -\nabla \cdot (e\mathbf{v}) - P \nabla \cdot \mathbf{v} + Q_{\text{visc}} + Q_J. \quad (1.8) \]

In the equations,

- \( \mathbf{B} \) is the magnetic field,
- \( \mathbf{v} \) the plasma velocity,
- \( \mathbf{E} \) the electric field,
- \( \mathbf{J} \) the electric current density,
- \( \rho \) the mass density,
- \( P \) the pressure,
- \( \mu \) is the magnetic permeability in vacuum,
- \( \mathcal{R} \) the gas constant,
- \( T \) is the plasma temperature,
- \( \mathbf{F} \) represents all other forces which may be present, such as the gravitational force
- \( \mathbf{R} \) in Eq. (1.7) represents a general non-ideal term, often taken to be \( \eta \mathbf{J} \), where \( \eta = 1/\sigma \) is the electrical resistivity and \( \sigma \) is the electric conductivity
- \( e \) is the internal energy,
• $Q_{visc}$ the viscous dissipation, and
• $Q_J$ the Joule dissipation.

For more details about these equations and their derivations, see Priest (1982).

1.1.2 The Induction equation

Using Ohm’s law together with Eqs. (1.4) and (1.6), we could eliminate the variable $E$ and $J$ to obtain the induction equation,

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) - \nabla \times (\eta' \nabla \times B),$$

where, $\eta' = \frac{1}{\mu \sigma}$ is the magnetic diffusivity and if it is taken to be constant (it generally depends on the plasma temperature and hence is expected to vary in space), we get

$$\frac{\partial B}{\partial t} = \nabla \times (v \times B) + \eta' \nabla^2 B,$$

which describes how the magnetic field $B$ evolves with time. The two terms on the right-hand side are known as the advection term and the diffusion term, respectively. The advection term describes how the magnetic field lines are carried along by the plasma velocity and hence the field lines are “frozen in” to the plasma if the advection term is dominant. That is, plasma elements which are initially on the same field line remain so for all later times. On the other hand, the magnetic field lines may slip through the plasma if the diffusion term is dominant see Figure (1.2).

In this thesis, the breakdown of the magnetic connection of the plasma elements will be used as a definition of magnetic reconnection, as discussed by Schindler et al. (1988). The framework of Schindler et al. (1988) distinguishes
Figure 1.2: Frames illustrate field line non-conservation (a local reconnection process). The straight black lines represent magnetic field lines, the blue shading shows the non-ideal region and the small red/blue circles represent plasma elements. As the plasma elements pass through the non-ideal region they may ‘slip’ between field lines.
Figure 1.3: Field line non-conservation. a) The flow lines cross the magnetic separatrix in the ideal region. The red circles represent plasma elements and the dashed lines plasma flow lines. b) Breakdown of magnetic connection (Global magnetic reconnection).

between global reconnection (where two line exchanging plasma elements that remain outside the diffusion region during the reconnection process can be found) and local reconnection (in which at least one of the plasma elements passes through the diffusion region). The local reconnection process is illustrated in Figure 1.2 while Figure 1.3 shows a global reconnection process.

Axford (1984) considered the localised breakdown of the ‘frozen-in’ field condition and the resulting changes of connection due to a localised non-idealness as the basis of magnetic reconnection, this non-idealness being localised inside the diffusion region ($D$). So, in order to know when these two different effects may take place, it is very important to determine the relative magnitude of these two terms. The ratio between the first and the second term on the right hand side of induction equation is known as the magnetic Reynolds number, $R_m$, given by

$$R_m = \frac{\|
abla \times (v \times B)\|}{|\eta \nabla^2 B|}$$

$$= \frac{Lv_0}{\eta} \quad \text{(1.11)}$$
where $L$ is a typical length scale and $v_0$ is a typical plasma velocity. The magnetic Reynolds number is very large almost everywhere in astrophysical plasmas, which means that the magnetic field is frozen to the plasma, or in other words all plasma elements lying along a given field line at a given time will lie on the same field line for all subsequent time. However, locations where the typical length scales are very small and hence the non-ideal effects become important, such as localized non-ideal regions (the magnetic field lines are able to slip through the plasma, and may break and rejoin) i.e., are localised where reconnection might take place. This process is fundamental in many areas of plasma physics.

1.1.3 Importance of magnetic reconnection

Magnetic reconnection is considered to play an important role in the early stage of many large-scale solar flares. It is responsible for many dynamic processes, whether in laboratories, the Earth’s magnetosphere, the Sun or in any astrophysical plasmas where magnetic fields exist (see, e.g., Priest and Forbes (2000), for a review). The importance of magnetic reconnection in these different environments comes because the plasma plays a key role in the generation of magnetic fields in these environments and has the ability to change the topology of the magnetic field and thus the release of stored magnetic energy and convert them into other various forms such as kinetic energy, heat and accelerated particles. In the following, we give some examples illustrating the role of magnetic reconnection in the above mentioned plasma environments:

- Heating the corona to its multi-million degree temperatures (e.g. Parker, 1983).
• Sudden violent events such as solar flares (Parker, 1963) and CMEs and the corresponding events on other stars.

• It is thought to be the process which is behind many other events on the Sun, such as X-ray bright points see (e.g. the ‘convering flux model’ of Parnell et al. (1994)).

• The Earth’s magnetosphere (where, uniquely for non-terrestrial events, in-situ spacecraft observations at reconnection sites have been made) as it interacts with the solar wind (Xiao et al., 2006), and similarly in other planetary magnetospheres (Huddleston et al., 1997).

• The laboratory, particularly in fusion devices see (e.g. Zweibel and Yamada (2009)).

The important role played by magnetic reconnection in these plasmas strongly led us to try to understand the fundamental physics of reconnection. Mathematically, we can say very good progress has been performed on both sides of the analytical and numerical understanding of the fundamental process of magnetic reconnection in recent years.

1.1.4 Two-dimensional (2D) magnetic reconnection

It is well known that, in 2D, reconnection can only take place at X-type magnetic null points, locations where the magnetic field \( B \) vanishes. A plasma flow transports magnetic flux towards the X-point, where the reconnection takes place, and the flow then transports the reconnected magnetic flux away from the X-point. The basic picture of reconnection was first formulated by Sweet (1958) and Parker (1957), resulting in a well-known reconnection mechanism called the Sweet-Parker
Figure 1.4: The Sweet-Parker mechanism for 2D reconnection, diffusion region and surrounding field with length $2L$ and width $2l$. Magnetic field lines are represented by the horizontal lines; the field is oppositely directed on either side of the diffusion region.

The Sweet-Parker mechanism for 2D reconnection, which provides an order-of-magnitude calculation of the energy which may be released when a current sheet is sandwiched between two regions of uniform oppositely-directed magnetic field (see Figure 1.4).

The inflow magnetic field and velocity are denoted by $B_i, v_i$. Simply by considering the principles of conservation of mass within and outside the sheet, and a steady balance of advection and diffusion, it can be shown that a (dimensionless) reconnection rate of

$$M_i = \frac{1}{\sqrt{R_{mi}}}$$

is obtained, where $R_{mi} = \frac{L_{v_{Ai}}}{\eta}$ is the magnetic Reynolds number based on the inflow Alfvén speed ($v_{Ai} = \frac{B_i}{\sqrt{\mu \rho}}$) and the sheet length ($L$). It can also be shown that half of the inflow magnetic energy is converted into kinetic energy, while the other half is released in the form of an ohmic heating. The main disadvantage of the Sweet-Parker mechanism is that the rate of energy release is not enough to explain the release of energy in solar flares, which was the main objective [Parnell].
The Petschek model for 2D steady reconnection (1964) suggested that slow MHD waves would significantly decrease the size of the diffusion region and, accordingly, increase the rate of reconnection. Thus he developed a model in which the length of the diffusion region may be considerably smaller than the global external length-scale. Also, the magnetic field in the inflow region is now no longer exactly uniform (see Figure 1.5).

Later, Priest and Forbes (1986) presented a set of new models for steady-state magnetic reconnection, including families of Almost Uniform and Non-Uniform (Priest and Lee, 1990) solutions. An in-depth review of 2D reconnection models can be found in Priest and Forbes (2000). Each of these models requires that an X-type null point of the magnetic field has collapsed to form a current sheet. This was then followed by a lot of research in 2D reconnection using both analytical and numerical techniques and not only considering collisional magnetic reconnection, but also studying other types of reconnection such as collisionless reconnection and Hall MHD reconnection.
1.1.5 Topology of 3D magnetic fields

One of the important things we need to know is where magnetic reconnection can occur? There are a number where proposed sites where reconnection may take place:

1. At 3D null points (points in space at which the magnetic field vanishes).

2. In the absence of null points.

In order to determine the nature of reconnection at a 3D null point, we must understand the structure of these null points. Identifying the structure around the point at which the magnetic field lines break and subsequently reform, known as the magnetic null point, is crucial to improving our understanding of reconnection. In later chapters, reconnection at three-dimensional null points, i.e where the field vanishes ($B = 0$), will be discussed, and so an introduction to them is given here.

If we assume that the magnetic field near a null point approaches zero linearly, we can approximate the components of the magnetic field using a first order Taylor expansion about the null point $(X_0, Y_0, Z_0)$. Consider the x component:

$$ B_X \approx B_X(X_0, Y_0, Z_0) + \frac{\partial B_X}{\partial X}_{X_0,Y_0,Z_0} (X - X_0) + \frac{\partial B_X}{\partial Y}_{X_0,Y_0,Z_0} (Y - Y_0) + \frac{\partial B_X}{\partial Z}_{X_0,Y_0,Z_0} (Z - Z_0) + \ldots $$

The first term is zero at the null by definition, and we reserve only the first order, linear terms:

$$ B_X \approx \frac{\partial B_X}{\partial X}_{(X_0,Y_0,Z_0)} (X - X_0) + \frac{\partial B_Y}{\partial Y}_{(X_0,Y_0,Z_0)} (Y - Y_0) + \frac{\partial B_Z}{\partial Z}_{(X_0,Y_0,Z_0)} (Z - Z_0). $$

Choose the null to be at the origin such that $X_0 = Y_0 = Z_0 = 0$

$$ B_X \approx \frac{\partial B_X}{\partial X}_{(0,0,0)} X + \frac{\partial B_Y}{\partial Y}_{(0,0,0)} Y + \frac{\partial B_Z}{\partial Z}_{(0,0,0)} Z. $$
Similarly for the y, z components:

\[
B_y \approx \frac{\partial B_y}{\partial X}(0,0,0) X + \frac{\partial B_y}{\partial Y}(0,0,0) Y + \frac{\partial B_y}{\partial Z}(0,0,0) Z,
\]

\[
B_z \approx \frac{\partial B_z}{\partial X}(0,0,0) X + \frac{\partial B_z}{\partial Y}(0,0,0) Y + \frac{\partial B_z}{\partial Z}(0,0,0) Z.
\]

Thus if we assume we are sufficiently close to the null, then the magnetic field may be expressed as

\[
B = \mathcal{M} \cdot r
\]

(1.13)

where \(\mathcal{M}\) is a matrix with the elements of the Jacobian of \(B\)

\[
\mathcal{M} = \begin{bmatrix}
\frac{\partial B_x}{\partial X} & \frac{\partial B_x}{\partial Y} & \frac{\partial B_x}{\partial Z} \\
\frac{\partial B_y}{\partial X} & \frac{\partial B_y}{\partial Y} & \frac{\partial B_y}{\partial Z} \\
\frac{\partial B_z}{\partial X} & \frac{\partial B_z}{\partial Y} & \frac{\partial B_z}{\partial Z}
\end{bmatrix},
\]

and \(r\) is the position vector \((X,Y,Z)^T\). The eigenvalues of \(\mathcal{M}\) sum to zero since \(\nabla \cdot B = 0\). Thus, each field line may be written in terms of eigenvalues and eigenvectors of matrix \(\mathcal{M}\). Since the sum of the eigenvalues is zero, there is always one eigenvalue whose real part has opposite sign to the other two, say for instance \(\lambda_1, \lambda_2 > 0, \lambda_3 < 0\). The skeleton of the null point is made up of a pair of field lines directed into (or out of) the null from opposite directions, known as the spine, and a family of field lines, which are directed out of (or into) the null lying in a surface, known as the fan plane. We find that the eigenvectors \(x_1\) and \(x_2\) with positive eigenvalues \(\lambda_1\) and \(\lambda_2\) define the plane of the fan, whilst the path of the spine is defined by the eigenvector \(x_3\) with negative eigenvalue \(\lambda_3\) (Priest and Titov, 1996).

Parnell et al. (1996) gave a general mathematical formula for a linear null point and classified its structure depending on the direction of the current \((J)\) with respect to the spine axis and fan plane, and its size. In the case of the current being zero \((J = 0)\) then the null point is called potential. Figure 1.6(a)
shows the potential null when the spine and fan are perpendicular, and field lines in the fan plane are purely radial, but they run parallel to the \(x\)-axis far from the null if \(0 < p < 1\) and parallel to the \(y\)-axis if \(p > 1\) [see Figures 1.6(c), 1.6(b) and 1.7(b) respectively]. All potential nulls have their spine and fan perpendicular to one another.

When the same sign eigenvalues of the null point are real, the field lines in the fan form the structure shown in Figure 1.6(a), while if we have the same sign complex eigenvalues, the field lines in the fan form a spiral structure, see Figure 1.7(a). If the current is directed parallel to the spine, we may have either of these structures depending on the level of current, while if the current is directed parallel to the fan, the spine and fan plane are tilted to each other (and the eigenvalues are real; for more details, see Parnell et al. (1996)). We will return to discuss these situations later in Chapters 3 and 5.

### 1.1.6 Non-null reconnection

As well as three-dimensional reconnection with null points that we have mentioned in the above, 3D current sheet formation and magnetic reconnection can also take place within regions of non-vanishing magnetic field, which is known as a non-null magnetic reconnection. Hornig and Priest (2003) analysed such a situation in a region of non-zero magnetic field, placing particular emphasis on the evolution of magnetic flux. An example of reconnection in the absence of null points is quasi-separatrix layers (QSLs), which are thin layers where there is a rapid change in field-line linkage Priest and Démoulin (1995). QSLs have proven extremely useful in identifying regions of 3D magnetic reconnection in theoretical configurations see, e.g., Titov and Hornig (2002).
Figure 1.6: The structure of magnetic field of 3D potential field with real eigenvalues where \( \mathbf{B} = \left( \frac{-2}{p+1} x, \frac{2p}{p+1} y, -2z \right) \). a) p=1. c) p=0.5. d) p=2.
Figure 1.7: The structure of 3D non potential magnetic field with null point with $\mathbf{B} = \left( \frac{2x}{p+1} - \frac{1}{2} jy, \frac{2py}{p+1} + \frac{1}{2} jx, -2z \right)$ and a complex eigenvalues. a) $p = 1$. b) $p = 2$ with $j = 1$.

1.1.7 Aims and outline

In this thesis, our aims are to understand the fundamental process of three dimensional magnetic reconnection. We aim to present three-dimensional solutions of different MHD models. In particular, we aim to study steady state MHD models to find solutions which describe 3D reconnection in the presence of null points with a localised non-ideal region, extending on the work started by Pontin et al. (2004, 2005). We start by giving a short overview of reconnection in two-dimensions and three-dimensions, and the differences between these two regimes in Chapter 2. In Chapter 3, we develop kinematic models of reconnection at a null point with current directed parallel to the fan plane. In Chapter 4 a numerical experiment is described to confirm the analytical solution of Chapter 3. In Chapter 5 we present new kinematic models for torsional “spine reconnection” and torsional “fan reconnection” within a current tube localised to the spine of
null or current localised around the fan surface, building on the work started by Pontin and Galsgaard (2007). In Chapter 6, numerical solutions to confirm the solutions which we have explained in the previous Chapter are presented. Finally, in Chapter 7 we introduce several examples to investigate the relationship between the current and the ratio of eigenvalues of the null point.
Chapter 2

The Nature of 2D and 3D Reconnection

2.1 Introduction

Before developing models for reconnection in 3D, it is important to understand several key differences between the behaviour of the magnetic flux in 2D reconnection and 3D reconnection. This is explained in the paper by Priest et al. (2003). Below we summarise the results of Priest et al. (2003).

2.1.1 Fundamental properties of 2D reconnection

1. A flux transport velocity $\mathbf{w}$ (Hornig and Schindler 1996; Hornig and Priest 2003), satisfying

$$ \frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}), $$

exists everywhere in 2D except at the X-point. This velocity, with respect to which the magnetic flux is frozen (by comparison with the ideal Ohm’s
law), can be given by

\[ w = \frac{E \times B}{||B||^2} \]  \hspace{1cm} (2.2)

which is possible in 2D since the electric field is perpendicular to the magnetic field. In the ideal regions \( w \) is the same as the plasma velocity \( v \), and, \( w \) is smooth and differentiable everywhere except at the null point.

2. As we have already mentioned, reconnection in 2D, and hence the change of the connectivity of the magnetic field lines takes place only at an \( X \)-type magnetic null point. That is, field line connections are preserved everywhere, even in the diffusion region, except at the \( X \)-point where they are cut and hence change their connection.

3. The mapping between field line footpoints is discontinuous. Consider the schematic \( X \)-point shown in Figure 2.1(a). For example, as the field line anchored at footpoint \( A_1 \) moves towards the separatrix, it is connected to \( B_2 \) on the opposite boundary. However, as \( A_1 \) moves across the separatrix to \( A_2 \) it suddenly becomes connected to a point \( D_2 \) on the same boundary as itself. This discontinuous mapping is a consequence of the fact the field lines break only a single point.

4. A flux tube which is partly within the non-ideal region moves with the plasma velocity \( v = \mathbf{w} \) everywhere outside the non-ideal region, while, it moves with a velocity \( \mathbf{v} \neq \mathbf{w} \) on the segment that lies in the diffusion region see Figure 2.1(b).

5. Reconnecting flux tubes rejoin perfectly Figure 2.1(c). That is, for any tube going to reconnect, there exists a corresponding flux tube on the opposite side of the \( X \)-point with which it will become perfectly rejoined after reconnection, such that two unique but differently connected flux tubes are
Figure 2.1: a) The mapping of field lines in 2D, as the plasma element $A_1$ moves towards $A_2$ it crosses a separatrix of the field. b) The behaviour of a flux tube in 2D when partly in a diffusion region. c) The breaking and perfect rejoining of flux tubes in 2D.
produced. This will henceforth be referred to as perfect reconnection of flux tubes.

2.1.2 Fundamental properties of 3D reconnection

In three-dimensional reconnection, the behaviour of magnetic flux is quite different from any of the properties listed above. It is therefore absolutely necessary to reform the way we think about reconnection occurring when we move to 3D. In the following, we will describe the process of reconnection in 3D.

1. In all cases, a flux tube velocity \( \mathbf{w} \) does not exist, or in other words no unique field line velocity \( \mathbf{w} \) exists. This is shown in the following theorem:

**Theorem 2.1.1.** For an isolated 3D diffusion region, a flux-conserving flow \( \mathbf{w} \) does not exist in general.

*Proof.* Suppose a finite diffusion region \( D \) and let us assume \( \mathbf{w} \) does exist. Then we have

\[
\frac{\partial \mathbf{B}}{\partial t} = \nabla \times (\mathbf{w} \times \mathbf{B}). \tag{2.3}
\]

Using Faraday’s law \( (1.6) \), then

\[
\nabla \times \mathbf{E} = -\nabla \times (\mathbf{w} \times \mathbf{B}).
\]

Now uncurl this equation to get

\[
\mathbf{E} + (\mathbf{w} \times \mathbf{B}) = \nabla \mathbf{F}, \tag{2.4}
\]

where \( \mathbf{F} \) is a scalar potential. Outside \( D \), \( \mathbf{w} = \mathbf{v} \) and \( \nabla \mathbf{F} = 0 \), and so without loss of generality we may assume \( \mathbf{F} = 0 \). However, taking the dot
Figure 2.2: Field line passes through the surface \((S)\) of a diffusion region \(D\) in 3D. After Priest et al. (2003)

product of \((2.4)\) with \(B\) we get

\[
B \cdot \nabla F = B \cdot E,
\]

\[
\Rightarrow \nabla F = \frac{B \cdot E}{|B|},
\]

which may be integrated to give

\[
F = \int E_{||}ds,
\]

where the integration is along a field line. Split the surface \((S)\) of the diffusion region into two parts, on one of which \((S_a)\) the magnetic field lines are entering \(D\) and on the other \((S_b)\) of which field lines are leaving \(D\). Suppose \(F = F_a = 0\) on the field lines before they enter \(D\) through \(S_a\) and \(F = F_b\) at the points where they leave through \(S_b\), see Figure 2.2. Then, in general, Eq. \((2.5)\) implies that \(F_b\) is non-zero and so \(F\) does not vanish on the field lines beyond \(S_b\), i.e., outside \(D\). But this is a contradiction and so we conclude that \(w\) does not in general exist.

The previous theorem shows that a flux-conserving flow may not exist in the presence of a localised non-ideal region. However, even in an ideal plasma
there are certain evolutions of the magnetic field for which no \( \mathbf{w} \) exists satisfying Eq. 7.1. The following theorem, due to Hornig and Schindler (1996), provides insight into which evolutions of the magnetic field are allowed in an ideal plasma in the vicinity of a magnetic null.

2. While in a 3D diffusion region, the field lines continually change their connection when they are moving in the non-ideal region, \( D \). In other words, field line conservation is violated everywhere in the volume of space defined by all field lines which thread \( D \). This means that a plasma element on one side of \( D \) is connected to a different plasma element on the opposite side of \( D \) at each instant in time.

3. In 3D, the mapping of field line footpoints is continuous everywhere except at separatrices, i.e. at the fans or spines of null points. Figure 2.3(a) shows schematically the mapping of field lines in \( \mathbf{B} \neq 0 \) reconnection. As we have mentioned in Chapter 1 the reconnection may take place in 3D either at null points or in the absence.

4. In 3D reconnection a flux tube does not generally break and reform perfectly to give two flux tubes see Figure 2.3(b).

5. A flux tube which passes into the non-ideal region immediately splits into two separate tubes, one defined by the set of field lines traced from the initial tube cross section on one side of \( D \), and the other defined by the set of field lines traced from the other cross-section on the other side of \( D \).

The table below summarises some of the differences between reconnection in 2D and reconnection in 3D.
Figure 2.3: The mapping of field lines in 3D. b) The breaking and imperfect rejoining of flux tubes. After [Priest et al. (2003)].


<table>
<thead>
<tr>
<th>Property of reconnection</th>
<th>Two Dimensions</th>
<th>Three Dimensions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Location</td>
<td>Only at an X-type null point</td>
<td>Anywhere in space, in the presence or absence of null-points.</td>
</tr>
<tr>
<td>Flux transport velocity</td>
<td>Exists everywhere in space except at the X-point</td>
<td>In general a unique velocity does not exist. Can be replaced by multiple transport velocities.</td>
</tr>
<tr>
<td>Change of connectivity</td>
<td>Occurs at the X-point</td>
<td>Occurs continually and continuously throughout the non-ideal region.</td>
</tr>
<tr>
<td>Counterpart</td>
<td>Unique reconnecting fieldline exists.</td>
<td>Generally no unique counterpart exists.</td>
</tr>
<tr>
<td>reconnecting fieldlines</td>
<td></td>
<td></td>
</tr>
<tr>
<td>Fieldline mapping</td>
<td>Discontinuous</td>
<td>Continuous (except at separatrices)</td>
</tr>
<tr>
<td>Rate of reconnection</td>
<td>Given by the electric field at the null point</td>
<td>Given by the maximum integrated parallel electric field across non-ideal region</td>
</tr>
</tbody>
</table>

### 2.1.3 3D Kinematic reconnection at a magnetic null point

As we have mentioned in Chapter 1, a null point is one of the locations that has been proposed for magnetic reconnection. Klapper et al. (1996); Bulanov and Sakai (1997) and Mellor et al. (2003) studied the generation of currents by a collapse of a 3D null point. This was followed up by Pontin and Craig (2005) who investigated the formation of current singularities in the vicinity of line-tied two- and three-dimensional null points using a numerical approach. The kinematics of
steady reconnection at three dimensional null points have been studied by [Priest and Titov (1996)] when $\eta = 0$, and a current-free magnetic null. They considered new types of three-dimensional magnetic reconnection, namely spine reconnection (when the cutting of field lines occurs across the fan plane when singular motion is driven at the spine axis) and fan reconnection (the cutting occurs across the spine, and singular motion is driven at the fan surface). Later, [Pontin et al. (2004, 2005)] improved this model by adding a finite resistivity, localised around the null point using a symmetric magnetic field. Two distinct cases were considered, in which the current ($J$) was directed parallel to first the spine and second the fan plane of the null, see Figure 2.4. The structures of the two solutions were found to differ greatly, and as a result, the reconnection rate, calculated by integrating the $E_{||}$ along field lines, represents very different behaviours of the flux for the two cases. The case in which $J$ is parallel to spine corresponds to one pair of complex conjugate eigenvalues, whereas when $J$ is parallel to the fan the eigenvalues are all real, see Figure 2.4.
In each of these investigations only the azimuthally symmetric case was con-
sidered, that is the case in which the magnetic field in the fan plane is isotropic. 
Pontin et al. (2004) were the first to show that the corresponding reconnection 
takes the form of a rotational slippage of magnetic flux threading the non-ideal region. (This is in contrast to the case where the current vector is parallel to 
the fan, in which case the magnetic flux is reconnected across the spine and fan 
(Pontin et al., 2005). The magnetic flux undergoes this rotational slippage in 
response to rotational flows in the ideal region in which the rate of flux transport 
in the azimuthal direction is different for field lines entering the non-ideal region 
than it is for field lines exiting the non-ideal region. The original model of Pontin 
et al. (2004) is based on the magnetic field 

\[ \mathbf{B} = B_0[r, jr/2, -2z] \]

in cylindrical polar coordinates. This field is associated with a spatially uniform 
current parallel to the spine (z-axis). The main part of this thesis, namely, 
Chapters 3, 4, 5, 6 and 7 deals with null magnetic reconnection, in which we 
build on the work started by Pontin et al. (2004, 2005), Rickard and Titov (1996); 
Galsgaard et al. (2003); Pontin and Galsgaard (2007); Pontin et al. (2007) have 
investigated the types of current concentrations that form self-consistently at 3D 
nulls in the dynamic regime. The results obtained encouraged Priest and Pontin 
(2009) to propose a new classification of 3D null point reconnection regimes.

2.2 Torsional Spine and Fan Reconnection

The old terminology ‘fan reconnection’ and ‘spine reconnection’ suggested by 
Priest and Titov (1996) does not adequately describe 3D null reconnection regimes.
Recent studies have revealed a number of characteristic ‘modes’ of reconnection that may occur at 3D nulls. These have recently been categorised by Priest and Pontin (2009) into ‘torsional spine reconnection’, ‘torsional fan reconnection’ and ‘spine-fan’ reconnection.

There are many numerical studies that have been conducted in order to overcome some limitations in analytical theory and to make the nature of reconnection at 3D nulls more obvious. First of all, Rickard and Titov (1996) investigated how a current accumulated along the spine axis and in the fan plane of null respectively. They used the linearised and cold magnetohydrodynamic (MHD) equations in cylindrical coordinates to analyse the perturbations of a single null point. This was demonstrated in the linear regime with a 2D simulation in the $rz$-plane.
Later, Galsgaard et al. (2003) proposed a rotational driving of the field lines around the spine and the perturbation propagates as a helical Alfvén wave towards the fan plane. They found a planar current layer develops on the fan plane. In addition to the above, Pontin and Galsgaard (2007) made use of a resistive magnetohydrodynamic (MHD) code in order to demonstrate how a rotational disturbance of field lines in the vicinity of either the spine or the fan plane can also produce strong currents along the spine when the rotation disturbs the fan plane, while rotation about the spine led to a current in the location of the fan near the null. The disturbance also propagates as an Alfvén wave, propagating along the background field lines. In the new categorisation of Priest and Pontin (2009), when the rotational slippage occurs in a tube of current aligned to the spine it is termed “torsional spine reconnection” while when the rotation forms a current layer in the fan plane the resulting reconnection is termed “torsional fan reconnection”. By contrast Pontin et al. (2007) found a current concentration forming at the null in response to shearing of the spine, where the fan eigenvalues are equal. The current flows through the null perpendicular to this shear plane, and thus parallel to the fan surface. They used a resistive MHD code to investigate the formation and dissipation of the current sheet in the null point when a local collapse of the field (the spine and fan collapse towards one another).

In all of the previous studies the perturbation was above a symmetric null point. So, these results encouraged me investigate what would happen when the symmetry of magnetic field is broken.
Chapter 3

Spine-Fan Magnetic Reconnection Kinematic Solution

3.1 Introduction

The kinematics of steady reconnection at three dimensional null points have been studied by Pontin et al. (2004, 2005) who improved the model of Priest and Titov (1996) by adding a finite resistivity, localised around the null point. The case in which the current \( \mathbf{J} \) was directed parallel to the fan plane of the null was considered by Pontin et al. (2005). It was found that magnetic flux is transported through the spine line and the fan plane. It can be shown that the reconnection rate gives a measure of the rate of flux transport across the separatrix surface of the null (Pontin et al., 2005). The case in which \( \mathbf{J} \) is parallel to the spine will be considered in a later chapter. In each of these investigations only the azimuthally symmetric case was considered, that is the case in which the magnetic field in the fan plane is isotropic. In this chapter, we focus on the case where \( \mathbf{J} \) is parallel to the fan surface (real eigenvalues), and for the first time consider
magnetic reconnection at a generic non-symmetric magnetic null point, i.e. a null for which the fan eigenvalues are not equal. In this chapter, we will investigate the effect of varying the symmetry of the initial null point field. The different modes of reconnection that occur in practice in a plasma (when the full set of MHD equations are considered) have recently been classified by Priest and Pontin (2009). In terms of the framework they have set up, the mode of reconnection considered here is termed *spine-fan reconnection*. The results in this chapter form the work described in Al-Hachami and Pontin (2010).

### 3.2 Method to Obtain the Solution

The subject of magnetic reconnection is a complex one, and its study is still in the early stages. Therefore, one approach that is used to try to understand the properties of this process is to consider a reduced set of the MHD equations. There are a number of analytical 3D solutions, which are described by Hornig and Priest (2003) and Wilmot-Smith et al. (2006, 2009), where there is no null point of the magnetic field, as well as the solutions in the presence of a null (Pontin et al., 2004, 2005; Priest and Pontin, 2009). These solutions are of kinematic reconnection, that is they satisfy Maxwell’s equations, as well as the induction equation. This approach can give great insight into the nature of a magnetic reconnection process occurring at an isolated diffusion region (Schindler et al., 1988). After investigating the properties of the solutions of this subset of the MHD equations, we then go on in the next Chapter to examine which properties survive when the full set of resistive MHD equations is solved.
We seek a solution to the kinematic, steady-state MHD equations in the locality of a magnetic null point. That is, we solve

\[ \mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}, \quad (3.1) \]
\[ \nabla \times \mathbf{E} = 0, \quad (3.2) \]
\[ \nabla \times \mathbf{B} = \mu_0 \mathbf{J}, \quad (3.3) \]
\[ \nabla \cdot \mathbf{B} = 0. \quad (3.4) \]

As discussed above, here we consider a null point with current directed parallel to the fan plane. We choose the magnetic field to be

\[ \mathbf{B} = \frac{B_0}{L} \frac{2}{p+1}(x, p y - j z, -(p+1) z), \quad (3.5) \]

where \( p \) is a parameter (here we restrict ourselves to the case \( j, p > 0 \)). This generalises the previous work by Pontin et al. (2005), who considered only the case where the field in the fan plane \( (z = 0) \) is azimuthally symmetric, corresponding to \( p = 1 \). For convenience we will write

\[ \frac{2B_0}{L(p+1)} = B'_0. \]

The current lies in the \( x \)-direction, and is given by

\[ \mathbf{J} = \frac{B'_0}{\mu_0}(j, 0, 0), \]

from Eq. (3.3). To find the local magnetic structure about a null point, we consider the magnetic field in the vicinity of the null point where the field vanishes \( (\mathbf{B} = 0) \). We consider the situation where all the eigenvalues are real. Examining the Jacobian matrix \( \mathbf{M} \) (as described in Chapter 1, Eq. 1.13), the eigenvalues of the null point are found to be

\[ \lambda_1 = B'_0, \quad \lambda_2 = p B'_0, \quad \lambda_3 = -(p+1) B'_0. \]
with corresponding eigenvectors

\[ k_1 = (1, 0, 0), \quad k_2 = (0, 1, 0), \quad k_3 = \left(0, 1, \frac{2p+1}{j}\right). \]

It is clear from the above that the fan plane is defined by \( k_1 \) and \( k_2 \) (since \( p > 0 \)), since they correspond to the same sign eigenvalues \cite{Parnell et al. 1996}. The fan plane of this magnetic null point is coincident with the plane \( z = 0 \) while the spine is not perpendicular to this, but rather lies along \( x = 0, y = jz/(2p + 1) \) (see Figure 3.1).

For the chosen magnetic field \textit{3.5}, closed-form expressions for the equations of the magnetic field lines can be found, by solving

\[
\frac{\partial \mathbf{X}(s)}{\partial s} = \mathbf{B}(\mathbf{X}(s)), \quad \text{(3.6)}
\]

where the parameter \( s \) runs along field lines, to give

\[
x = x_0 e^{B_0's} \quad \text{(3.7)}
\]
\[
y = \left(y_0 - \frac{jz_0}{2p+1}\right) e^{pB_0's} + \frac{jz_0}{2p+1} e^{-B_0'(p+1)s} \quad \text{(3.8)}
\]
\[
z = z_0 e^{-B_0'(p+1)s}. \quad \text{(3.9)}
\]

The inverse of Eqs. (3.7, 3.8, 3.9) are

\[
x_0 = xe^{-B_0's} \quad \text{(3.10)}
\]
\[
y_0 = \left(y - \frac{jz}{2p+1}\right) e^{-pB_0's} + \frac{jz}{2p+1} e^{B_0'(p+1)s} \quad \text{(3.11)}
\]
\[
z_0 = ze^{B_0'(p+1)s} \quad \text{(3.12)}
\]

which describes the equations of the magnetic field lines in terms of some initial coordinates \( \mathbf{X}_0 = (x_0, y_0, z_0) \).

We proceed to solve Eqs. (3.1-3.4) as follows. From Eq. (3.2) we can write, in general,
Figure 3.1: The structure of the magnetic null point with $j = 1$ and different values of $p$: (a) $p = 0.5$, (b) $p = 1$ and (c) $p = 2$. The shaded cylinder shows the shape of the diffusion region.
where $\phi$ is a scalar potential. Then the component of Eq. (3.1) parallel to $B$ is $-(\nabla \phi)_{||} = \eta J_{||}$ since

\[
\mathbf{E} + \mathbf{v} \times \mathbf{B} = \eta \mathbf{J}
\]

\[
\Rightarrow \quad \mathbf{E} \cdot \mathbf{B} + (\mathbf{v} \times \mathbf{B}) \cdot \mathbf{B} = \eta \mathbf{J} \cdot \mathbf{B}
\]

\[
\Rightarrow -\nabla \phi \cdot \mathbf{B} = \eta \mathbf{J} \cdot \mathbf{B}.
\]

Now, since \( \frac{d\phi}{ds} = \frac{d\phi}{dx} \frac{dx}{ds} \) by the chain rule and \( \frac{dx}{ds} = B \), we have

\[
\frac{d\phi}{ds} = \nabla \phi \cdot \mathbf{B}
\]

\[
\frac{d\phi}{ds} = -\eta \mathbf{J} \cdot \mathbf{B}
\]

and we can calculate $\phi$ by integrating along magnetic field lines:

\[
\phi = - \int \eta \mathbf{J} \cdot \mathbf{B} \, ds + \phi_0 \quad (3.14)
\]

where $\phi_0$ is a constant of integration. Note that while $\phi_0$ must be independent of $s$, it may be a function of $x_0, y_0, z_0$, i.e., it may vary from one field line to another.

By substituting the Eqs. (3.7, 3.8, 3.9) into the integrand of Eq. (3.14), we can perform this integration to obtain $\phi(X_0, s)$. Once this is done, we use Eqs. (3.10, 3.11, 3.12) to eliminate $s, x_0$ and $y_0$ to obtain $\phi(X)$, treating $z_0$ as a constant (see below). The electric field can subsequently be found from Eq. (3.13) and we then find the plasma velocity perpendicular to the magnetic field, $v_\perp$, by taking the vector product of Eq. (3.1) with $B$ to obtain

\[
v_\perp = \frac{(\mathbf{E} - \eta \mathbf{J}) \times \mathbf{B}}{B^2} \quad (3.15)
\]
where \( \mathbf{v}_|| \) is arbitrary because \((v_|| \hat{\mathbf{B}}) \times \mathbf{B} = 0 \) (where \( \hat{\mathbf{B}} \) is a unit vector along \( \mathbf{B} \)).

Now, in order to investigate the properties of magnetic reconnection in a fully 3D system, we impose a diffusion region that is spatially localised in 3D. This is the case relevant to astrophysical plasmas, which are known to be effectively ideal except in very small regions where energy release occurs. The diffusion region is chosen to be localised around the null point itself, in line with the results of past work which have shown that shearing motions tend to focus current in the vicinity of the null [Rickard and Titov, 1996; Pontin and Galsgaard, 2007; Pontin et al., 2007]. We choose the resistivity to be localised, and take it to be of the form

\[
\eta = \eta_0 \begin{cases} 
  f_1(R_1, z) & R_1 < a, \ (z^2)^{2p/2p+1} < b^2, \ p \geq 1 \\
  f_2(R_2, z) & R_2 < a^{1/p}, \ (z^2)^{2p/2p+1} < b^2, \ 0 < p < 1 \\
  0 & \text{otherwise}
\end{cases}
\]

(3.16)

with

\[
f_1(R_1, z) = \left( \left( \frac{R_1}{a} \right)^2 - 1 \right)^2 \left( \frac{(z^2)^{2p/2p+1}}{b^2} - 1 \right)^2
\]

and

\[
f_2(R_2, z) = \left( \left( \frac{R_2}{a^{1/p}} \right)^2 - 1 \right)^2 \left( \frac{(z^2)^{2p/2p+1}}{b^2} - 1 \right)^2
\]

where \( R_1 = \sqrt{(x^2)^p + (y - jz/(2p + 1))^2} \) and \( R_2 = \sqrt{x^2 + ((y - jz/(2p + 1))^2)^{1/p}} \),

where \( \eta_0, a \) and \( b \in \mathbb{R}^+ \). This is done in order to localise the product \( \eta \mathbf{J} \), and hence the diffusion region, since we have not yet discovered a way to proceed with our analytical method with localised \( \mathbf{J} \). The exact mathematical form for \( \eta \) is not expected to affect the qualitative structure of the solution, and is chosen in order to render the equations tractable. The crucial property for the structure of the solution is the localisation of the diffusive term \( \eta \mathbf{J} \). The dependence of \( \eta \) on \( p \) is chosen differently for \( p \geq 1 \) and \( 0 < p < 1 \) to ensure that \( \eta(x, y, z) \) is always differentiable, and is chosen in such a way as to maintain consistency in the dependence of the diffusion region size in the \( x \)-direction on \( p \), which is shown later to be an important property. Here \( \eta_0 \) is the value of \( \eta \) at the null point, and the diffusion region is a tilted cylinder centered on the spine axis, extending to
\[ z = \pm b^{(p+1)/2p} \text{ when } p \geq 1 \text{ and } z = \pm b^{(p+1)/2} \text{ when } p < 1. \]

The cross-section of the diffusion region in the \( z = 0 \) plane is circular with radius \( a \) when \( p = 1 \), but when \( p \neq 1 \), it extends to \( x = \pm a^{1/p} \) and \( y = \pm a \). In order to integrate Eq. (3.14), we must choose a surface on which to start our integration (i.e. on which to set \( s = 0 \)) that intersects each field line once and only once see Figure (3.2), in order that \( \phi \) is single-valued. We choose surfaces above and below the fan surface, \( z = \pm z_0, \text{ constant.} \) To simplify the mathematical expressions, and without loss of generality, we assume \( z_0 = b \). Performing the calculation of \( \phi(X) \) as described above yields two expressions for \( \phi \), for \( z > 0 \) and \( z < 0 \). In order to match these two expressions at the fan plane, that is for \( \phi \) to be smooth and continuous, and thus physically acceptable, we must set the value of \( \phi \) at \( z = \pm z_0 \) (i.e. \( \phi_0 \) in Eq. 3.14) to be

\[
\phi_0 = \frac{2B_0}{L(p+1)} \frac{\eta_0 j}{\mu_0} x_0 \left\{ \begin{array}{ll}
\frac{1}{7} b^{4(p-1)/p+1} - \frac{2}{3} b^{2(p-1)/p+1} - 1, & 0 < p < 1, \\
\frac{1}{(8p-1)} b^{4(p-1)/p+1} - \frac{2}{(4p-1)} b^{2(p-1)/p+1} - 1, & p \geq 1.
\end{array} \right.
\] (3.17)

Now \( \phi(X), \ E \) and \( \mathbf{v}_\perp \) can be obtained from (3.14), (3.13) and (3.15), as described earlier. The mathematical expressions are too lengthy to show here but can be calculated using a symbolic computation package. Here we have used Maple v.12 and the commands used can be found in Appendix A.1

### 3.3 Properties of the Solution

#### 3.3.1 Nature of reconnection

In order to determine the structure of the magnetic reconnection process, we will examine the plasma velocity perpendicular to the magnetic field (\( \mathbf{v}_\perp \)). This
velocity transports the magnetic flux outside the diffusion region. The flow does not cross the spine in the $x$-direction ($v_{\perp x}(0, y, z) = 0$), so that $v_{\perp x}$ is negligible for the reconnection process. However, in the $yz$-plane, the plasma flow crosses both the spine and the fan. Note that this is qualitatively the same as the situation described by Pontin et al. (2005) in the case of $p = 1$. The nature of the plasma flow in a plane of constant $x$ with different values of $p$ is shown in Figure 3.4. Note that the qualitative structure of the stagnation-point flow is not affected a great deal by varying $p$. However the general trend is that as $p$ tends to zero, the plasma flow across the fan plane becomes weaker see Figure 3.3. We will return to discuss this behaviour below.
Figure 3.3: $v_z$ flow with $p$, for $\eta_0 = \mu_0 = B_0 = j = a = b = L = 1$, $x = z = 0$ and $y = 1$.

Figure 3.4: Structure of the plasma flow (arrows) across the spine and fan (black lines) in a typical plane of constant $x = 0$, where the grayed area is the diffusion region, for (a) $p = 2$, (b) $p = 0.9$, (c) $p = 0.5$, for parameters $\eta_0 = \mu_0 = B_0 = j = a = b = L = 1$. 
### 3.3.2 Reconnection rate

It is generally accepted that magnetic reconnection plays a fundamental role in many types of explosive astrophysical phenomena, for example solar flares. Yet what determines the reconnection rate is still a major problem and this is an important aspect of any reconnection model. In general, the reconnection rate in 3D is defined by the maximal value of

\[ F = \int E_\parallel dl, \quad (3.18) \]

along any field line threading a spatially localised diffusion region \( D \) (e.g. Schindler et al., 1988). By symmetry, in this case

\[ F = \int_{C_2} E_\parallel dl, \quad (3.19) \]

where the curve \( C_2 \) lies along the \( x \)-axis, as shown in Figure 3.5. Since the fan is a flux surface, the integral may equally well be performed along the curve \( C_1 \) lying in the fan perpendicular to \( B \), see Figure 3.5. Now, since the curve \( C_1 \) lies outside \( D \) and therefore, along it \( \mathbf{v} \times \mathbf{B} = -\mathbf{E} \), we can write

\[ F = -\int_{C_1} \mathbf{v} \times \mathbf{B} \cdot dl, \quad (3.20) \]

from which it is clear that this reconnection rate measures the rate at which flux is transported across the fan surface by the flow in the ideal region (Pontin et al., 2005).

From Eq. (3.18), we have

\[ F = \int_{-a^{1/p}}^{a^{1/p}} E_x dx \]

\[ = \frac{2B_0}{L(p+1)} \frac{\eta_0 j}{\mu_0} \left\{ \begin{array}{ll}
16 a^{1/p}, & 0 < p < 1, \\
2a \left(1 + \frac{a^{4(p-1)}}{(4p+1)} - \frac{2a^{2(p-1)}}{(2p+1)}\right), & p \geq 1.
\end{array} \right. \quad (3.21) \]
Figure 3.5: The curves C1 and C2 joining two points on the x-axis, where the grayed area is the diffusion region, the arrows indicate the direction of field lines.

Note as a point of verification that this reduces to the expression found by Pontin et al. (2005) when $p = 1$. Here we consider the dependence of the reconnection rate on the parameter $p$ in two distinct cases—see Figure 3.6. First we set the parameter $j$ to be a constant, $j = 2j_0$ say, which results in a current which is dependent on $p$. We then go on to consider the case where we set $j = j_0(p + 1)$, so that the current ($\mathbf{J} = 2B_0j/(L(p+1)) \hat{x}$) is independent of $p$.

We will also consider the effect, in each of these cases, of taking different values for the parameter $a$, which controls the dimensions of the diffusion region. When $a = 1$, the diffusion region is symmetric for all $p$, having circular cross-section in any plane of constant $z$. However, as stated above, the boundary of the diffusion region intersects each of the 3 coordinate axes at

$$
\begin{align*}
    x &= \pm a^{1/p}, \quad y = \pm a, \quad z = \pm b^{(p+1)/2}, \quad 0 < p < 1, \\
    x &= \pm a^{1/p}, \quad y = \pm a, \quad z = \pm b^{(p+1)/2p}, \quad p \geq 1.
\end{align*}
$$

Thus the diffusion region becomes asymmetric in the $xy$-plane when $a \neq 1$ and $p \neq 1$. It will be seen later that this property is advantageous when comparing with the results of a numerical simulation.
Figure 3.6: Dependence of the reconnection rate on $p$, for $a = 1.5$ (solid curve), $a = 1$ (dash-dotted curve), $a = 0.5$ (long dashed), for (a) $j = 2j_0$, (b) $j = j_0(p+1)$, and parameters $\eta_0 = b = \mu_0 = B_0 = j_0 = L = 1$.

Reconnection rate as $p \to \infty$

In the limit $p \to \infty$, we observe from (3.22) that the diffusion region becomes approximately symmetric (exactly symmetric if $a = 1$). The following all holds for all values of $a$. For the two choices of dependence for our parameter $j$ stated above, evaluating Eq. (3.21) we find

$$\lim_{p \to \infty} F_{j=2j_0} = 0$$

$$\lim_{p \to \infty} F_{j=j_0(p+1)} = \frac{4j_0B_0\eta_0}{L\mu_0}.$$  

Consider first the case where the parameter $j$ is chosen to be a constant, $j = 2j_0$ (so that the current $\mathbf{J} = 4B_0j_0/(L(p+1)\mu_0) \hat{x}$). The magnetic field in this case is $\mathbf{B} \to (2B_0/L)(0, y, -z)$ as $p \to \infty$. That is, the magnetic field approaches a 2D X-point structure with zero current, and so the result above ($F \to 0$) is as expected.
By contrast, the reconnection rate approaches a constant finite value as $p \to \infty$ when $j = j_0(p + 1)$ (so that $J = 2B_0j_0/L\mu_0 \hat{x}$). In this case the magnetic field is $B \to (2B_0/L)(0, y - j_0z, -z)$. So the configuration is that of a 2D X-point with a uniform current (proportional to $j_0$). As the diffusion region has only a finite extent along the direction of the current ($\hat{x}$), the reconnection rate is finite. Note that as expected it is proportional to the parameters $\eta_0, B_0/L$ and $j_0$, where $\eta_0$ is the resistivity at the null, and $2B_0j_0/L\mu_0$ is the current modulus.

**Reconnection rate as $p \to 0$**

We now turn to the opposite limit; $p \to 0$. Note that our two parameter choices $j = 2j_0$ and $j = j_0(p + 1)$ clearly reduce to the same situation (with $j_0$ replaced by $2j_0$) as the limit is approached. Setting $p = 0$ the magnetic field is $B = (2B_0/L)(x, -j_0z, -z)$. We note that this field contains a neutral line in 3D (along $y$) which is orthogonal to the direction of current flow—not a configuration associated with 2D reconnection. In fact the limit of Eq. (3.21) is not well defined for all choices of our parameters. Therefore we consider that $p = 0$ is not a physically relevant parameter choice and consider only the limit $p \to 0$.

As $p \to 0$, the magnetic field in the fan plane parallel to the current vector becomes strong, while the $\hat{y}$-component becomes weak. Correspondingly, the flow across the fan surface becomes isolated to a small region near the fan, and weakens, see Figure 3.4. Furthermore, in this case the diffusion region $D$ is highly anti-symmetric.

Evaluating Eq. (3.21) we find

$$\lim_{p \to 0} F = \begin{cases} 
0, & a < 1 \\
\frac{32B_0j_0\eta_0}{15L\mu_0}, & a = 1 \\
\infty, & a > 1 
\end{cases}$$
(taking $j = 2j_0$). For $a < 1$ the extent of $D$ along the $x$-axis (direction of current flow) shrinks to zero. The result of the weak flow across the fan for small $p$ is therefore that the reconnection rate also approaches zero when $p \to 0$. By contrast, for $a > 1$ the boundaries of $D$ stretch to infinity along $x$ (see Eq. 3.22). Correspondingly, for $a > 1$ the reconnection rate $F \to \infty$. Although the flow is still very weak across the fan, the diffusion region now has much larger extent in the $x$-direction, and so although the flux reconnected per unit time per unit length in that direction decreases, the total flux reconnected increases. When $a = 1$, $D$ is symmetric, because the boundary is at $x = \pm a$, and the reconnection rate approaches a constant value as $p \to 0$.

### 3.4 Conclusion

In this Chapter we have investigated the effect of the symmetry of the magnetic field on magnetic reconnection at an isolated null point. We concentrate on the so-called spine-fan mode of 3D null point reconnection (Priest and Pontin 2009). In the following chapters we will go on to consider the ‘torsional spine’ and ‘torsional fan’ modes, which involve a current flowing parallel to the spine of the null point.

In this Chapter, we discussed a steady solution to a subset of the resistive MHD equations, where the magnetic null point was defined by $B=B'_0(x, py - jz, -(p + 1)z)$. This magnetic field has current aligned to the fan surface of the null point, and [Pontin et al.] (2005) investigated this situation in the non-generic symmetric case $p = 1$ (repeated eigenvalues). In this work we use $p$ as a parameter. By necessity, as the dynamics of the system are not included in this steady-state kinematic solution, a current is imposed, which has the same
orientation at the null as found in the simulations (the orientation of $\mathbf{J}$ at the null has been shown to be the crucial quantity in determining the magnetic structure of the reconnection process \cite{Pontin2004, Pontin2005}). In order to have a localised diffusion region around the null, we artificially imposed a localised resistivity. We found the nature of the plasma flow, and the resulting qualitative structure of the reconnection process, to be the same as found in the symmetric case. Specifically, we found plasma flow across both the spine line and fan plane of the null for all values of $p$. However, the results discussed above show that depending on our choice of parameters there are various different ways in which the diffusion region dimensions and the reconnection rate may depend on the asymmetry of the field ($p$). We will now go on in the next Chapter to describe simulations in the resistive MHD regime, in order to investigate which of these dependencies is relevant in a dynamically evolving plasma.
Chapter 4

Spine-Fan Magnetic
Reconnection Simulation

4.1 Introduction

In this Chapter, we describe the results of a resistive magnetohydrodynamic (MHD) numerical simulation. The aim of this numerical experiment is to test the analytical solution which was described in Chapter 3. The contents of this chapter have been published in the paper by Al-Hachami and Pontin (2010).

4.2 Computational Setup

We now proceed to test the results of the mathematical model presented in the previous chapter by performing numerical simulations, which solve the full set of
resistive MHD equations. We solve the MHD equations in the following form

\[ \mathbf{E} = -\mathbf{v} \times \mathbf{B} + \eta \mathbf{J} \]  \hspace{1cm} (4.1)

\[ \mathbf{J} = \nabla \times \mathbf{B} \]  \hspace{1cm} (4.2)

\[ \frac{\partial \mathbf{B}}{\partial t} = -\nabla \times \mathbf{E} \]  \hspace{1cm} (4.3)

\[ \frac{\partial \rho}{\partial t} = -\nabla \cdot (\rho \mathbf{v}) \]  \hspace{1cm} (4.4)

\[ \frac{\partial (\rho \mathbf{v})}{\partial t} = -\nabla \cdot (\rho \mathbf{v} + \tau) - \nabla P + \mathbf{J} \times \mathbf{B} \]  \hspace{1cm} (4.5)

\[ \frac{\partial e}{\partial t} = -\nabla \cdot (e \mathbf{v}) - P \nabla \cdot \mathbf{v} + Q_{visc} + Q_J, \]  \hspace{1cm} (4.6)

where \( \mathbf{v}, \mathbf{B}, \mathbf{E}, \eta, \mathbf{J}, \rho, \tau, P, e, Q_{visc}, Q_J \) are the velocity, magnetic field, electric field, resistivity, electric current, density, viscous stress, pressure, internal energy, viscous dissipation and Joule dissipation, respectively. Here we provide a brief explanation of the method used for the numerical simulations. We run simulations that are similar to those described by Pontin et al. (2007). For more details on the numerical method, see Nordlund and Galsgaard (1997); Pontin and Galsgaard (2007). All simulations use numerical resolution of \( 128^3 \) grid cells, a uniform resistivity model, and a so-called ‘hyper viscosity’ model. This is calculated using a combined 2nd- and 4th-order method, which effectively ‘switches on’ the viscosity only where grid-scale features develop in \( \mathbf{v} \), in order to maintain code stability. In this way, the effect of viscosity is minimised, and we focus on the effect of the resistivity (see Nordlund and Galsgaard (1997)).

We consider an isolated three-dimensional null point within our computational volume, which is driven from the boundary. We begin initially with a potential magnetic field

\[ \mathbf{B} = \frac{B_0}{L} \frac{2}{p+1} (x, py, -(p+1)z). \]  \hspace{1cm} (4.7)

Taking \( B_0=L=1 \) and \( \eta=0.0007 \), constant, throughout. The computational domain has dimensions \([x, y, z] = [\pm 3, \pm 3, \pm 0.5]\), with the magnetic field being line...
tied (i.e, \( v=0 \)) on all boundaries. In the kinematic model (described in the previous Chapter), the spine and fan plane are not orthogonal, but in the simulation at the outset they are orthogonal, which means the plasma is in equilibrium (we assume a uniform plasma at \( t = 0 \) such that the pressure gradient is zero we set \( \rho = 1 \) and \( e = 0.025 \) everywhere at \( t = 0 \)). At \( t = 0 \), the spine of the null point lies in \( z \)-direction, and the fan plane in the \( z = 0 \) plane see Figure 4.1(a).

A driving velocity is then assumed on the \( z \)-boundaries, which advects the spine footpoints in opposite directions on the opposite boundaries. The spine is driven until the resulting disturbance reaches the null, resulting in the formation of a current sheet as the magnetic field becomes stressed and distorted. The resulting configuration shares key properties with the configuration considered in the kinematic model: the spine and fan have collapsed toward one another generating a current parallel to the fan surface, and furthermore a localised non-ideal region is present around the null. After some time the driving velocity is reduced back to zero. The explicit form taken for the driving velocity is defined by the streamfunction

\[
\psi(z = \pm 0.5) = \pm 0.01 \left( \left( \frac{t - 1.8}{1.8} \right)^4 - 1 \right)^2 \sin \left( \frac{\pi x}{3} \right) \cos^2 \left( \frac{\pi y}{6} \right) e^{-8.9(x^2+y^2)},
\]

(4.8)

\( 0 \leq t \leq 3.6 \). This drives the spine footpoints in the \( \pm \hat{y} \) direction, but has return flows at larger radius (see Figure 4.1(b)). The driving is slow compare with typical Alfvén times of the system. More details can be found in the paper by Pontin et al. (2007). The authors performed simulations where the null was perturbed by shearing motions at the spine boundary; as a result, a strong current was found concentrating at the null. The spine and fan of the null point become closer to one another with time, since they were initially perpendicular (when the driving is finished the field starts to relax towards the initial configuration). Furthermore, they found that the current sheet will spread along the fan of the null based on
Figure 4.1: a) Schematic of the initial 3D null point in the computational domain. The black arrows indicate the direction of the boundary driving. b) Boundary driving flow in $z = 0.5$ plane.

the strength of driving. Below we describe our simulation results.

### 4.3 Results

#### 4.3.1 Current sheet

In order to simplify the discussion we will initially explain the behaviour of the current at one value of $p$ ($p = 2$). We first examine the temporal evolution of current in the volume. In the beginning, the spine and fan are orthogonal, but then the angle between them begins to change, reaching a minimum value once the current sheet forms. In other words, the null collapses from a perpendicular $X$-type null point, with the angle between the $X$ becoming greatly reduced, see
After the boundary driving ceases the current begins to decrease again, and the spine and fan relax back towards their initial perpendicular state, see Figure 4.4(a).

We now discuss how the current sheet formation, as described above, depends on the value of $p$. Figure 4.3 illustrates the dimensions of the current sheet for various values of $p$ at the time when the current modulus is of maximum value. For the case investigated previously, $p = 1$, the sheet was found to be approximately of equal dimensions along $x$ and $y$, the two coordinate directions associated with the fan surface. Looking at Figure 4.3 we see a large difference between the geometry of the current sheet at $p = 0.1$ and at $p = 10$ at the time of maximum current. We find the current sheet at $p = 0.1$ is large, being very extended along the $x$-axis, that is, the direction along which $J$ and the parallel electric field lie. However, this length decreases when $p$ is increased. The results suggest that when the value of $p$ approaches zero, the current sheet will grow indefinitely in the plane perpendicular to the shear, i.e. the direction of current flow through the null ($x$-direction). Note that with respect to the field strength in the fan plane, decreasing $p$ corresponds to weaker magnetic field strength along the $x$-direction. Thus the extension of the current sheet could be attributed to the fact that the weak field region extends in that direction and the magnetic field becomes less able to resist the collapse to form the current layer. That is, when the magnetic field parallel to the current becomes weaker there is less magnetic pressure associated with this ‘guide field’ component in the current sheet away from the null, and the current sheet is able to extend further away from the null.
4.3.2 Maximum current attained

Figure 4.4(a) illustrates the evolution of the current modulus maximum within the domain in time, for runs with different values of the parameter \( p \). We notice in each case that the peak current grows sharply in time to reach a maximum value, and then decreases again, as discussed above. Figure 4.4(b) shows the maximum value in time of the current plotted against \( p \). We observe that there is a positive correlation between \( p \) and the maximum current – in other words, when \( p \) increases then the maximum value of the current that is attained also increases. Furthermore, there is a negative correlation between the size of the current sheet and the value of \( p \), see Figure 4.3. Thus, although the current becomes more localised as \( p \) increases, it also becomes more intense. One important point to note is that of course the effective value of \( p \) will change during the simulations as the magnetic field is deformed. This can be confirmed be calculating the eigenvalues of \( \nabla B \) at the null as the simulations proceed. We find that the relative change is small – of order 1\% for \( p = 0.1 \) and order 10\% for \( p = 10 \). Thus the ordering of the values of \( p \) that we selected for our simulations is preserved and the trends that are observed for the \( p \)-dependence will be unaffected.

4.3.3 Reconnection rate

The nature of the plasma flow, and the resulting qualitative structure of the reconnection process, are found to be independent of the value of \( p \). Specifically, we find plasma flow across both the spine line and fan plane of the null for all values of \( p \). Figure 4.5 shows the plasma flow for four different values of \( p \). Comparing with Figure 3.4 we see that the trend for the geometry of the flow is the same as in the kinematic solution. Specifically, for large \( p \), the flow exhibits a relatively symmetric stagnation structure (in the \( x = 0 \) plane). For smaller \( p \)
the flow across the fan becomes confined to a narrower region, and comparatively weaker with respect to the flow across the spine.

Here we calculate the reconnection rate, i.e. the amount of flux transported across the fan surface, as before by integrating the electric field component parallel to the magnetic field ($E_\parallel$). Similarly to the reconnection rate calculation in Chapter 3, by symmetry, the integral is performed along the field line lying along the $x$-axis where, since we are in the resistive MHD regime, we have $E_\parallel = \eta \mathbf{J} \cdot \mathbf{B} / |\mathbf{B}|$.

In Figure 4.6 we show the evolution of the reconnection rate in time for different values of $p$. Initially the rate clearly stays constant (zero) in time, i.e. during the early evolution, between $t = 0$ and $t = 1$. Later, it starts to develop until it gains its maximum value, and then begins to decrease. This follows the same pattern as the evolution of the current, being indicative of the fact that the null point collapses to form the current sheet and reconnection occurs, and then the system relaxes once the driving ceases. It is clear from Figure 4.6 that the maximum reconnection rate attained increases as the value of $p$ is decreased. It is worth emphasising here that although our intuition tells us that there is a positive correlation between current and reconnection rate, by contrast in this study, we notice the inverse is true, i.e. when the peak current increases the reconnection rate decreases. This is because the diffusion region stretches when $p$ tends to zero in the direction where the $E_\parallel$ lies. Therefore, the rate increases even though the current decreases, since the integrand in Eq. (3.18) is non-zero over a much larger portion of the $x$-axis.
4.4 Comparison with Incompressible Models

We finally compare our results with those of the incompressible model of Craig and Fabling (1998). They were studying steady-state reconnection solutions (fan and spine reconnection), and their model involves the flow of material across one separatrix (fan or spine), while the other is contiguous with the current sheet but has no flow across it. Then a flow is imposed uniformly on all planes i.e., imposed incompressible (divergence-free) velocity field everywhere. The remarkable feature of their system is the symmetry in the velocity and magnetic fields that is broken only by the resistive term. In particular, the momentum equation and induction equation are

\[
(v \cdot \nabla) \omega - (\omega \cdot \nabla)v = (B \cdot \nabla)J - (J \cdot \nabla)B
\]

\[
(v \cdot \nabla)B - (B \cdot \nabla)v = \eta \nabla^2 B
\]

where the magnetic and velocity fields satisfy

\[
\nabla \cdot B = 0, \quad \nabla \cdot v = 0,
\]

and the current density and fluid vorticity are given by

\[
J = \nabla \times B, \quad \omega = \nabla \times v,
\]

with very small resistivity \( \eta \). The assumptions are made in order to facilitate their method of solution. The disturbance field is represented by plane waves propagating along the spine axis. We find their results differ from ours in terms of the dependence of the peak current on \( p \). In particular, they found that (in terms of our parameters) the maximum current decreases when \( p \) goes to infinity. This may be down to the very different geometries of the current sheet in the two models (the current sheet in their incompressible model is planar and extends to infinity in all directions along the fan for all values of \( p \)). However, it is of interest
to note that we actually find the same dependence of the reconnection rate on $p$, i.e. as $p$ decreases the reconnection rate increases (since in fact we find a negative correlation between $J_{\text{max}}$ and the reconnection rate as $p$ is varied).

4.5 Conclusions and Comparison Between Simulations and Kinematic Solution

We described the results of a computational resistive MHD simulation in which we investigated for different values of the parameter $p$ (the ratio of the fan eigenvalues). Since in this case the full set of MHD equations was solved self-consistently, we began with an equilibrium potential magnetic null point (with $\mathbf{J} = 0$). The system was then driven away from this equilibrium in such a way as to induce a local collapse of the null leading to current sheet formation and spine-fan magnetic reconnection. The resulting configuration shares key properties with the analytical solution described in Chapter 3: the spine and fan are non-orthogonal with a current flowing parallel to the fan surface, and a localised diffusion region is focussed at the null. Also, in both cases the flow in the $yz$-plane exhibits a stagnation-point structure. There is in agreement between the model and the simulations, in that for large $p$ the stagnation structure is relatively symmetric, while for smaller $p$ the flow across the fan becomes confined to a narrower region, and weaker compared with the flow across the spine.

One of the major results that arises from the sequence of simulations is that both the peak intensity and the dimensions of current sheet are strongly dependent on the symmetry/asymmetry of the field in the fan surface, or in other words, on the value of $p$. In terms of the sheet dimensions, the length along the direction of current flow at the null increases when $p$ goes to zero, i.e. the diffusion region
is stretched in the $x$-direction when $p$ tends to zero. In the kinematic solution it was also possible by choosing the correct parameters to have the diffusion region dimensions have such a $p$-dependence. In order for our kinematic solution, to be physically relevant, this implies that the parameter $a$ in our solution should be chosen such that $a > 1$. Furthermore, as there is little difference in the size of the diffusion region in $z$ for different $p$ in the simulations, we should take $b = 1$ in our mathematical model.

In addition to the current sheet at the null, we examined the reconnection rate in both cases. In order to compare the results, in light of the discussion above, we consider the parameter regime $a > 1$ in the kinematic solution. When $a > 1$ the reconnection rate $\rightarrow \infty$ as $p \rightarrow 0$. On the other hand, as $p \rightarrow \infty$ the reconnection rate approaches either zero or a constant finite value, depending on whether the current falls to zero or remains fixed, respectively, as $p$ is increased (see Figure 3.6). Turning to the simulations, as shown in Figure 4.6 the reconnection rate indeed becomes very large as $p \rightarrow 0$, in agreement with the kinematic model. In addition, as $p$ becomes large the current at the null falls, and the reconnection rate appears to asymptotically approach some small value, also in agreement with the kinematic model. Whether this value is finite or zero is not possible to tell within the restrictions of the present simulations. The results of both the mathematical model and simulations reveal that the symmetry/asymmetry of the magnetic field in the vicinity of a null can have a profound effect on the geometry of any associated reconnection region, and the rate at which the reconnection process proceeds.
Figure 4.2: Structure of the magnetic field for the run with $p = 2$ (a) at $t = 0$, and (b) at the time of maximum current ($t = 3.0$), once the magnetic field has locally collapsed to form a current sheet. The black field lines are traced from around the spine for $z > 0$, while the grey field lines are traced from $z < 0$. (c) Contour plots showing $|\mathbf{J}|$ in the $x = 0$ plane at $t=1.8$, 2.7, 3, 4.
Figure 4.3: Isosurfaces of $|\mathbf{J}|$, at 50% of the maximum at that time, showing dimensions of the current sheet for different $p$ at the time of maximum $|\mathbf{J}|$: (a) $p = 0.1$ ($t = 3.8$), (b) $p = 0.5$ ($t = 3.5$), (c) $p = 1$ ($t = 3.3$), (d) $p = 2$ ($t = 3.0$) and (e) $p = 10$ ($t = 3.2$). Plot dimensions are $[x, y, z] = [\pm 2.7, \pm 1, \pm 0.17]$. The origin is located at the lower back left-hand corner of the cuboid shown.
Figure 4.4: (a) Evolution of the maximum value of $|\mathbf{J}|$ in time with different $p$; $p = 0.01$ (dotted), $p = 0.05$ (long dashed), $p = 0.1$ (dash dot dot), $p = 0.5$ (dash dot), $p = 1$ (dotted), $p = 2$ (dashed) and $p = 10$ (solid). (b) Peak spatial and temporal value of $|\mathbf{J}|$ for different $p$. 
Figure 4.5: Plasma velocity in the $x = 0$ plane for $[y, z] = [\pm 0.3, \pm 0.3]$ and (a) $p = 0.5$ (b) $p = 1$ (c) $p = 2$ (d) $p = 10$. Background shading shows the current density. In each case the image is shown for the time frame in which the value of $|J|$ reaches a maximum.
Figure 4.6: (a) Reconnection rate at different values of $p$, where the dotted curve is the reconnection rate at $p = 0.1$, the long dashed curve at $p = 0.5$, the solid curve at $p = 1$, the dashed dot curve at $p = 2$ and the dashed dot dot curve at $p = 10$. (b) Variation of the maximum reconnection rate with the parameter $p$. 
Chapter 5

Torsional Spine and Fan
Kinematic Solutions

5.1 Introduction

In the previous two Chapters we have considered reconnection at a null in which the current is directed parallel to the fan plane. Here we move on to consider the case where the current is directed parallel to the spine of the null. Pontin et al. (2004) studied the kinematics of steady reconnection at a 3D null point when a finite resistivity localised around the null point and the current (J) was directed parallel to the spine. Furthermore, they considered a simple spiral null point, as we mentioned earlier in Chapter 2. They found counter-rotational flows centred on the spine, and showed that the reconnection takes the form of rotational slippage. At the beginning of this Chapter, we will examine magnetic reconnection at a generic non-symmetric magnetic null point, i.e., at one in which the fan eigenvalues are not equal, similar to the case considered in Chapter 3. As a first step we make a direct generalisation of the work by Pontin et al. (2004) by considering
a localised resistivity around the null. In the previous chapter we investigated the effect of varying the symmetry of the background null point field, and shown that while the basic properties of the reconnection are unchanged, the dimensions and intensity of the current layer, and the reconnection rate, are strongly dependent on the degree of asymmetry. In this Chapter, we are hoping to understand the properties of torsional spine and torsional fan magnetic reconnection at a 3D null point, with respect to their dependence on the symmetry of the magnetic field around null. We therefore go on to introduce two additional analytical kinematic solutions. In these solutions, we have been able to construct solutions in which the current density is localised, thus allowing a spatially uniform resistivity to be used. In each of our solutions, the structure chosen for the magnetic field and the resulting current layer is based on the results of resistive MHD simulations in which the dynamic formation of these current layers has been observed. In each case, we investigate the effect on our solution of varying the symmetry of the magnetic field. The torsional spine and torsional fan reconnection modes do not act to transfer magnetic flux between these topological domains, but rather permit a rotational slippage of the magnetic flux within the domains lying close to the nulls and therefore the domain boundaries. Owing to this fact, these null point reconnection modes are unlikely to be involved in energetic events that involve a large-scale restructuring of the magnetic flux between topological domains as the coronal field seeks a lower energy state (e.g. during solar flares). Rather, they are a mechanism to dissipate energy and reduce stress associated with the dynamic perturbation of the coronal field by the turbulent boundary driving from the photosphere.

Parts of the work presented in this chapter form a part of the manuscript by Pontin et al. (2011) which has been submitted for publication.
5.2 Kinematic Model for Reconnection with Spine-Aligned Current: Uniform Current

We first consider the most direct generalisation of the model of Pontin et al. (2004). We take a model with uniform current. Since we want an isolated diffusion region, \( \eta J \) should be localised in space. Consequently, we impose a localisation of the resistivity. As in Chapter 3, we seek a solution to the kinematic, steady, resistive MHD Eqs. (3.1-3.4) in the locality of a magnetic null point. Whereas in Chapter 3 we examined the kinematic behaviour around a magnetic null point whose associated current was parallel to the fan plane, here we are moving to investigate the behaviour of the magnetic field in the vicinity of a null point where the current \( J \) lies parallel to the spine. We choose the magnetic field to be

\[
B = \frac{B_0}{L_0} \left( \frac{2x}{p + 1} - \frac{1}{2} jy, \frac{2py}{p + 1} + \frac{1}{2} jx, -2z \right)
\]  

(5.1)

where \( p \) is a parameter (here we restrict ourselves to the case \( p > 0 \) without loss of generality). The special case \( p = 1 \) corresponds to rotational symmetry about the null point which was studied by Pontin et al. (2004). The spine lies in the \( z \)-direction, and the current is \( J = (B_0/\mu_0)(0, 0, j) \) from Eq. (3.3), and is also directed along the \( z \)-axis, (i.e., we have a null with current directed parallel to the spine line). It is initially useful to define \( Q = \frac{L}{2(p+1)} \), \( L = \sqrt{4p^2 - 8p + 4 - j^2p^2 - 2j^2p - j^2} \) such that, from the matrix \( \mathcal{M} \) (see Eq. 1.13), the eigenvalues are

\[
\lambda_1 = \frac{B_0}{L_0} (1 + Q), \quad \lambda_2 = \frac{B_0}{L_0} (1 - Q), \quad \lambda_3 = -2\frac{B_0}{L_0}.
\]

In line with Parnell et al. (1996) it is useful to define a threshold current,

\[
\dot{j}_{\text{threshold}} = \sqrt{\frac{4(p-1)^2}{(p+1)^2}}.
\]
Therefore, \( Q = \frac{1}{2} \sqrt{j_{\text{threshold}}^2 - j^2} \), which leads to three different cases of eigenvalues we must consider.

1- If \( j^2 > j_{\text{threshold}}^2 \), we have complex conjugate eigenvalues \((\det M > 0)\) see Figures 5.1(a) and 5.1(b).

2- If \( j^2 < j_{\text{threshold}}^2 \), the eigenvalues become real \((\det M < 0)\) see Figure 5.1(d).

3- If \( j^2 = j_{\text{threshold}}^2 \), we have repeated eigenvalues see Figure 5.1(c).

In addition to these three cases we also have that the eigenvalues are always complex when \( j > 2 \) for all \( p \).

In the following, we will discuss in detail each of the above cases separately, because they require different treatments.

5.2.1 Case 1: \( j^2 > j_{\text{threshold}}^2 \)

To start with, let us consider the situation where the level of the component of current parallel to the spine is greater than that of the threshold current. This implies that the two eigenvalues corresponding to the eigenvectors that define the fan plane are complex and field lines in this plane will be spirals (the fan and spine are perpendicular).

The eigenvalues of the null point in this case are

\[
\lambda_1 = \frac{B_0}{L_0} (1 + Q), \quad \lambda_2 = \frac{B_0}{L_0} (1 - Q), \quad \lambda_3 = -2 \frac{B_0}{L_0}.
\]
Figure 5.1: The magnetic field configurations of 3D non-potential fields (the fan plane $xy$-plane) with $j = 1$ and different values of $p$: (a) $p = 1$, (b) $p = 2$, (c) $p = 3$ and (d) $p = 5$, in the fan plane.
where $Q$ is imaginary, with corresponding eigenvectors

$$k_1 = \begin{pmatrix} 1 \\ -2p+2-L \\ j(p+1) \end{pmatrix}, \quad k_2 = \begin{pmatrix} 1 \\ -2p+2+L \\ j(p+1) \end{pmatrix}, \quad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$ 

Since $Q$ is imaginary, obviously the eigenvectors relating to the eigenvalues will also be complex conjugates (the real and imaginary part of $k_1$ and $k_2$ will define the fan surface).

For the magnetic field defined as in Eq. (5.1), closed-form expressions for the equations of magnetic field lines can be found, as in Chapter 3, by using Eq. (3.6) where the parameter $s$ runs along field lines, to give

$$x = x_0 e^{\frac{\beta_0}{\tau_0} s} \cos Qs + \frac{(y_0 j p + y_j + 2x_0 p - 2x_0)e^{\frac{\beta_0}{\tau_0} s}}{L}, \quad (5.2)$$

$$y = \frac{e^{\frac{\beta_0}{\tau_0} s}(-L \sin Qs + \cos Qs(2p - 2))}{j(p+1)} + \frac{1}{L j(p+1)}(y_0 j(p+1)$$

$$+ 2x_0(p - 1)(\sin Qs(2p - 2) + \cos QsL)e^{\frac{\beta_0}{\tau_0} s},$$

$$z = z_0 e^{\frac{\beta_0}{\tau_0} s}, \quad (5.4)$$

with the inverse mapping $X_0(x_0, s)$ given by

$$x_0 = e^{-\frac{\beta_0}{\tau_0} s} \left( \sin(Qs)(y_j p + y_j - 2x + 2p x) + L \cos(Qs) \right) \frac{1}{L}, \quad (5.5)$$

$$y_0 = e^{-\frac{\beta_0}{\tau_0} s} \left( \cos(Qs)(-y_j p - y_j + 2x - 2p x) + L \sin(Qs) \right) \frac{1}{L}, \quad (5.6)$$

$$z_0 = z e^{2\frac{\beta_0}{\tau_0} s}. \quad (5.7)$$

We now proceed to solve equations (3.1-3.4) by using the general method described in Chapter 3.

In order to have a localised non-ideal term $\eta \mathbf{J}$ we have to localise the resistivity as we mentioned before, since $\mathbf{J}$ is constant. Note that if we prescribe $\eta$ we can always calculate $\phi$ from the component of Eq.(3.1) parallel to $\mathbf{B}$, $-(\nabla \phi)_\parallel =$
by using Eq. (3.14). Substituting equations (5.2-5.4) into the integrand of Eq. (3.14), we can perform this integration to obtain \( \phi(X_0, s) \). Once this is done, we use Eqs. (5.5-5.7) to eliminate \( s, x_0 \) and \( y_0 \) to obtain \( \phi(X) \), treating \( z_0 \) as a constant. In general we define \( \eta \) in a piecewise manner to be able to solve for \( s \) on the boundary of diffusion region (this implies also that all field lines must cut each of the top surface and side surface of the cylindrical non-ideal region once and only once). However, in this case we were unable to find any such surfaces to bound the diffusion region which satisfy these conditions. We therefore take \( \eta \) to be of the form

\[
\eta = \eta_0 e^{-\frac{R^2}{a^2} - \frac{z^2}{b^2}}
\] (5.8)

where \( R = \sqrt{x^2 + py^2} \) and \( \eta_0, a \) and \( b \in \mathbb{R}^+ \). We use Maple’s inbuilt newtoncotes6 method to solve Eq. (3.14) where \( 101^3 \) gridpoints are used over the domain \(-2 \leq x, y \leq 2, 0 \leq z \leq 2 \) [see Appendix A.2 and B.1]. The exact profile of \( \eta \), given by Eq. (5.8), is chosen such that the dimension of the diffusion region is controlled by the parameters \( a \) and \( b \), where \( a \) controls the radius and \( b \) the height. The functional dependence of \( R \) on \( p \) is chosen to match the behaviour in numerical simulations presented in the next Chapter. The electric field \( E \) can subsequently be found by using a finite difference (we use a five point finite difference to compute the numerical derivative of \( \phi \) to calculate the electric field \( E \)) method and the plasma velocity perpendicular to the magnetic field, \( v_\perp \), by using Eq. (3.15).

### 5.2.2 Case 2: \( j^2 < j^2_{\text{threshold}} \)

In the case where the magnitude of the component of current parallel to the spine is less than that of the threshold current, all three eigenvalues are real and we
have corresponding eigenvectors

\[ k_1 = \begin{pmatrix} 1 \\ \frac{-2p+2-L}{j(p+1)} \\ 0 \end{pmatrix}, \quad k_2 = \begin{pmatrix} 1 \\ \frac{-2p+2+L}{j(p+1)} \\ 0 \end{pmatrix}, \quad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}, \]

so the fan and spine are perpendicular and the field lines in the plane of the fan become parallel to the line,

\[ y = \frac{(-2p + 2 + L)}{j(p+1)} x, \]

close to the null, and

\[ y = \frac{(-2p + 2 - L)}{j(p+1)} x \]

when far from the null. From Equation (3.6) the field line equations are:

\[
x = \frac{x_0(L - 2p + 2) - y_0(jp + j)}{2L} e^{\frac{\mu_0}{\tau_0 s}(1+Q)} + \frac{y_0(jp + j) + x_0(2p - 2 + L)}{2L} e^{-\frac{\mu_0}{\tau_0 s}(1-Q)} \]

\[
y = -\frac{(2p - 2 + L)(x_0(L - 2p + 2) - y_0(jp + j))}{2j(p+1)L} e^{\frac{\mu_0}{\tau_0 s}(1+Q)} + \frac{(-2p + 2 + L)(y_0(jp + j) + x_0(2p - 2 + L))}{2j(p+1)L} e^{-\frac{\mu_0}{\tau_0 s}(1-Q)} \]  

\[ z = z_0 e^{-2B_0 s} \]  

(5.9) \hspace{1cm} (5.10) \hspace{1cm} (5.11)

with the inverse mapping \( X_0(x_0, s) \), given by

\[
x_0 = \frac{x(-2p + 2 + L) - y(jp + j)}{2L} e^{-(1+\frac{1}{Q})\frac{\mu_0}{\tau_0 s}} + \frac{x(-2 + L + 2p) + y(jp + j)}{2L} e^{(-1+\frac{1}{Q})\frac{\mu_0}{\tau_0 s}} \]  

\[ y_0 = \frac{x(4 + 4p^2 - L^2 - 8p) + y(2p^2 - 2 + jL + jLp)}{2jL(p+1)} e^{-(1+\frac{1}{Q})\frac{\mu_0}{\tau_0}} \]

\[ - \frac{x(4p^2 - 8p + 4L^2) + y(2p^2j - 2j - jL - jLp)}{2jL(p+1)} e^{(-1+\frac{1}{Q})\frac{\mu_0}{\tau_0}} \]  

\[ z_0 = z e^{2\frac{\mu_0}{\tau_0 s}}. \]  

(5.12) \hspace{1cm} (5.13) \hspace{1cm} (5.14)
Note that here all the terms are real, since we have real eigenvalues. Now it is possible, by the method that was explained already in Section 5.2.1 of this section, to find $E$ and $v_{\perp}$.

### 5.2.3 Case 3: $j^2 = j_{\text{threshold}}^2$

In the case where the current parallel to spine and threshold current are equal, we find that the two of the eigenvalues are repeated so have only one associated eigenvector, such that

$$\lambda_{1,2} = \frac{B_0}{L_0}, \quad \lambda_3 = -\frac{2B_0}{L_0},$$

and the eigenvectors are, respectively,

$$k_{1,2} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \quad k_3 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$  

So the field lines lying in the plane of the fan directed away from the null and form what looks like a spiral null (critical spiral) [Parnell et al., 1996]. The field lines in the plane of the fan become parallel to the line,

$$y = -x,$$

both as they approach the null and as they approach infinity. From Eq. (3.6) we can find the field line equations, which are

$$x = x_0 e^{\frac{B_0}{L_0} s} - \frac{p-1}{p+1} (x_0 + y_0) e^{\frac{B_0}{L_0} s}, \quad (5.15)$$

$$y = -x_0 e^{\frac{B_0}{L_0} s} + \frac{p-1}{p+1} (x_0 + y_0) s e^{\frac{B_0}{L_0} s}, \quad (5.16)$$

$$z = z_0 e^{-\frac{B_0}{L_0} s}. \quad (5.17)$$
The inverse of Eqs (5.15-5.17) are:

\[ x_0 = e^{-\frac{B_0}{t_0} s} \left( \frac{p y_s + x p + x p s + x + s y - x s}{p + 1} \right), \]

\[ y_0 = e^{-\frac{B_0}{t_0} s} \left( -\frac{p y_s - y p + x p s - y - s y - x s}{p + 1} \right), \]

\[ z_0 = z e^{2\frac{B_0}{t_0} s}. \]

The method used to calculate \( E, v_\perp \) is the same as before.

### 5.3 Kinematic Solution-Analysis

#### 5.3.1 The plasma flow

We examine in this section the nature of the solution in the three different cases. To integrate Eq. (3.14) we choose to set \( s = 0 \) on \( z = \pm z_0 \) (\( z_0 = 2b \)), therefore \( \phi \) is constant for \( z > b \). The plasma velocity for \( 0 < z < b \) is rotational around the spine within the diffusion region which has an elliptical cross-section when \( p \neq 1 \). Again there is no flow across either the spine or the fan. In order to show the results more clearly, we will use

\[ v = v_\perp - \frac{(v_\perp)_z}{B_z} B. \]

This is convenient to show plots of the vector field in the plane of constant \( z \), without suppressing any information, since now the velocity has only \( x \) and \( y \) components (see Figure 5.2). It is clear from Figure 5.2 shows that the diffusion region \( D \) extends in the \( x \)-direction when \( p > 1 \). We can use the same technique as for \( z > 0 \) to find a solution for \( z \leqslant 0 \) by integrating from \( z = -b \).
Figure 5.2: Plasma flow in the plane $z = 0.5$, with different $p$ (a) at $p = 1$, (b) $p = 2$, (c) $p = 3$, (d) $p = 5$, for the parameters $B_0=1$, $a = 1$, $b = 2$, $\eta_0=1$, $j = 1$. The red curve marks the boundary of the non-ideal region.
5.3.2 Reconnection rate

The rate of reconnected flux, in general, is given by the maximal integral of the parallel electric field $E_\parallel$ along any field line threading a spatially localised diffusion region $D$. In two-dimensional models the reconnection line is the extension of the hyperbolic null point along the invariant direction. We will be considering just the flux reconnection in the half-space $z > 0$. When the current is parallel to the spine, as here, this quantifies a rotational slippage of flux (Pontin et al., 2004). Therefore from Eq. (3.18) the reconnection rate, $F$, is the integral over the parallel electric field along the spine axis, is given by

$$F = \int_0^\infty E_z \, dz = \phi(x = y = 0, z = \infty) - \phi(x = y = 0, z = 0) = \frac{\sqrt{\pi}}{2} b_j B_0 \eta_0.$$

In this work, the reconnection takes the form of a rotational slippage of magnetic flux threading the non ideal region, we found the reconnection rate is completely different to the situation considered in the previous work (Al-Hachami and Pontin, 2010), where $J$ is parallel to the fan plane of null point, and there is an effect of the diffusion region and parameters $a$ and $p$ on the rate of reconnection. Here the result of reconnection rate is independent of the parameter $b$.

5.4 Kinematic Model for Torsional Spine Reconnection with Localised Current

Several of the previous 3D analytical models of reconnection which have helped to increase our understanding of the process have been kinematic (Pontin et al., 2004, Pontin et al. (2005), Al-Hachami and Pontin (2010)). A typical feature of reconnection in astrophysical plasma is that non-ideal regions are localised in
3D as a result of intense current concentration. Therefore, in this section we describe a kinematic model for torsional spine reconnection within a current tube that is localised to the spine of the null point. The form of the magnetic field and resulting current structure is chosen to match behaviour observed in the numerical simulations described by Pontin and Galsgaard (2007). This is one of the features that distinguishes our model from previous kinematic models where a localised non-ideal region was obtained through using an artificially localised “anomalous” resistivity. Here, we have been able to construct a solution in which the current density is localised, thus allowing a spatially uniform resistivity to be used. This adds a degree of physical plausibility to the models, since in practice in an astrophysical plasma a localised non-ideal region is associated with a localised current layer. As in the previous section we are looking for a solution to the kinematic, steady resistive MHD equations (3.1-3.4) in the location of a magnetic null point. The resistivity $\eta$ is taken to be uniform. We consider first the azimuthally symmetric case before going on to consider situations where the symmetry is broken.

The magnetic field is taken to be of the form

$$\mathbf{B} = \frac{B_0}{L_0} (\mathbf{B}_P + \mathbf{B}_J)$$

(5.22)

where

$$\mathbf{B}_P = [r, 0, -2z]$$

(5.23)

and

$$\mathbf{B}_J = \begin{cases} 0, jr \left(1 - \left(\frac{r}{a}\right)^2m\right)^{2\mu} \left(1 - \left(\frac{z}{b}\right)^2n\right)^{2\nu}, 0 \end{cases} \quad \text{if } r < a \text{ & } |z| < b$$

$$[0,0,0] \text{ otherwise}$$

(5.24)

in cylindrical polar coordinates $(r, \theta, z)$ where $j, a, b \in \mathbb{R}^+$ and $n, m, \mu, \nu \in \mathbb{N}$. Here $\mathbf{B}_P$ defines the potential background null point component, and $\mathbf{B}_J$ defines the non-potential component of $\mathbf{B}$ associated with the tubular current structure,
which extends to radius \( r = a \) and to \( z = \pm b \). In accordance with previous results, we assume an extended tube of current aligned to the spine so that typically \( b \gg a \). Using Eq. \((3.3)\) we find

\[
\mathbf{J} = \frac{B_0}{L_0 \mu_0} \left[ \frac{4r j n_\nu \left(1 - \left(\frac{r}{a}\right)^{2m}\right)^{2\mu} \left(1 - \left(\frac{z}{b}\right)^{2n}\right)^{2\nu-1} z^{2n-1}}{b^{2n}}, 0, \right.
\]

\[
-2j \left(1 - \left(\frac{z}{b}\right)^{2n}\right)^{2\nu} \left(-1 + \left(\frac{r}{a}\right)^{2m}\right)^{2\mu} + 2m \mu \left(1 - \left(\frac{r}{a}\right)^{2m}\right)^{2\mu} \frac{r}{a} \right]
\]

\[(5.25)\]

so that the current peaks at the origin and vanishes at the boundary \((r = a, z = \pm b)\) and outside the diffusion region. The resulting magnetic field and current density are represented in the plots in Figure 5.3(b). For \( m = n = \mu = \nu = 1 \), \( \mathbf{B} \) and \( \mathbf{J} \) are continuous and differentiable. However, in order that all physical quantities in the final solution are continuous and differentiable it is necessary to choose higher values for these constants. We take \( m = 3, \mu = n = 2 \) and \( \nu = 1 \). The resulting magnetic field and current density are represented in the plots in Figure 5.3.

Parametric equations for the magnetic field lines associated with Eqs. \((5.22)\) \((5.24)\) can be found in a straightforward way by solving \( \partial \mathbf{X}(s)/\partial s = \mathbf{B}(\mathbf{X}(s)) \) where the parameter \( s \) runs along field lines, to give

\[
\theta = j r_0 \left( \frac{-6r_0^{12} z_0^4 e^{5s \frac{B_0}{L_0}}}{5a^{12} b^4} - \frac{r_0^6 e^{-8}}{3a^6 b^8} - \frac{2r_0^{12} z_0^4 e^{13s \frac{B_0}{L_0}}}{11a^{18} b^4} + \frac{r_0^{12} z_0^4 e^{13s \frac{B_0}{L_0}}}{a^{12} b^8} + \frac{6r_0^6 e^{-4}}{a^6 b^4} + \frac{2r_0^4 e^{-7s \frac{B_0}{L_0}}}{7b^4} - \frac{r_0^{18} e^{19s \frac{B_0}{L_0}}}{19a^{18}} \right.
\]

\[
-1 - \frac{3r_0^6 e^{-4}}{7a^6} + \frac{3r_0^{12} e^{13s \frac{B_0}{L_0}}}{13a^{12}} - \frac{12r_0^{12} z_0^4 e^{-13s \frac{B_0}{L_0}}}{5a^{12} b^4} + \frac{6r_0^{12} z_0^4 e^{-8s \frac{B_0}{L_0}}}{a^{12} b^8} + \frac{6r_0^6 z_0^4 e^{-s \frac{B_0}{L_0}}}{a^6 b^4} + \frac{2r_0^4 e^{-9s \frac{B_0}{L_0}}}{3a^6 b^8} + \frac{6r_0^2 e^{-6s \frac{B_0}{L_0}}}{11a^{18} b^4}
\]

\[
\left. + \frac{2z_0^4 e^{-15s \frac{B_0}{L_0}}}{7b^4} - \frac{10r_0^6 z_0^4 e^{-3s \frac{B_0}{L_0}}}{3a^{18} b^8} - \frac{10r_0^6 z_0^4 e^{-3s \frac{B_0}{L_0}}}{3a^{18} b^8} + \frac{3r_0^6}{7a^6} - \frac{3r_0^{12}}{13a^{12}} + \frac{r_0^{18} z_0^4 e^{11s \frac{B_0}{L_0}}}{19a^{18}} + \frac{\theta_0}{j r_0} + \frac{z_0^8}{15b^8} \right) \quad (5.26)
\]
Figure 5.3: a) Magnetic field lines for the torsional spine model defined by Eqs. (5.22, 5.30, 5.31) for $a = 1, b = 5, j = 3$. The shaded surface shows a current isosurface $|J| = 0.1$.  b) Current vectors in the $y = 0$ plane for $a = 1, b = 2$.  c) Current vectors in the $y = 0$ plane with the parameters $a = 1$ and $b = 4$. 
\[ z = z_0 e^{-2s \frac{B_0}{\tau_0}}, \]  

(5.27)

with the inverse transformation given by

\[ r_0 = re^{-\frac{B_0}{\tau_0} s}, \]  

(5.28)

\[ \theta_0 = j r \left( -1 + \frac{2e^{7s \frac{B_0}{\tau_0} z^4}}{7b^4} - \frac{6r^6 e^{\frac{B_0}{\tau_0} z^4}}{a^6 b^4} + \frac{r^{18}}{19a^{18}} + \frac{r^6 e^{9s \frac{B_0}{\tau_0} z^8}}{3a^{18} b^8} - \frac{r^{18} z^8 e^{-3s}}{3a^{18} b^8} - \frac{2r^{18} z^4}{11a^{18} b^8} \right) \]

\[ - \frac{e^{s \frac{B_0}{\tau_0} z^8}}{15b^8} + \frac{r^{12} z^8}{15a^{12} b^8} - \frac{r^{6} z^8}{3a^{6} b^8} + \frac{6r^6 z^4}{a^6 b^4} - \frac{3r^{12}}{13a^{12}} + \frac{\theta}{jr} - \frac{3r^6 e^{-7s \frac{B_0}{\tau_0}}}{7a^6} - \frac{r^{18} e^{-19s \frac{B_0}{\tau_0}}}{19a^{18}} \]

\[ + \frac{z^8}{15b^8} + \frac{3r^{12} e^{-13s \frac{B_0}{\tau_0}}}{13a^{12}} + e^{-s \frac{B_0}{\tau_0}} + \frac{3r^6}{7a^6} + \frac{2r^{18} z^4 e^{-11s \frac{B_0}{\tau_0}}}{11a^{18} b^4} - \frac{r^{12} e^{-3s \frac{B_0}{\tau_0}} z^8}{a^{12} b^8} + \frac{r^{18} z^8}{3a^{18} b^8} \]

\[ - \frac{2z^4}{7b^4} - \frac{6r^{12} z^4 e^{-5s \frac{B_0}{\tau_0}}}{5a^{12} b^4} + \frac{6r^{12} z^4}{5a^{12} b^4} \right) \)

\[ z_0 = z e^{\frac{B_0}{\tau_0} s}. \]  

(5.29)

To find \( \phi \) in Eq. (3.14) we should choose in which direction along the magnetic field lines to integrate. Since we are assuming a rotational current driving across the fan of a null point, we will integrate down towards the null from \( z = z_0 \) and up towards the null from \( z = -z_0 \), where \( z_0 = b \) is a constant, setting \( s = 0 \) on these surfaces for each half-space. As in the previously-described solutions, \( \phi_0 \) at \( z = \pm z_0 \) must be chosen such that \( \phi \) is continuous and smooth at the fan plane \( (z = 0) \). In this work we choose \( \phi_0 = 0 \) which satisfies these conditions. This means we are starting with \( \phi \) constant for \( z > b \), therefore from Eqs. (3.13) and (3.15) \( \mathbf{E} \) and \( \mathbf{v}_\perp \) are zero for \( z > b \). We can obtain the solution for \( z < 0 \) in a similar manner as for \( z > 0 \) by integrating from \( z = -b \).

The results of performing the above analysis are as follows. Rotational plasma flows are still present around the spine axis of the null as in the previous solutions. Using the freedom of arbitrary flow parallel to \( \mathbf{B} \) in the model, for illustrative
purposes we can choose to add a component $\mathbf{v}_\parallel$ such that $v_z = 0$, by using Eq. 5.21.

A purely azimuthal flow is found. The flow is clockwise in some $z$-planes and anti-clockwise in others (see Figure 5.4) and, in addition, may change its sense of rotation even within a given $z=\text{constant}$ plane (as shown in Figure 5.5). This is due to the presence of a return current close to $r=a$ (see Figure 5.3). The field lines’ change of connectivity is in the form of a counter-rotational slippage, as previously found.

We would now like to investigate how the properties of the solution vary when the rotational symmetry of the above system is broken. We may break the symmetry either in the potential component $\mathbf{B}_P$ defining the magnetic null or in the component $\mathbf{B}_J$ defining the current tube. Our new potential component of the magnetic field is given by

$$
\mathbf{B}_P = \left[ \frac{2}{p+1} x, \frac{2p}{p+1} y, -2z \right] \quad (5.30)
$$

in Cartesian coordinates where $p > 0$ is a parameter. Thus as $p$ varies the magnetic field along the spine direction is fixed while the ratio between the fan eigenvalues (associated with the eigenvectors along the $\hat{x}$ and $\hat{y}$ directions) varies. We choose to break the symmetry in $\mathbf{B}_J$ by converting to Cartesian coordinates and setting

$$
\mathbf{B}_J = \begin{cases} 
  j \left( 1 - \left( \frac{R}{a} \right)^6 \right)^4 \left( 1 - \left( \frac{qy}{b} \right)^4 \right)^2 [qy, x, 0] & R \leq a \  \& \ |z| \leq b \\
  [0, 0, 0] & \text{otherwise}
\end{cases} \quad (5.31)
$$

where $R^2 = x^2 + qy^2$ (note that this reduces to expression (5.24) when $q = 1$). This has the effect of distorting the current into a cylinder with elliptical cross-section, with major and minor axes along the $x$- and $y$-axes, extending to $x = \pm a$, $y = \pm a/\sqrt{q}$ (see the images in Figures 5.5, 5.6).
Figure 5.4: a) Vector-field representing the flow velocity at $z = 2$ b) $z = 3.5$ c) $v_\theta$ at $r = 0.5$, all the plots with the parameters $a = 1, b = 4, j = 1, B_0 = 1, L_0 = 1$. The red curve marks the boundary of the non-ideal region.
Figure 5.5: Plasma flow vectors in the $z = 2$ plane for the torsional spine kinematic solution. The parameters are $\eta_0 = \mu_0 = B_0 = L_0 = a = j = 1$ and $b = 4$ for a) $p = q = 1$. b) $p = q = 1.5$, c) $p = q = 2$, d) $p = q = 3$, e) $p = q = 5$ and f) $p = q = 10$. The red curve marks the boundary of the non-ideal region.
Figure 5.6: Current density for the torsional spine kinematic solution. The plane $z=2$ is shown and the parameters are (a) $p = q = 1$, (b) $p = q = 1.5$, (c) $p = q = 2$, (d) $p = q = 3$, (e) $p = q = 5$, (f) $p = q = 10$. The parameters are $\eta_0 = \mu_0 = B_0 = L_0 = a = j = 1$ and $b = 4$. 
When the rotational symmetry is lost it is no longer possible to find closed-form expressions for the field line mapping. We therefore numerically integrate $B$ to find field lines and solve Eqs. (3.14, 3.13 and 3.15) along those field lines to find $\phi$. $\phi$ is then interpolated onto a rectangular grid with $81^3$ gridpoints in each direction covering the volume $-2 < x, y < 2, 0 < z < 4$ with the solution being symmetric about $z = 0$. The package we used is *ode23* which is an implementation of an explicit Runge-Kutta method in MATLAB. Trapezoidal numerical integration method is used for the numerical integration of $\phi$, and five-point finite differences are used to compute the numerical derivative of $\phi$ to calculate the electric field ($E$) see Appendix B.2. Throughout this section we set $B_0 = L_0 = \eta_0 = j = 1, a = 1, b = 4$.

Pre-empting the results of the next Chapter, we present here results for $p = q$, such that as $p$ increases the current tube narrows along the direction associated with the large fan eigenvalue, i.e. the strong field direction in the fan. We restrict our attention to the range $p \geq 1$, which simply selects the $\hat{y}$ direction as the strong field direction in the fan.

The results of the above analysis are presented in all Figures (5.5-5.12). As $p$ is increased, and the current tube shrinks in the $y$-direction, with the dominant current component $J_z$ intensifying in the part of the tube close to the $y$-axis (i.e. the direction of the short axis of the ellipse) – see Figure 5.6. The strongest current in this region results in an enhanced plasma flow speed. The direction of the flow is distorted from the circular pattern found at $p = q = 1$ case, but continues to flow on closed elliptical paths around the spine ($z$-axis) see Figure 5.5. As the fan plane is approached the radius of the elliptical shells of positive and negative azimuthal flow increase, owing to the hyperbolic nature of the field structure – see Figure 5.7. We find $v_\theta$ from Eq. 5.21 such that $v_r = v_z = 0$ when there is rotational symmetry ($p = q = 1$), as shown in the first frame of
Figure 5.7: Intensity map of $v_x$ in the $x = 0$ plane for the kinematic torsional spine solution with $p = q = 1$, $a = 1$, $b = 4$.

Figure 5.8(a) The magnitude of max $v_\theta$ decreases as the distance from the central plane along the $z$-direction increases and vanishes outside the diffusion region, see Figures (5.9(a), 5.9(b)) it is clear from these Figures that the max $v_\theta \to \infty$ as $z \to 0$. However, this occurs at progressively larger $r$ as $z$ decreases, and because we do not have a realistic magnetic field (our magnetic field strength increases indefinitely with radius, and we must consider that it is embedded in some global region where the magnetic field is bounded) and we have looked for a locally valid solution. However, if $p \geq 1$ then $v_r \neq 0$ producing an asymmetry in the $x$–direction in proportion to the value of $p$ and $q$ as detailed in Figures (5.8(b), 5.8(c), 5.8(d), and 5.8(e)). Furthermore, it can be seen from Figure (5.9(c)) the relationship between the maximum value of $v$ with $p$.

In order to determine the reconnection rate we calculate $\Psi$ from

$$\Psi = \int \mathbf{E} \cdot \mathbf{B} / |\mathbf{B}| \, ds.$$  \hfill (5.32)

Due to the breaking of the symmetry it is no longer clear that the maximal value
Figure 5.8: Contours of $v_\theta$ and $v_r$ with different values of $p = q$. a) $v_\theta$ at $p = 1$. b) $v_\theta$ at $p = 2$. c) $v_r$ at $p = 2$. d) $v_\theta$ at $p = 5$. e) $v_r$ at $p = 5$ with $z = 0.5$ and $\eta_0 = j = a = 1, b = 4$. 
Figure 5.9: We show the maximum (a) and minimum (b) values of $v_\theta$ along the flux surface $R = \sqrt{\frac{a^2 b}{z}}$, (c) gives the maximum of $|v|$ with $p$ in the $z = 0.1$ plane. The parameters $p = q = \eta_0 = j = a = B_0 = L_0 = 1$, $b = 4$ have been taken throughout.
of $\Psi$ should occur along the spine field line, as was found in previous studies (note that the current modulus has maximum value away from the spine for large $p$). However, it turns out that indeed the maximum occurs along field lines asymptotically close to the spine for all $p$, as demonstrated in Figure [5.10]. Note that the elliptical pattern is distorted for $\Psi$ due to the spiralling field lines and the fact the $\Psi$ is an integrated quantity see Figure (5.11).

Figure [5.12] displays the peak value of the current density (which we impose) and the reconnection rate as a function of the degree of asymmetry. It is clear that the peak current scales linearly with $p(= q)$ and that correspondingly the reconnection rate scales linearly with $p$. Note however that all of the above solutions were obtained with a fixed value of the parameter $j$ which also contributes to controlling the peak current density, and that the velocity and reconnection rate will increase proportional to this parameter. However, as we shall see next Chapter this peak current is dependent on $p$ for the background null, which will undoubtedly affect the described $p$-dependence.

### 5.5 Kinematic Model for Torsional Fan Reconnection With Localised Current

In the previous section we have given a solution which models the situation when an imposed rotation of the fan plane of the null point, drives a current along the spine and gives rise to torsional spine reconnection. We now turn our attention to modelling the torsional fan reconnection mode, which involves rotational slippage of field lines is a current layer localised around the fan surface. We proceed to solve Eqs. (3.1-3.4) in the same way as described in Section 5.4. Again, we first analyse a model for the cylindrically symmetric case, in which for the first time a
Figure 5.10: Contours of reconnection rate $\int \mathbf{E} \cdot \mathbf{B} / |\mathbf{B}| ds$ plotted in the plane $z = 1$ for $B_0 = L_0 = j = a = 1, b = 4$ and a) $p = q = 1$, b) $p = q = 1.5$, c) $p = q = 2$, d) $p = q = 3$, e) $p = q = 5$, f) $p = q = 10$. 
Figure 5.11: Field lines in the fan plane for (a) $p = q = 1$ and (b) $p = q = 5$ for the parameters $\eta_0 = J = a = 1 = B_0 = L_0$, $b = 4$.

Figure 5.12: Dependence on the anisotropy parameter $p$ of the maximum values of $|J|$ and the reconnection rate $\Psi_{\text{max}}$ for kinematic torsional spine model.
Figure 5.13: a) Magnetic field lines for the torsional fan model defined by Eqs. (5.22, 5.30, 5.33) for $a = 5; b = 1; j = 50; p = 1.5; q = 1$. The shaded surface shows a current isosurface. b) Vector-field illustration of the localised axisymmetric current with the parameters $a = 3, b = 1, j = 1, B_0 = 1, L_0 = 1, y = 0$.

Localised current layer is included in the fan plane around the null. The structure of the magnetic field is chosen by comparing with the results of the numerical simulations of Galsgaard et al. (2003) and Pontin and Galsgaard (2007). We again construct our magnetic field as the sum of a potential part ($B_p$) and non-potential part ($B_J$) such that $B = \frac{B_0}{L_0}(B_P + B_J)$ with $B_P = [r, 0, -2z]$, and this time

$$B_J = \begin{cases} 
0, jrz \left(1 - \left(\frac{z}{a}\right)^2\right)^{2\mu} \left(1 - \left(\frac{z}{b}\right)^2\right)^{2\nu}, 0 & \text{if } r \leq a \\
[0, 0, 0] & \text{and } |z| \leq b
\end{cases} \quad (5.33)$$

see Figure (5.13). Note that $B$ is given in cylindrical coordinates and $B_\theta$ is now odd in $z$, and since we are modelling a current layer focused on the fan plane we assume that $b \ll a$. From Eq. (3.3) we find
\[
J = \frac{B_0}{L_0 \mu_0} \left[ r_j \left( 1 - \left( \frac{r}{a} \right)^{2m} \right)^{2\mu} \left( -1 + \left( \frac{z}{b} \right)^{2n} \right)^{2\nu} + 4n\nu \left( 1 - \left( \frac{z}{b} \right)^{2n} \right)^{2\nu - 1} \left( \frac{z}{b} \right)^{2n} , 0, \right. \\
\left. -2jz \left( 1 - \left( \frac{z}{b} \right)^{2n} \right)^{2\nu} \left( -1 + \left( \frac{r}{a} \right)^{2m} \right)^{2\mu} + 2m\mu \left( 1 - \left( \frac{r}{a} \right)^{2m} \right)^{2\mu - 1} \left( \frac{r}{a} \right)^{2m} \right].
\]

(5.34)

We again choose the integers \( m, n, \mu, \nu \) in such a way that all physical quantities in our solution are continuous and differentiable, specifically \( m = 3, \mu = 2, n = 2, \nu = 6 \). The field line mapping for \( B \) is again given by

\[
r = r_0 e^{B_0 s},
\]

(5.35)

\[
z = z_0 e^{-2B_0 s},
\]

(5.36)

and the inverse mapping is

\[
r_0 = r e^{-B_0 s},
\]

(5.37)

\[
z_0 = z e^{B_0 s},
\]

(5.38)

along with a lengthy expression for \( \theta(r_0, \theta_0, z_0, s) \) which is not required to obtain the solution. Figure (5.14) shows the structure of the field lines, outside the diffusion region we see the field lines straighten out as the field becomes more potential in nature. Note that the field lines are traced from a cylindrical surface \( r = r_0 \) surrounding the spine.

Due to the fact that the current is assumed to be localised and built up in the fan, created by rotation around the spine, we have used another technique to solve Eqs. (3.14, 3.13 and 3.15), i.e., the approach used here does not follow that we used in the spine reconnection case of the preceding section. This time we integrate Eq. (3.14) from \( r = r_0 \) (where \( r_0 \) is constant, \( r_0 > a \)) and set \( s = 0 \) at \( r = r_0 \). We therefore begin with \( \phi \) constant for \( r > a \), and therefore the electric
Figure 5.14: Top and side views of field lines in Eq. 5.33 plotted for parameters $\eta_0 = \mu_0 = B_0 = L_0 = b = j = 1$ and $a = 4$.

field and velocity are zero for $r > a$. We choose to do this because it leads to non-zero flow for $|z| > |z_0|$, which is consistent with the observed result from the simulations that this reconnection mode is set up by rotational driving flows in the regions around the spine footpoints.

The new torsional fan solution is represented in Figures 5.15(a), 5.15(b), 5.16 and 5.17(a). The structure of the plasma flow is essentially rotational, as found by Pontin et al. (2004). That is, when we subtract a component of $v$ parallel to $B$ such that $v_z = 0$, then the remaining flow is non-zero only in the azimuthal direction (see Figure 5.15). Thus, field lines traced from comoving footpoints in the ideal region at $z > z_0$ (or $z < -z_0$) rotate around the spine at a fixed radius, while field lines traced from the ideal region at $r > r_0$ remain fixed ($v = 0$ there), and we have a change of connectivity in the form of a counter-rotational slippage. Owing to the fact that $J_r$ changes sign for different levels of $z$, the rotational flow within the current layer has regions of both clockwise and anti-clockwise rotation,
Figure 5.15: Plasma flow vectors for the torsional fan kinematic solution. Plotted for parameters $\eta_0 = \mu_0 = B_0 = L_0 = b = j = 1$ and $a = 4$ with a) $p = q = 1$, b) plasma flow at $z = -0.5$ and $p = q = 1$, c) $p = q = 2$, d) $p = q = 3$, e) $p = q = 5$ and f) $p = q = 10$. The flow for all figures other than (b) is shown in the $z = 0.5$ plane.
Figure 5.16: Intensity map of \( v_x \) in the \( x = 0 \) plane for the kinematic torsional fan solution for parameters \( \eta_0 = \mu_0 = B_0 = L_0 = b = j = b = p = q = 1 \) and \( a = 4 \).

As before, we would now like to investigate the dependence of the properties of our solution on the symmetry of the magnetic field. We proceed as in Section 5.4 to solve Eqs. (3.14, 3.13 and 3.15) using the semi-analytical method described there see Appendix B.3. \( 81^3 \) gridpoints are used over the domain \(-4 \leq x, y \leq 4, 0 \leq z \leq 2\), and we use parameters \( B_0 = L_0 = j = \eta_0 = 1, b = 1, a = 4 \). We take the potential component of our magnetic field \( (\mathbf{B}_P) \) to be given by Eq. (5.30), with the non-potential component taken to be

\[
\mathbf{B}_J = \begin{cases} 
  jz \left(1 - \left( \frac{R}{a} \right)^6 \right)^4 \left(1 - \left( \frac{z}{b} \right)^4 \right)^{12} [y, qx, 0] & R \leq a \& |z| \leq b \\
  [0, 0, 0] & \text{otherwise} 
\end{cases} 
\]  

where \( R^2 = qx^2 + y^2 \) (which reduces to Eq. (5.33) when \( q = 1 \)). The current layer now has the shape of an elliptical disc, with major and minor axes along the \( x \)- and \( y \)-axes, extending to \( x = \pm a/\sqrt{q}, y = \pm a \). Pre-empting the results of the following Chapter, we present here results for \( p = q \), such that as \( p \) increases the
current disc shrinks along the direction associated with the small fan eigenvalue, i.e. the weak field direction in the fan. As shown in Figure 5.17, the current density is enhanced in the regions around the short axis of the ellipse.

As in the torsional spine solution, if we set the parallel flow in such a way as to eliminate $v_z$, then the plasma flow in the $xy$-plane follows closed elliptical paths, being strongest in magnitude where the current is enhanced (compare Figures 5.15 and 5.17). Furthermore, it can be seen from Figure 5.16(a) there exist regions of both negative and positive azimuthal flow around the spine.

Examining the dependence of the peak current and reconnection rate on the degree of asymmetry, we find that both increase with increasing $p = q$, as shown in Figure 5.18.

5.6 Conclusion

Let us now discuss our major results. Firstly, we have reviewed exact analytic solutions (kinematic solutions) describing magnetic reconnection in three dimensions where the magnetic null point was defined by

$$B = \frac{B_0}{L_0} \left( \frac{2x}{p+1} - \frac{1}{2} j y, \frac{2py}{p+1} + \frac{1}{2} j x, -2z \right).$$

This magnetic field has current aligned to the spine line of the null point, and Pontin et al. (2004) studied this situation in the non-generic symmetric case $p = 1$ (complex eigenvalues). In this work, we consider $p$ as a parameter. Our new model exhibits the same structure of plasma flow as previous torsional spine reconnection models i.e., only one sign of rotational flow due to the fact that we have a uniform current. In addition we found that the reconnection rate is independent of $p$. We then went on to present analytical models for torsional spine and torsional fan magnetic reconnection reconnection at 3D null points, which included for the first time fully
Figure 5.17: Contours of the current magnitude for the torsional fan kinematic solution plotted for parameters $\eta_0 = \mu_0 = B_0 = L_0 = b = j = 1$ and $a = 4$ with a) $p = q = 1$, b) $p = q = 1.5$, c) $p = q = 2$, d) $p = q = 3$, e) $p = q = 5$ and f) $p = q = 10$. The $z = 0$ plane is taken.
localised current layers that determine the boundary of the non-ideal region, thus alleviating the requirement in previous models to have an artificially localised (‘anomalous’) resistivity. We also for the first time investigated the generic case where the null point is not radially symmetric, i.e. where the fan eigenvalues are not equal. We have shown that the geometry of the current layers within which torsional spine and torsional fan reconnection occur is strongly dependent on the symmetry of the magnetic field defining the null point. Torsional spine reconnection still occurs in a narrow tube around the spine, but with elliptical cross-section when the fan eigenvalues are different. For torsional fan reconnection, the reconnection occurs in a planar disk in the fan surface, which is again elliptical when the symmetry of the magnetic field is broken. The short axis of the ellipse is along the weak field direction, with the current being peaked in these weak field regions. The peak current and peak reconnection rate in this case are clearly dependent on the asymmetry, with the peak current and the reconnection rate are increasing as the degree of asymmetry is increased.
Chapter 6

Torsional Spine and Fan Simulations

6.1 Introduction

In the previous Chapter we developed kinematic solutions to model reconnection at a null point with current directed parallel to the spine. We began by considering a model with uniform current, before presenting models with a current layer localised in the vicinity of the null point, spatially localised around the spine or fan. In both cases we analysed models exhibiting symmetry/asymmetry of the magnetic field. In this Chapter, as a complement to what we have done in Chapter 5, we investigate numerical experiments of torsional spine and fan reconnection at 3D null points. We generalise previous studies by considering rotational perturbations of a generic asymmetric magnetic null point. Resistive MHD simulations have demonstrated that the form of the current layer is different depending on whether the rotational perturbation primarily disturbs the fan field lines or field lines around the spine. The perturbation behaves essentially
as an Alfvén wave, travelling along the magnetic field lines which, due to the hyperbolic structure of the field around the null, leads to the perturbation accumulating either in the vicinity of the spine or the fan. When the fan field lines are subjected to a rotational disturbance, torsional spine reconnection takes place in a tubular current structure that forms at the spine [Rickard and Titov 1996, Pontin and Galsgaard 2007]. Within this tube, the magnetic field spirals around the spine line. The radius of the tube decreases, and the current intensifies, until the twisting of the field being driven by the perturbation is balanced by rotational slippage facilitated by resistive diffusion. When field lines in the vicinity of the spine line are disturbed, a current layer forms on the fan surface, within which torsional fan reconnection takes place [Rickard and Titov 1996, Galsgaard et al. 2003]. Again field lines spiral within the current layer, whose magnitude intensifies as the twisting of the field is concentrated in an increasingly narrow sheet over the fan surface. Once the sheet becomes sufficiently thin resistive diffusion dissipates the twist leading once again to a rotational slippage of field lines.

Parts of the work presented in this chapter form a part of the manuscript by Pontin et al. (2011) which has been submitted for publication.

6.2 Torsional Spine Reconnection Simulations

6.2.1 Computational setup

To complement the kinematic model solution we now describe the results of numerical simulations of the full system of resistive MHD equations. The code that we use is the same as the one used for the simulations presented in Chapter 4. The equations are solved on a numerical grid of $256^3$ gridpoints over
\[ [x, y, z] \in [\pm 0.75, \pm 0.75, \pm 3] \] with uniform \( \eta = 10^{-4} \). All boundaries are closed and line-tied. We repeat the simulations described by Pontin and Galsgaard (2007), in which a localised rotational perturbation of the magnetic field is imposed on a background equilibrium null point. Specifically, we begin with a potential magnetic field given by Eq. (5.30), and initialise the plasma density and thermal energy to be spatially uniform with values \( \rho = 1 \) and \( e = 0.025 \), respectively, and the velocity to be zero. We have checked that the results are not affected by the choice that the strong/weak field directions are parallel to the mesh directions. i.e., if we rotate the background field by e.g. \( \theta = \pi/4 \) about the \( z \)-axis and repeat the simulations we find the same results. In general, there are three cases to consider:

- \( p = 1 \): This case is discussed by (Pontin and Galsgaard, 2007). This magnetic null has cylindrical symmetry about the spine axis.

- \( p > 0, p \neq 1 \): This case describes field lines that rapidly curve such that they run parallel to the \( x \)-axis if \( 0 < p < 1 \) and parallel to the \( y \)-axis if \( p > 1 \).

- \( p = 0 \): In this case equation (5.30) reduces to the \( X \)-point potential field in the \( xz \)-plane and forms a null line along the \( y \)-axis through \( x = z = 0 \).

In this Chapter, we only consider \( p \geq 0 \) without loss of generality. In addition, we impose at \( t = 0 \) a magnetic field perturbation composed of a ring of magnetic flux centred on the null point and lying in the fan plane. The magnetic field of the flux ring is given in cylindrical coordinates \((r, \theta, z)\) with the only non-zero component being

\[
B_\theta = b_0 \exp \left( -\frac{(r - r_0)^2}{\alpha^2} - \frac{z^2}{\zeta^2} \right),
\]

with \( b_0 = 0.05, r_0 = 0.18, \alpha = 0.08, \zeta = 0.06 \) (see Figure 6.1).
Figure 6.1: Perturbed component of the magnetic field line, a rotation within the fan plane. After Pontin and Galsgaard (2007).

(We have also performed simulations where this initial perturbation is elliptical rather than circular, but found no change to the qualitative results – thus here we confine our discussion to the circular perturbation.)

For $\ell > 0$ the perturbation splits, with wavefronts travelling both toward and away from the null see Figure (6.2). We focus on the behaviour of the ingoing pulse, which for $p = 1$ gradually stretches out to form a cylindrical tube of intense current around the spine. When $p \neq 1$, the azimuthal symmetry of the perturbation wavefront is broken as soon as the evolution begins. The Alfvén speed in the radial direction now depends on $x$ and $y$, and so the wavefront travels toward the spine faster along the $y$-axis where $|B|$ is stronger. Propagation in the $z$-direction is essentially unaffected, and the current associated with the perturbation forms into a cylinder with elliptical cross-section, whose length and eccentricity both increase as the pulse steepens towards the spine and the peak
Table 6.1: Data on the simulations of torsional spine reconnection.

| $p$ | $|J|_{\text{max}}$ | $(J_z > 0)_{\text{max}}$ | $\Psi_{\text{max}}$ $^a$ | $L_x$ $^b$ | $L_y$ $^b$ | $L_z$ $^b$ |
|-----|-----------------|-----------------|-----------------|---------|---------|---------|
| 1   | 0.34            | 0.24            | $4.2 \times 10^{-5}$ | 0.012   | 0.012   | 3.9     |
| 2   | 0.36            | 0.27            | $5.7 \times 10^{-5}$ | 0.041   | 0.0098  | 3.9     |
| 3   | 0.29            | 0.31            | $6.0 \times 10^{-5}$ | 0.059   | 0.0079  | 3.3     |
| 5   | 0.31            | 0.34            | $6.1 \times 10^{-5}$ | 0.085   | 0.0075  | 2.5     |
| 10  | 0.31            | 0.36            | $5.8 \times 10^{-5}$ | 0.11    | 0.0072  | 1.9     |

$^a$ peak integrated parallel electric field attained.

$^b$ current layer dimensions measured at the time when $J_z > 0$ reaches its temporal maximum.

Figure 6.2: Travelling of wavefronts towards and away from the null point in the $x, z$ and $y = 0$ plane, at times $t = 0.6$ and $t = 1.8$ and $p = 1$. 
current correspondingly intensifies (see Figure 6.3).

6.2.2 Current Evolution

In the following sections we will describe the results of numerical simulations run with different values of $p$ ($p = 1, 1.5, 2, 3, 5, 10$). As shown in Figure 6.4, there is a strong (at least 100%) increase in the peak current density as the disturbance reaches the spine for each of the runs. It is also clear from the plot that this occurs earlier for larger $p$ due to the increased speed of propagation along the $y$-axis. In general, the peak current is higher for larger $p$. However, this is complicated by a competing effect – namely that there are two distinct maxima in $|\mathbf{J}|_{\text{max}}$ during the simulations, one corresponding to the localisation of the leading edge of the pulse and the other to the trailing edge of the pulse. As can be readily seen in Figure 6.3, they correspond to opposite signs of $J_z$. Thus, while the maximum positive value of $J_z$ strictly increases as $p$ increases, it is found that for $p = 1, 2$ the trailing edge of the pulse (corresponding to $J_z < 0$) dominates, see Figures 6.4 and 6.5.

The dimensions of the current layer are shown in the final three columns of Table 6.1. They are measured at a time corresponding to the localisation of the leading edge of the pulse, i.e. the time when $J_z > 0$ reaches its temporal maximum. The dimensions in the $xy$-plane demonstrate the elliptical nature of the current tube, centred on the spine, with eccentricity increasing with $p$ as shown in Figure 6.6. The length in the $z$-direction ($L_z$) decreases with increasing $p$, again as a simple consequence of the changing speed of propagation of the disturbance. Examining Eq. (5.30), one can see that the background magnetic field in the $z$-direction is independent of $p$. Thus, since the pulse localises at an earlier time for larger $p$ as explained above, the disturbance will not have travelled.
Figure 6.3: Time sequence showing contours of $J_z$ in the $x = 0$ (top) and $z = 0$ (bottom) planes for torsional spine reconnection with a) $p = 2$, b) $p = 3$ and c) $p = 5$. 
Figure 6.4: Evolution of the peak value of $|J|$ for the different torsional spine simulations: $p = 1$ (solid line), $p = 2$ (dotted), $p = 3$ (dashed), $p = 5$ (triple-dot-dashed) and $p = 10$ (dot-dashed).

such a large distance in this direction. It is also worth noting from Figures 6.6, 6.7 that for $p \neq 1$ the maximum current is attained not exactly on the spine, but in two locations displaced symmetrically from the spine along the $x$-axis.

### 6.2.3 Parallel electric field and reconnection

We turn now to consider the reconnection rate, calculated as described in Eq. 5.32. The maximal value of $\Psi$ is found over all field lines that thread the current layer, these being field lines that pass close to the null and its spine and fan. For $p = 1$, due to symmetry the maximum can be found on any field line which runs asymptotically close to the null. However, for $p > 1$ the current is strongest in the weak field regions around the $x$-axis, and so field lines that pass through these regions attain the highest values of $\Psi$. The evolution of the reconnection rate is
Figure 6.5: Plots of $J_z$. Each sub-figure shows $J_z$ at the time of maximum $J_z$ (left) and minimum $J_z$ (right). The $p$-values are (a) $p = 1$, (b) $p = 2$, c) $p = 3$, d) $p = 5$ and e) $p = 10$. All plots are in the $y = 0$ plane for $[x, z] = [\pm 0.3, \pm 3]$. 
Figure 6.6: Current isosurface at 50% of maximum value, for the torsional spine simulations with, a) $p = 1$, b) $p = 2$. c) $p = 3$. d) $p = 5$ and e) $p = 10$. Taken in each case at the time when the positive value of $J_z$ reaches a maximum.
plotted for the different simulations in Figure 6.8. The maximum value attained does not depend strongly on $p$ (see the third column of Table 6.1), except that it is significantly lower for $p = 1$. However, this is likely to be down to the fact that for $p = 1$ the disturbance extends all the way to the $z$-boundaries, and so we ‘miss’ some of the length of the current layer. Note that the reconnection rate calculated here is a net effect of integrating through regions of both positive and negative $\mathbf{E} \cdot \mathbf{B}$ – we have different senses of reconnection (rotational slippage) occurring on the leading and trailing edges of the pulse, and here we measure the net effect.

### 6.2.4 Discussion and comparison with kinematic model

The numerical simulations discussed above demonstrate the localisation of a rotational perturbation towards the spine of a non-symmetric linear 3D null. The resulting current intensification is associated with torsional spine magnetic reconnection. The current tube that forms around the spine has a structure that is closely matched by the kinematic steady-state model described in Sect. 5.4. In particular, the current is dominated by the component parallel to the spine ($J_z$),
Figure 6.8: Evolution of the reconnection rate for the different torsional spine simulations: $p = 1$ (solid line), $p = 2$ (dotted), $p = 3$ (dashed), $p = 5$ (triple-dot-dashed) and $p = 10$ (dot-dashed).

and is localised within a tube of elliptical cross-section. The short axis of the ellipse is aligned with the weak field direction in the fan plane, along which the current is most intense. The eccentricity of the ellipse increases as the magnetic field asymmetry increases.

The plasma flow also has a similar qualitative structure in the kinematic model and simulations. This structure is that of a rotation along elliptical paths around the spine, with these elliptical paths closely following the current density isosurfaces (see Figure 6.9). One difference between the model and simulations is that both signs of rotational are only seen during a certain period in the simulations. Specifically, as the twist associated with the perturbation propagates towards the null it drives flow predominantly in the positive rotational sense, and when the reconnection process is completed ($t \gtrsim 3.5$) the field then untwists leading to a large-scale flow in the opposite direction. It is only approximately between the times that $J_z$ reaches its maximum positive and negative values (approximately $1.5 < t < 3$, see Figure 6.4) that the Lorentz force accelerates the plasma in
opposite rotational senses at different distances from the null. By contrast, in the steady model where momentum balance is neglected, rotational plasma flows of both senses are required to maintain the steady state see Figure 6.9.

In the numerical simulations, it is found that the peak reconnection rate is approximately independent of $p$. We can understand why this should be the case by noting that, while $J_z$ increases with increasing $p$, the length of the current layer along $z$ decreases with increasing $p$. Thus for larger $p$ we have a larger integrand in Eq. (5.32), but it is non-zero over a shorter distance. Note also that the reconnection rate is found to be given by the integral of $E_\parallel$ along field lines lying asymptotically close the spine. Since $\mathbf{J}$ is dominated by $J_z$ and the current layer has minimal extent in the $xy$-plane, then it is natural that the reconnection rate should not depend strongly on the $\mathbf{B}_{xy}$ field components away from the null. In order to match these simulation results for the $p$-dependence of the reconnection rate, the magnetic field $\mathbf{B}_J$ in the kinematic model in Chapter 5 could be normalised by a factor proportional to the current modulus.

The behaviour of the disturbance – spreading out along the spine in all of the simulations – suggests that the dominant wave mode is an Alfvén wave. However, it is highly likely that other wave modes are present. Owing to the differing wave speeds approaching the null from different directions, one may speculate that if the disturbance were sufficiently strong as to be considered non-linear, then some of the effects discussed by McLaughlin et al. (2009) would be present. Those authors considered the propagation of a non-linear fast-mode wave towards a 2D X-point, and observed the formation of cusp-shaped structures as the wavefront collapsed, and the formation of both fast and slow mode shock waves. However, since our simulation is three-dimensional, it is not feasible at present to use the size of numerical grid required to properly resolve such features.
Figure 6.9: Plasma flow in the (x,y) plane for \( z = 0.05 \) and (a) \( p = 1 \), b) \( p = 2 \) and c) \( p = 5 \).
Each image is at the time of maximum current.
Table 6.2: Data on the simulations of torsional fan reconnection.

| p | $|J_{xy}|_{\text{max}}$ | $\Psi_{\text{max}}$ a |
|---|----------------|----------------|
| 1 | 0.94 | $12.7 \times 10^{-5}$ |
| 2 | 1.20 | $7.4 \times 10^{-5}$ |
| 3 | 1.33 | $5.7 \times 10^{-5}$ |
| 5 | 1.49 | $4.5 \times 10^{-5}$ |
| 10 | 1.55 | $3.8 \times 10^{-5}$ |

6.3 Torsional Fan Reconnection Simulations

6.3.1 Computational setup

We now perform numerical simulations similar to those described in Section 6.2, designed to investigate the effect of the background field asymmetry on the torsional fan reconnection mode. We take a numerical grid of $256^3$ gridpoints over $[x, y, z] \in [\pm 2.5, \pm 2.5, \pm 0.5]$ with uniform $\eta = 10^{-4}$ and closed and line-tied boundaries. As before we begin with the potential magnetic field given by Eq. (5.30), and initialise the plasma density and thermal energy to be spatially uniform with values $\rho = 1$ and $e = 0.025$, respectively, and the velocity to be zero. This time we perturb the initial equilibrium by applying a rotational driving velocity on the boundaries around the spine footpoints both above and below the null, as in Galsgaard et al. (2003). The sense of rotation is opposite above and below the fan plane. Specifically, we apply the following azimuthal velocity in the $z = \pm 0.5$ planes;

$$v_\theta = A \left( \left( \frac{t - \tau}{\tau} \right)^4 - 1 \right)^2 r (1 + \tanh(1 - C^2 r^2)) \quad t \leq 2\tau$$  \hspace{1cm} (6.2)

where $r = \sqrt{x^2 + y^2}$, $C = 10$, $\tau = 1.6$ and $A = \mp 0.1$ at $z = \pm 0.5$. 
6.3.2 Results

For the symmetric case \( p = 1 \), the evolution of the system following the initiation of the driving velocity is described in detail by Galsgaard et al. (2003). A disturbance that is dominated by a torsional Alfvén wave is launched from the driving \( z \) boundaries towards the null. The wave front spreads along the \( x \)- and \( y \)-directions travelling along the field lines, with its velocity in the \( z \)-direction being independent of \( x \) and \( y \). The wavefront steepens as it approaches the fan surface due to the hyperbolic structure of the field, eventually forming a planar current layer in the fan. This process is demonstrated in Figure 6.10 (see also Figure 6.11), from which one can see that the current associated with the current layer is orientated parallel to the fan surface and flows radially inwards toward the null (note that in this \( x = 0 \) plane, \( J_x \) is zero by symmetry). It is worth noting that there also remains a large distributed current near the boundaries where the twisting was applied – observed as strong concentrations of \( J_z \) – with the modulus of the current in this region being approximately equal to that of the current in the layer at the fan. We have repeated the simulation with different values of \( \eta \), and found that for lower \( \eta \) the peak current density at the fan is stronger relative to the concentration near the boundary. Hence, if we were able to run the simulation with a realistic value of \( \eta \) for an astrophysical plasma the current layer at the fan would dominate.

Owing to the high resistivity that we must use – and the resulting dominance of the \( J_z \) component near the driving boundaries (as discussed above), it is most clear to observe the formation and evolution of the fan current layer by plotting evolution of the peak value of \( |J_{xy}| \), as in Figure 6.12 (the dominant current component in the current is parallel to the fan surface for all \( p \)). We can see that for \( p = 1 \) this quantity rises steadily as the pulse approaches the fan, after which
Figure 6.10: Contours of $J_y$ (left) and $J_z$ (right) in the $x = 0$ plane over $[y, z] \in [\pm 2.5, \pm 0.5]$ for $t = 0.8$ (top), $t = 1.8$ (middle) and $t = 4.2$ (bottom), for the torsional fan simulation with $p = 1$. 
Figure 6.11: Contours of $|\mathbf{J}|$ for the torsional fan simulations, plotted in the $z = 0$ (fan) plane over $[x, y] \in [\pm 2.5, \pm 2.5]$ for a) $p = 1$, b) $p = 2$, c) $p = 5$ and d) $p = 10$. Taken in each case at the time when the value of $J_{xy}$ reaches a maximum.

there is a period of around 1.5 Alfvén times when the value remains steady, after which it decreases. Note that this steady period is consistent with the period during which the driving velocity remains steady – see the crosses in the figure.

The simulation described above has been repeated for different degrees of symmetry in the initial magnetic field (values of $p$ in Eq. (5.30)). We consider values of $p > 1$, so that the magnetic field strength increases more quickly along the $y$-direction than along the $x$-direction. As a result, when the boundary driving is initiated the azimuthal symmetry of the wavefront that propagates into the domain is broken. The wavefront remains approximately planar ($z$ constant), spreading into an elliptical shape with long axis along the $y$-direction (along which the Alfvén speed is greater) and short axis along $x$. However, as the wavefront gets closer to the null, it becomes steadily more inhomogeneous, with the current density focussing in the weak field regions near the $x$-axis. That is, although the current layer that forms eventually at the fan is more extended along $y$ (as a simple consequence of mapping a circular driving region on the boundary along $\mathbf{B}$), this current is most intense along the short axis of the ellipse, as shown in Figure 6.11.

For different $p$ the dominant current component in the current sheet is always
Figure 6.12: Evolution of the maximum of $J_{xy}$ for the different torsional fan simulation runs: $p = 1$ (solid line), $p = 2$ (dashed), $p = 3$ (dotted), $p = 5$ (dot-dashed) and $p = 10$ (triple-dot-dashed). For comparison, the time variation of the amplitude of the driving velocity is also represented, with the crosses.

parallel to the fan surface, so to observe the formation of this sheet it is again instructive to examine the evolution of the peak values of $J_{xy}$. This is plotted for the runs with different $p$ in Figure 6.12. It is clear that the overall qualitative behaviour is similar between the simulations. However, the overall peak current attained is largest when the initial field is most asymmetric ($p = 10$) – see also Table 6.2. As $p$ is increased, this increasing current maximum is localised in a gradually narrower ‘channel’ around the $x$-axis. Note that for the simulations with the largest values of $p$ the peak current does not remain steady for such a long period as for small $p$. This may be influenced by the fact that the disturbance interacts with the $y$-boundaries at a later time for large $p$.

We turn now to consider the rate of reconnection in the different simulations. As discussed above, at the limited magnetic Reynolds number we are able to use the fan current layer does not dominate over the distributed current near
the driving boundaries – though indications are that it would for more realistic astrophysical parameters. Therefore, in order to examine only the effect of reconnection in the thin current layer, we calculate the reconnection rate by integrating $E_\parallel$ along the magnetic field line in the fan plane that passes through the peak of the current density – which in practice lies very close the $x$-axis by symmetry. When the current layer has fully formed at the fan this genuinely measures the reconnection rate associated with the dynamically forming fan current layer. Clearly, at earlier stages when the perturbation is yet to reach the fan there will still be a modest amount of reconnection, which is not measured by this approach. Therefore, when analysing the plots of reconnection rate versus time shown in Figure 6.13, one should bear in mind that the curves do not accurately portray the maximum reconnection rate prior to $t \approx 2.5$. However, they do provide information on the peak reconnection rate in the system over time. We see that this occurs in each of the simulations at $t \approx 4$, just before the peak current density starts its decline. A clear pattern emerges that the reconnection rate is greatest for the symmetric case with $p = 1$, and steadily decreases for simulations with larger $p$.

6.3.3 Discussion and comparison with kinematic model

The numerical simulations discussed above demonstrate the localisation of a rotational perturbation towards the fan of a non-symmetric linear 3D null, resulting in torsional fan magnetic reconnection. The planar current structure that forms around the fan qualitatively resembles that found in the kinematic steady-state model described in Section 5.4. In particular, the current is dominated by a component parallel to the fan where the current vector is directed (at its maximum
intensity when \( p \neq 1 \) radially towards the null. Current isosurfaces have an elliptical shape, with the short axis of the ellipse aligned with the weak field direction in the fan plane, along which the current is most intense. The eccentricity of the ellipse increases as the magnetic field asymmetry increases. The plasma flow also has a similar qualitative structure in the kinematic model and simulations. This structure is that of a rotation along elliptical paths around the spine, with these elliptical paths closely following the current density contour see Figure 6.14.

In the torsional fan reconnection simulations – in contrast to the torsional spine case – the peak reconnection rate depends strongly on the magnetic field asymmetry (\( p \) parameter). This is because in the torsional fan case, the contribution to the integrand in the reconnection rate definition (5.32) comes largely from field lines closely aligned to the \( xy \)-plane, which is the plane in which the magnetic field is varying as \( p \) is varied. The interpretation of the reconnection
Figure 6.14: Plasma flow in the $x, y$ and $z = 0.1$ plane for torsional spine numerical solution with a) $p = 2$ and b) $p = 5$. Each image is at the time of maximum current.

rate when $p \neq 1$ should be the same as that described in Section 5.5. Perhaps counter-intuitively, the scaling of the reconnection rate with $p$ is opposite to the dependence of the peak current on $p$ (as $p$ increases the peak current increases while the reconnection rate decreases – see Figures 6.12, 6.13). This is because, unlike in 2D, the reconnection rate is not a local quantity defined at a point, but rather is defined as an integral along a field line. We see from Figure 6.11 that while the peak current is higher for large $p$, it is localised along the field line (along $x$) to a much greater extent. So the peak value of the integrand in Eq. (5.32) may be larger for larger $p$, but the integrand is non-zero over a much shorter section of the field line, resulting in a lower net value for the integral. Note that this was also observed in the study of spine-fan reconnection carried out by Al-Hachami and Pontin (2010) and discussed in Chapters 3 and 4.

Note that in the simulations after the current layer that formed dynamically at the fan has started to dissipate, the dominance of the current near the driving
boundaries dominates to a steadily greater extent, since the large-scale current concentrations dissipate only very slowly. This is consistent with the results of Pontin and Craig (2005), who found that only when shear perturbations of the spine/fan are made does the lowest energy state of the system involve a current sheet at the null – when rotational perturbations are performed around the spine then the lowest energy state is achieved when the twist of the field is distributed along the field lines, in concentrations that extend outwards from the driving boundaries.

Finally, note that here we have chosen to model an instance of torsional fan reconnection in which the vorticity of the driving flow around the spine has opposite sign on the opposite boundaries. As a consequence, a strong current – directed predominantly in the radial direction – is present at the fan surface. It is worth noting that in our kinematic model, we could have equally chosen $B_\theta$ to be even in $z$ to model the situation where the driving flows have the same sign of vorticity, which lead to cancellation at the fan plane of the currents generated (Galsgaard et al., 2003). It should also be noted that the return currents present close to $z = \pm b$ are not present in the simulations as we drive only in one sense, but could be induced by reversing the sign of the driving velocity at some intermediate time in the simulations.
6.4 Numerical Simulations of Torsional Fan Reconnection: Alternative Method

6.4.1 Method

In this section, we will compare and test our theoretical results of the mathematical model that we presented in the previous Chapter. In other words, we would like to numerically test the validity of the steady-state 3D reconnection solutions discussed in Section 5.5. We consider a rotation about the spine of the isolated 3D null point within our computational volume, which is disturbed by perturbing the magnetic field in the same way as in Section 6.2. This allows a more direct comparison between the two reconnection modes via the simulations. To obtain fan reconnection numerical results, which can be compared to the steady-state results, we run simulations that are similar to those described by Pontin and Galsgaard (2007). That is, we set the boundary velocity to zero and apply the same perturbed magnetic field form to that used in rotation about fan

\[ B_\theta = b_0 \exp \left( -\frac{r^2}{\alpha^2} - \frac{(z - z_0)^2}{\zeta^2} \right) \]  

(6.3)

where \( z_0 = 0.15, \zeta = 0.06, \alpha = 0.05, b_0 = 1, \eta = 0.0002 \) and the domain size is chosen to be \([x, y, z] = [\pm1.25, \pm1.25, \pm0.75] \) (see Figure 6.15).

In the previous section, we have investigated the propagation of a helical Alfvén wave toward the fan plane, launched by a rotational driving of the field lines around the spine, but the one major difference in this simulation is that there we imposed a driving velocity at the (line-tied) boundaries, starting initially with zero velocity in the domain and having a peak velocity on the boundaries, whereas
Figure 6.15: A schematic illustration showing the fan and spine of a null together with the perturbation magnetic field (circular regions with arrows). The perturbation represents a rotation around the spine line.

here we perturb the magnetic field within the domain (using an internal magnetic perturbation). Note that we begin initially with a potential magnetic field that describes a 3D null point located at the origin, exactly the same as that given in Eq. (4.7)

6.4.2 Current evolution and plasma flow

Now we present the results. We wish to examine how the current evolves, spreading out along the fan plane, as described by Pontin and Galsgaard (2007) for case $p = 1$. In Figure (6.16) the current is seen to be localised in three-dimensions around the origin and has its greatest strength along the fan plane. Note that this gives a close match to the pattern of the current in the torsional fan kinematic solution presented in the last chapter – see Figure 5.13. As before, as the
Figure 6.16: Vector plots of the current flow at $t = 0.9$ and $p=1$ in the $z = 0$ plane.

perturbation evolves, it generates a current which spreads out along the diverging field lines on both sides of the fan plane.

Since we superpose two disturbances of the magnetic field perturbation given in Eq. 6.3 with $z_0$ of opposite sign, in this case the current $j_z$ has an opposite sign and no current will pass through the null point. Instead the current concentrates in the fan plane. In other words, the current from the two perturbations above and below the fan will combine into one strong current located in the fan plane when the perturbation reaches the fan. This is true for all values of $p$, see Figure (6.17).

It is clear from Figure (6.18) that the maximum current increases when $p$ increased. This agrees well with the analytical investigation. In fact the maximum $J_x$ increases when $p$ increases, while the peak of $J_y$ decreases when $p$ increases. It is clear that $J_z$ becomes weak when the disturbance propagates towards the
Figure 6.17: For a rotation of the spine; (a) current modulus in the $y = 0$ plane at times $(t = 0, 0.45, 1.21)$ and $p = 1$, (b) at times $(t = 0.45, 1.2)$ and $p = 2$, (c) at times $(t = 0.45, 1.2)$ and $p = 3$, (d) at times $(t = 0.45, 1.2)$ and $p = 5$ and (e) at times $(t = 0.45, 1.2)$ and $p = 10$. 
Figure 6.18: (a) Evolution of the maximum values of each current components: solid line maximum $|J|$, dash dot $J_x$, dotted $J_y$, dashed $J_z$ for $p = 1$, (b) $p = 1.5$, (c) $p = 2$, (d) $p = 3$, (e) $p = 5$, (f) $p = 10$. (g) Evolution of the maximum values of $J_x$ with different values of $p$. h) Maximum values of $J_x, J_y, J_z$ with different values of $p$, such that solid line $J_x$, dotted $J_y$, dashed $J_z$. 
fan plane and null see Figure 6.18. This is because $\Delta z$ is associated with the length scale of the evolving perturbation which is decreasing as the disturbance is squeezed in towards the fan plane. The current eventually reaches a maximum value as the pulse localises. In Figures (6.19(a); 6.19(b)) it is seen that the current possesses two components, namely a “black” component and “white” component, while they have an opposite sign, since the black represents the negative part and the white represents the positive. Again the current layer at the fan plane extends along $y$, and is most intense along the short axis of the ellipse, see Figure 6.19(c). However, we note that no current is growing at the null point itself. The plasma flow in this case is, as expected, of a rotational nature, with similar form to that found for the simulations described in the Section 5.5. Furthermore, the flow pattern extends along the $y$-direction forming an elliptical pattern when $p$ goes to the infinity as in the simulations described in Section 6.3. Again there is no flow across the fan or spine.

We know that the maximum integrated electric field parallel to a magnetic field line $E_{||}$, where this maximum is taken over all field lines which thread the non-ideal region (assumed to be localised) will measure the reconnection rate in 3D (Schindler et al. (1988); Hornig and Priest (2003)). Therefore, to calculate the rate of reconnection, we systematically trace a set of field lines that thread the current concentrations, and seek the maximum in the integrated parallel electric field. The results are shown in Figure 6.20. Note that here, the reconnection rate first decreases and then increases again. This is because the reconnection rate is associated with $J_z$, which itself contained in the main current of the system, and $J_z$ becomes weak when the disturbance propagates towards the fan plane, as we have mentioned before.
Figure 6.19: (a) $J_x$ at time of maximum current. Both (a) and (b) are shown in the $y = 0$ plane, for $[x, z] = [\pm 0.45, \pm 0.45]$. (c) Current modulus in the $z = 0$ (fan) plane when $J_{xy}$ reaches a maximum. From left to right the panels represent $p = 1, 1.5, 2, 3, 5, 10$. 
6.4.3 Discussion

In this section and Section 6.3 we have perturbed the system by using different methods. In the two previous sections two methods of initiating torsional fan reconnection were discussed namely perturbing the magnetic field within the domain at \( t = 0 \), and by driving for a finite time from the boundaries. Each method has its own advantages. We have only performed the torsional spine simulations for one of these methods for the following reasons. First, the torsional spine simulations (Section 6.2) are performed with an internal perturbation within the domain since driving from the boundary in this way in our Cartesian simulation domain would involve driving around the corners of the domain which is difficult to do in a smooth and stable way. Second, we choose to consider also the case where we drive from the boundary in the torsional fan case since by applying an internal perturbation, as in Section 6.3, it is difficult to develop a significant current increase at the fan (see Figure 6.18). This is because of the large value of \( \eta \) that we must use owing to our numerical resolution – the internal perturbation
(which must be well localised initially) suffers significant diffusion before reaching the fan, as discussed by Pontin and Galsgaard (2007). Note that this is not such an issue for the case where the fan field lines are rotated (Section 6.2) since there the incoming pulse spreads only in one direction (\(z\)) owing to the hyperbolic nature of the field while it contracts in two \((x, y)\) – however, for the case of a spine rotation the in-coming pulse spreads in two directions \((x, y)\) and contracts only in one \((z)\).

### 6.5 Conclusion

Here we have presented numerical models for torsional spine and fan magnetic reconnection at 3D null points. The numerical models involved solving resistive MHD equations. We investigated the process of current accumulated along the spine or fan, in particular, investigating the effect of breaking the symmetry of the initial null point field. Torsional spine reconnection still occurs in a narrow tube around the spine, but with elliptical cross-section when the fan eigenvalues are different. The eccentricity of the ellipse increases as the degree of asymmetry increases, with the short axis of the ellipse being along the strong field direction. Furthermore, the current profile is not azimuthally symmetric around the spine, but peaks in these strong field regions. The numerical simulations suggest that the spatiotemporal peak current, and the peak reconnection rate attained, do not depend strongly on the degree of asymmetry. For torsional fan reconnection, the reconnection occurs in a planar disk in the fan surface, which is again elliptical when the symmetry of the magnetic field is broken. The short axis of the ellipse is along the weak field direction, with the current being peaked in these weak field regions. The peak current and peak reconnection rate in this case are clearly dependent on the asymmetry, with the peak current increasing but the
reconnection rate decreasing as the degree of asymmetry is increased.
Chapter 7

Ideal and Non-Ideal Evolution at a 3D Null

7.1 Introduction

In the previous chapters, we have developed models for reconnection at 3D nulls. In this chapter, we address the question: Why might reconnection occur at 3D nulls?

To understand the behaviour of astrophysical plasmas we should know where magnetic reconnection may take place in 3D, and therefore what are likely locations of energy release. So in order to identify these sites, we must understand where current sheets form. As discussed in Chapter 1, there are a number of locations proposed for the current configuration:

1. Null points, where the magnetic field vanishes.

2. Separator lines, field lines that run from one null point to another.
3. In the absence of nulls, e.g. at a quasi-separatrix layer.

In this Chapter, we are interested in null points. An ideal evolution of a magnetic field may be written

$$\frac{\partial B}{\partial t} = \nabla \times (w \times B).$$

(7.1)

Uncurl this equation to get

$$E + (w \times B) = \nabla F,$$

(7.2)

(e.g. Hornig and Schindler (1996)). Where $w$ is the magnetic flux velocity. The evolution of a magnetic field is said to be ideal if $B$ can be viewed as being frozen into some ideal flow, i.e if there exists some $w$ which satisfies Eq. (7.1) everywhere with $w$ continuous and smooth, such that it is equivalent to a real plasma flow. Examining the component of Eq. (7.2) parallel to $B$, it is clear that in configurations containing closed field lines, the constraint that $\oint E \cdot dl = 0$ must be satisfied in order to satisfy Eq. (7.2). However, even when no closed field lines are present, there are still configurations in which it may be impossible to find a smooth velocity $w$ satisfying Eq. (7.2). Priest et al. (2003) proved that in 3D a single flux tube velocity $w$ does not generally exist, see Theorem 2.1.1.

**Theorem 7.1.1.** For an ideal evolution, the ratio of eigenvalue pairs must be time-independent i.e there is no topology conserving, differentiable flow, which can change the ratio of eigenvalues at a null point.

In the following, we will try to relate the above theorem to the behaviour of the electric current at the null. We will present examples which demonstrate the following:

**Theorem 7.1.2.** If the ratio of eigenvalues of the Jacobian matrix of a magnetic field $B$ evaluated at a 3D null point, depends on time 't', then the current also varies with time. However the converse is not necessarily true.
7.1.1 Examples

We illustrate this with a number of examples. In Chapter 1 we have given an introduction to the topology of 3D magnetic fields. In general, each of three field components of a 3D linear null may be written in terms of three constants, making nine in total. However, Parnell et al. (1996) showed, by using $\nabla \cdot \mathbf{B} = 0$, and by normalising and rotating the coordinate axes, these may be reduced to four constants, namely $p, q, j_\parallel, j_\perp$ such that (Priest and Forbes, 2000)

$$
\mathbf{B} = \begin{pmatrix}
1 & \frac{1}{2} (q - j_\parallel) & 0 \\
\frac{1}{2} (q + j_\parallel) & p & 0 \\
0 & j_\perp & -(p + 1)
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
$$

with the current given by

$$
\mathbf{J} = \frac{1}{\mu_0} (j_\perp, 0, j_\parallel)
$$

(7.3)

where $p$ and $q$ are parameters, $j_\parallel$ is the component of current parallel to the spine, $j_\perp$ is the component of current perpendicular to the spine. The solenoidal constraint $\nabla \cdot \mathbf{B} = 0$, leads to the trace of $\mathbf{M}$ being zero (Parnell et al., 1996). In other words, the magnetic field $\mathbf{B}$ near a null point is expressed as $\mathbf{B} = \mathbf{M} \cdot \mathbf{r}$ where $\mathbf{M}$ is a matrix with the elements of the Jacobian of $\mathbf{B}$ and $\mathbf{r}$ is the position vector $(x, y, z)^T$. We will relax these constraints slightly and consider a new magnetic field dependent on time, taking the form

$$
\mathbf{B} = \begin{pmatrix}
1 & \frac{1}{2} (q - f(t)) & 0 \\
\frac{1}{2} (q + f(t)) & p & -h(t) \\
0 & g(t) & -(p + 1)
\end{pmatrix}
\begin{pmatrix}
x \\
y \\
z
\end{pmatrix}
$$

(7.4)

where $f(t) = j_\parallel$ and $g(t) + h(t) = j_\perp$. A magnetic field is said to be structurally stable if the fundamental characteristics of the topology are not affected by any slight change in the field, that is, if the elements of its skeleton are preserved. On the contrary, a magnetic field is structurally unstable if an arbitrary change
in the field causes a change in the topology. Thus, in 3D, isolated linear nulls are structurally stable, but null lines are structurally unstable. More generally, null points are stable if the determinant $\det(\nabla B) \neq 0$. Therefore, there is no eigenvalue zero to allow for stability, or vice versa a zero eigenvalue is enough for instability. A bifurcation is one that involves a change in the nature of null points. While it is debatable whether such a change of eigenvalues is already a change of the topology of the field lines, undoubtedly as a result the reconnection will occur.

We will present several cases to prove Theorem 7.1.3 as shown in Table 7.1.

**Example 7.1.1. Case 1: Both the current and ratio of eigenvalues are independent of time**

First we present an example that demonstrates, if the current is independent of time, then the ratio of eigenvalues is also independent of time. The magnetic field is $B = (x, py - jz, -(p + 1)z)$, where $p$ is a parameter, with current given by

$$J = \frac{1}{\mu_0} (j, 0, 0).$$

(7.5)
The Jacobian matrix is
\[
\mathcal{M} = \begin{bmatrix}
1 & 0 & 0 \\
0 & p & -j \\
0 & 0 & -(p + 1)
\end{bmatrix},
\]
where \( f(t) = g(t) = q = 0 \) in Eq. (7.4) and the eigenvalues are
\[
\lambda_1 = 1 \ , \ \lambda_2 = p \ , \ \lambda_3 = -(p + 1).
\]

We notice that the ratio of eigenvalues and current both are independent of time.

The presence of current parallel to the fan at the null point is very important because it is responsible for the relative angle of the fan and the spine. In the next example we consider an example in which this angle changes with time.

**Example 7.1.2. Case 2: Both the current and the eigenvalue ratios are changing with time**

The magnetic field is \( B = (x, py - h(t)z, -(p+1)z + g(t)y) \), such that, without loss of generality, the current lies in \( x \)-direction, and is given by
\[
\mathbf{J} = \frac{1}{\mu_0} (g(t) + h(t), 0, 0).
\] (7.6)

The fan and spine both move with time, and the fan plane of this magnetic null point is coincident with the plane \( y = \frac{\frac{1}{2} - \frac{1}{2} \sqrt{\frac{(1+2p)^2 - 4g(t)h(t) + p}{h(t)}}}{h(t)}z \), while the spine is not perpendicular to this, but rather lies along \( x = 0, z = \frac{\frac{1}{2} + \frac{1}{2} \sqrt{\frac{(1+2p)^2 - 4g(t)h(t) + p}{h(t)}}}{h(t)}y \) (see Figure [7.1]). The Jacobian matrix is
\[
\mathcal{M} = \begin{bmatrix}
1 & 0 & 0 \\
0 & p & -h(t) \\
0 & g(t) & -(p + 1)
\end{bmatrix}.
\]

We notice \( f(t) = q = 0 \). The eigenvalues are
\[
\lambda_1 = 1 \ , \ \lambda_2 = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{(1+2p)^2 - 4g(t)h(t)}{h(t)}} , \ \lambda_3 = \frac{1}{2} - \frac{1}{2} \sqrt{\frac{(1+2p)^2 - 4g(t)h(t)}}.
\]
We assume that $|H(t)| < \frac{1}{4}(1 + 2p)^2$ and $0 < p < 1$, where $H(t) = g(t)h(t)$, in order to preserve the nature of the null. Note that this was the magnetic field considered by Pontin et al. (2007). There are some conditions that must be taken to choose the magnetic field they are as follows: $B$ should be linear to fit our assumptions. In addition, if we want to consider evolution that could be a “local process” i.e., can occur when boundary conditions are fixed, then the potential part of $B$ must be independent of time. In order to ensure this, we will need to ‘decompose’ into $B = B_{pot} + B_{non-pot}$. First consider the case where the variation in the current comes from the terms $g(t)$ and $h(t)$ in Eq. (7.4). Now we want to decompose the above magnetic field to two parts; a unique potential part (independent of time) and unique non-potential part (which may be dependent on time). In the present example, the potential part is $B_{pot} = (x, py, -(p + 1)z)$ and the non-potential part $B_{non-pot} = (0, -h(t)z, g(t)y)$. The decomposition of the original magnetic field can be achieved by other ways, for example, as follows.

- $B = \left( \frac{x}{2}, \frac{py}{2}, -\frac{(p+1)z}{2} \right) + \left( \frac{x}{2}, \frac{py}{2} - h(t)z, -\frac{(p+1)z}{2} + g(t)y \right)$, but this way is unsuccessful because we can decompose the non-potential part to two parts, potential and non-potential parts.

- $B = (x + yz, py + xz, -(p + 1)z + xy) + (-yz, -xz - h(t)z, -xy + g(t)y)$, also unsuccessful, because it contains non-linear terms. Therefore, $B = B_{pot} + B_{non-pot}$ is the unique way to decompose this magnetic field.

If we examine the total magnetic field, we find that the ratio of eigenvalues and current are both dependent on time. We have decomposed the magnetic field into two parts (potential and non-potential) because we want a potential part independent of time since the variation of $B_{pot}$ can be considered as a global effect, but we want to study a local non-ideal process (reconnection).
Figure 7.1: The structure of the $B_y, B_z$ components of the magnetic null $\mathbf{B} = (x, py - h(t)z, -(p + 1)z + g(t)y)$, in the $y$ and $z$ plane, where the solid lines are refer to spine and fan of null. (a) at $g(t) = h(t) = 0$, (b) at $g(t) = h(t) = 0.5$, (c) $g(t) = h(t) = 1$, with the parameter $p = 1$ throughout.
Example 7.1.3. Case 3: The current is time-dependent and the eigenvalue ratios are fixed

Let us consider the case where the variation in the current comes only from the term \( g(t) \) in Eq. (7.4). The magnetic field is chosen to be

\[
\mathbf{B} = (x, py, -(p + 1)z + g(t)y),
\]

with the current given by

\[
\mathbf{J} = \frac{1}{\mu_0}(g(t), 0, 0). \tag{7.7}
\]

The Jacobian matrix is

\[
\mathcal{M} = \begin{bmatrix}
1 & 0 & 0 \\
0 & p & 0 \\
0 & g(t) & -(p + 1)
\end{bmatrix},
\]

and \( p > 0, \: q = h(t) = 0, \: f(t) = 0 \) (the component of current parallel to the spine) and \( g(t) \neq 0 \) is (the component of current perpendicular to the spine). The eigenvalues of the null point are

\[
\lambda_1 = 1, \: \lambda_2 = p, \: \lambda_3 = -(p + 1),
\]

and are clearly independent of time. If we decompose the magnetic field into two unique parts by the same previous manner. Then the unique potential component is \( \mathbf{B}_{pot} = (x, py, -(p + 1)z) \). The non-potential magnetic field component is \( \mathbf{B}_{non-pot} = (0, 0, g(t)y) \). This example shows that there is not a direct one-to-one relationship between time-dependence of the eigenvalue ratios and time-dependence of the current.

Now we will show another example where the \( f(t) \) in Eq. (7.4) is non-zero but \( g(t) = h(t) = 0 \). The magnetic field is \( \mathbf{B} = (x - f(t)y, py, -(p + 1)z) \), such that, without loss of generality, the current lies in \( z \)-direction, and is given by

\[
\mathbf{J} = \frac{1}{\mu_0}(0, 0, f(t)) \tag{7.8}
\]
and the eigenvalues are
\[ \lambda_1 = 1, \lambda_2 = p, \lambda_3 = -(p + 1) \]

with corresponding eigenvectors
\[ k_1 = (1, 0, 0), \quad k_2 = \left(1, \frac{1-p}{f(t)}, 0\right), \quad k_3 = (0, 0, 1). \]

In this example the magnetic field in the fan plane of this magnetic null point is changing with time, while the locations of the fan and spine are fixed (see Figure 7.2). We notice the current and the eigenvalue ratios have the same behaviour as in the previous example.

**Example 7.1.4. Case 4: The ratio of eigenvalues depends on time and the current is fixed**

In this Section, we seek a magnetic field, such that the ratio of the eigenvalues is dependent on time but the current is independent of time. Consider the magnetic field \( B = (x + f(t)x, py - f(t)y, -(p + 1)z) \). This corresponds to a zero current \( (J = 0) \), but the ratio of eigenvalues are dependent on time:

\[ \lambda_1 = 1 + f(t), \lambda_2 = 1 - f(t), \lambda_3 = -(p + 1). \]

However, here the potential part of the magnetic field is clearly time-dependent far from the null. Here we would like to determine whether some local process can lead to a time-dependent eigenvalue ratio but constant current i.e., the time-dependence must appear in the non-potential part. However, this magnetic field does not satisfy this requirement. We would like to know in general if it is possible to find an example that does satisfy the condition. Now we consider the Parnell et al. (1996) generalisation. The eigenvalues are:

\[ \lambda_{1,2} = \frac{p + 1 \pm \sqrt{(p-1)^2 + q^2 - j_{\|}^2}}{2}, \quad \lambda_3 = -(p + 1). \]
Figure 7.2: The structure of the $B_x, B_y$ components of the magnetic null $\mathbf{B} = (x - f(t), py, -(p + 1)z)$, in the $x$ and $y$ plane where the solid line shows the fan eigenvector direction, (a) at $f(t) = 0.5$, (b) at $f(t) = 1$, (c) at $f(t) = 1.5$, with the parameter $p = 1$.

Let us assume the ratio of eigenvalues are time-dependent

$$\frac{\lambda_2}{\lambda_3} = F(t)$$

$$\Rightarrow \lambda_2 = \lambda_3 F(t).$$

Now, since,

$$\lambda_1 + \lambda_2 + \lambda_3 = 0$$

$$\Rightarrow \lambda_1 + \lambda_3 F(t) + \lambda_3 = 0$$
First substituting $\lambda_1$ and $\lambda_3$ in Eq. (7.9) we get

$$j|| = \pm \sqrt{(-F(t)^2 - F(t))4p^2 + (-4 - 8F(t) - 8F(t)^2)p + 4(F(t) + F(t)^2) + q^2}.$$  

(7.10)

In order for this to be independent of $t$ we require $p = -\frac{1+F(t)}{F(t)}$ or $-\frac{F(t)}{F(t)+1}$, and substituting this into (7.10), we obtain $j|| = \pm q$. Note that we could also choose $q$ in such a way as to remove the time-dependence in $j||$. However, this requires that $q$ is a function of $p$, whereas these should be two independent parameters, according to this framework.

Hence the magnetic field is

$$\mathbf{B} = \left( x, qx - \frac{F(t) + 1}{F(t)} y, \left( \frac{F(t) + 1}{F(t)} - 1 \right) z \right),$$

and the current is

$$\mathbf{J} = (0, 0, q).$$

Now if we decompose this magnetic field into potential and non-potential components as before, we get

$$\mathbf{B}_{pot} = \left( x, -\frac{F(t) + 1}{F(t)} y, \left( \frac{F(t) + 1}{F(t)} - 1 \right) z \right),$$

and

$$\mathbf{B}_{non-pot} = (0, qx, 0),$$

but this is a contradiction because we want to have only the non-potential part time-dependent. Therefore $q$ should be dependent on time i.e., the current dependent on time. This implies that if the ratios of eigenvalues are time-dependent then in order for the current to be fixed we require a time-varying potential component to the field, which we argue as before is not relevant to studying a local non-ideal process. Therefore we can exclude Case 4 in Table 7.1.
7.2 Discussion and Conclusions

Now, we want to know the relationship between Case 2 and Case 3 since it appears that in both examples we have a similar configuration (the relative angle between the spine and fan changes in time) but a different behaviour of the eigenvalues. Therefore, we transform the magnetic field in Case 2, such that the spine should be always lying in the same-direction. The magnetic field before transformation is \( B = (x, py - h(t)z, -(p + 1)z - g(t)y) \). The method is chosen here the same technique used by [Priest and Forbes 2000]. Now we have a new magnetic field after transformation, \( B = (u, \frac{1}{2}v(L - 1), -\frac{1}{2}w(L + 1) + 2g(t)v) \), where \( P = 1 + 2p \) and \( L = \sqrt{P^2 - 4g(t)h(t)} \). The coordinate transformation is given by

\[
x = u
y = \frac{v}{\sqrt{1 + \frac{4h(t)g(t)}{(1 + \sqrt{1 + 4p^2 + 4p - 4h(t)g(t) + 2p})^2}}}
+ \frac{g(t)w}{\sqrt{h(t)g(t) + \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4p^2 + 4p - 4h(t)g(t) + p}\right)}^2}
z = \frac{-2g(t)v}{\sqrt{1 + \frac{4h(t)g(t)}{(1 + \sqrt{1 + 4p^2 + 4p - 4h(t)g(t) + 2p})^2}}(1 + \sqrt{1 + 4p^2 + 4p - 4h(t)g(t) + 2p}) + \frac{(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4p^2 + 4p - 4h(t)g(t) + p})w}{\sqrt{h(t)g(t) + \left(\frac{1}{2} + \frac{1}{2}\sqrt{1 + 4p^2 + 4p - 4h(t)g(t) + p}\right)}^2}.
\]

The current is still the same as before the transformation,

\[
J = \frac{1}{\mu_0} (2g(t), 0, 0), \quad (7.11)
\]

and the eigenvalues are:

\[
\lambda_1 = 1, \quad \lambda_2 = \frac{L - 1}{2}, \quad \lambda_3 = \frac{-L - 1}{2}
\]
with corresponding eigenvectors

\[ k_1 = (1, 0, 0), \quad k_2 = (0, 1, \frac{2g(t)}{L}), \quad k_3 = (0, 0, 1) \]

such that the spine is defined by \( k_3 \), whilst the plane of the fan is defined by
the eigenvectors \( k_1 \) and \( k_2 \). It is clear the after transformation the spine is
fixed and only the fan is time-dependent (see Figure 7.3). If we decompose
this magnetic field into two parts, potential and non-potential magnetic fields,
then the potential part is \( B_{\text{pot}} = (u, -\frac{1}{2}v, -\frac{1}{2}w) \) and the non-potential part is
\( B_{\text{non-pot}} = (0, \frac{1}{2}vL, -\frac{1}{2}wL) \).

If we look to the non-potential part, the eigenvalues are

\[ \lambda_1 = 0 \quad \lambda_2 = \frac{L}{2} \quad \lambda_3 = -\frac{L}{2} \]

which clearly shows that the ratio of eigenvalues is still dependent on time even
though we have now fixed the location of the spine. This is the case even though
the two cases (after and before transformation) do not have the same behaviour;
the current and eigenvalues ratios have the same properties (i.e., both dependent
on time).

In this work, we have presented a few examples to show the relationship be-
tween the ratio of eigenvalues and current. As a first step we reviewed the general
matrix of three dimensional null points. After that we went on to investigate the
relationship between the ratios of the eigenvalue and current, and the collective
result of all these examples clearly demonstrates that if the ratio of eigenvalues
is dependent on time then the current should also be dependent on time, but the
converse is not necessarily true.
Figure 7.3: The structure of the $B_u, B_w$ components of the magnetic null $\mathbf{B} = (u, \frac{1}{2}v(L - 1), -\frac{1}{2}w(L + 1) + 2g(t)v)$ after transformation, in the $u$ and $w$ plane, where the solid line refer to the fan. (a) at $g(t)=h(t)=0$, (b) at $g(t)=h(t)=0.5$, (c) $g(t)=h(t)=1$, with the parameter $p = 1$ throughout.
Chapter 8

Summary and Future Work

8.1 Summary

Solutions of the magnetohydrodynamic (MHD) equations are very important for modelling laboratory, space and astrophysical plasmas. Realistic models should be three dimensional and hence progress towards more realistic geometries in MHD is very important for our understanding of plasmas in these different environments. However, only a few analytical solutions of the MHD equations exist in three dimensions and most work consists of numerical simulations. In this thesis, we have presented both analytical and numerical solutions of three dimensional MHD models in two different areas.

In Chapter 3 the steady kinematic MHD equations were solved in order to determine the nature of 3D reconnection at an isolated non-ideal region. A resistive non-ideal term ($\eta \mathbf{J}$) was included in Ohm’s law, and was localised by imposing a localised resistivity, where the magnetic null point was defined by $\mathbf{B}=B_0(x, py - jz, -(p + 1)z)$. This magnetic field has current aligned to the fan
surface of the null point, and Pontin et al. (2005) investigated this situation in the non-generic symmetric case $p = 1$ (repeated eigenvalues). In Chapter 3 we used $p$ as a parameter. We found the nature of the plasma flow, and the resulting qualitative structure of the reconnection process, to be the same as found in the symmetric case. Specifically, we found plasma flow across both the spine line and fan plane of the null for all values of $p$.

In Chapter 4 we described the results of a related resistive MHD numerical simulation in which we investigated different values of the parameter $p$ (the ratio of the fan eigenvalues). The system was then driven away from equilibrium in such a way as to induce a local collapse of the null leading to current sheet formation and spine-fan magnetic reconnection. The resulting configuration shared key properties with the analytical solution described in Chapter 3: the spine and fan are non-orthogonal with a current flowing parallel to the fan surface, and a localised diffusion region is focussed at the null. Also, in both cases the flow in the $yz$-plane exhibited a stagnation-point structure. There is agreement between the model and the simulations, in that for large $p$ the stagnation structure is relatively symmetric, while for smaller $p$ the flow across the fan becomes confined to a narrower region, and weaker, compared with the flow across the spine. One of the major results that arises from the sequence of simulations is that both the peak intensity and the dimensions of current sheet are strongly dependent on the symmetry/asymmetry of the field in the fan surface, or in other words, on the value of $p$.

In Chapter 5, at the beginning, we have reviewed exact analytic solutions (kinematic solution) describing magnetic reconnection in three dimensions where the magnetic null point was defined by $B = \frac{B_0}{L_0} \left( \frac{2x}{p+1} - \frac{1}{2} jy, \frac{2py}{p+1} + \frac{1}{2} jx, -2z \right)$. This magnetic field has current aligned to the spine line of the null point, and Pontin et al. (2004) studied this situation in the non-generic symmetric case $p = 1$. 
(complex eigenvalues). In this work, we considered $p$ as a parameter. The model exhibits the same structure of plasma flow as previous torsional spine reconnection models i.e., only one sign of rotational flow due to the fact that we have a uniform current. In addition we found that the reconnection rate is independent on $p$. Secondly, we have introduced our new analytical models, named “torsional spine” and “torsional fan reconnection”, and then we generalised existing models for torsional spine and torsional fan null point reconnection as follows. We began in each case by introducing a new kinematic analytical solution for the corresponding reconnection mode in which a localised current layer is present at the null. We then went on to consider the effect of varying the symmetry of the background magnetic field, by varying the ratio of the fan eigenvalues of the null. The results we obtained are as follows. Torsional spine reconnection still occurs in a narrow tube around the spine, but with elliptical cross-section when the fan eigenvalues are different. For torsional fan reconnection, the reconnection occurs in a planar disk in the fan surface, which is again elliptical when the symmetry of the magnetic field is broken. The short axis of the ellipse is along the weak field direction, with the current being peaked in these weak field regions. The peak current and peak reconnection rate in this case are clearly dependent on the asymmetry, with the peak current increasing but the reconnection rate decreasing as the degree of asymmetry is increased.

In Chapter 6 we performed numerical simulations of the full system of MHD equations in which nulls with varying degrees of symmetry are subjected to rotational disturbances, to complement the analytical models discussed in Chapter 5. We investigated the process of current accumulation along the spine or fan, in particular investigating the effect of breaking the symmetry of the initial null point field. It was found that torsional spine reconnection occurs in a cylindrical region with elliptical cross-section around the spine. The eccentricity of the
ellipse increases as the degree of asymmetry increases, with the short axis of the ellipse being along the strong field direction. Furthermore, the current profile is not azimuthally symmetric around the spine, but is peaked in these strong field regions. The numerical simulations suggest that the spatiotemporal peak current, and the peak reconnection rate attained, do not depend strongly on the degree of asymmetry. For torsional fan reconnection, the reconnection occurs in a planar disk in the fan surface, which is again elliptical when the symmetry of the magnetic field is broken.

Finally, in Chapter 7 we have discussed the relationship between the ratio of eigenvalues and current, and we found if the ratio of eigenvalues is dependent on time then the current should also be dependent on time but the converse is not necessarily true.

8.2 Future work

Some immediate questions which have arisen during the course of this work are as follows.
In Chapter 4 we have tested our spine-fan reconnection analytical results by numerical solving the full MHD equations, with a transient driving profile. In the future we can run our simulation by using continuous driving. In Chapter 5 and 6 we have investigated torsional spine and fan reconnection. While we have relaxed the the rotational symmetry of the magnetic field in these studies, the field structure of a linear null is still relatively simple. In the future it will be important to understand how these reconnection modes are modified – and how they release the energy associated with imposed stresses – when the null point is embedded in more realistic coronal geometries. One recent study suggests that
other features present in the magnetic field may in some cases attract the current preferentially over the nulls and therefore inhibit the formation of the torsional spine current layers (Santos et al., 2011). The importance of a number of other parameters such as the magnitude of the perturbation, the plasma-\( \beta \) and the resistivity \( \eta \) are also yet to be explored. Each of these extensions could also be investigated for spine-fan reconnection simulations described in Chapter 4. In addition, we can perturb the magnetic field by applying a rotational driving velocity on the boundaries around the fan plane by using a cylindrical geometry (or developing the numerical method to allow driving around corners) and by using a continuous driving for torsional fan reconnection. Moreover, a greater effort put into understanding wave mode properties, such as in McLaughlin et al. (2009) for the 2D case, may yield interesting results.
Appendix A

Maple commands

A.1 Input for Maple worksheet

The following gives the Maple commands used in the calculation described in Chapter 3, Section 3.2 (Maple version 12).

restart:with(linalg):with(plots):

> B := [B0 * x, B0 * (p * y − j * z), −(p + 1) * B0 * z]
> R1z := ((x^2)^p + (y − j * z/(2 * p + 1))^2) * z^((2p/(p+1))):
> simplify(dotprod(grad(R1z, [x, y, z]), B, orthogonal)):
> J := curl(B, [x, y, z]):
> JdotB := dotprod(J, B, orthogonal):
> eta − inner := eta0 * ((R1z/a)^2 − 1)^2 * (((z^2)^((2p/(p+1)))/b^2) − 1)^2:
> eta := piecewise(R1z < a^2 and((z^2)^((2p/(p+1)))) < b^2, eta − inner, 0):
> R1 := sqrt((x^2)^p + (y − j * z/(2 * p + 1))^2) :
\[ R_0 \left( x^2 \right) = \sqrt{\left( (y_0 - j * z_0) / (2 * p + 1) \right)^2} \]

\[ etain1 := \left( y = x0 * e^{Bo} \right), z = z0 * e^{(-p+1)Bo}, y = e^{Bo} \cdot (y_0 - j * z_0) / (2 * p + 1) + j * e^{-Bo} \cdot z_0 / (2 * p + 1), eta - inner \) : \]

\[ integrand1 := -etain1 * subs(x = x0 * e^{Bo}, intB) : \]

\[ integrand := -etain1 * \left( Bo * \left( (x0^2 * (e^{Bo})^2 + (e^{Bo})^2 * (y_0 - j * z_0) / (2 * p + 1))^2 / \left( z_0^{(4p/p+1)} * \left( e^{-2pBo} \right)^2 / b^2 - 1 \right)^2 \right) \right) : \]

\[ phix0 := -\left( 32 * b^8 * p^2 + 16 * b \cdot \frac{2s(5p+3)}{(5p+1)} * p + 4 * b * \left( \frac{4(3p+1)}{(p+1)} \right) - b^8 * p + b^8 - 2 * b \right) \cdot etain1 \cdot Bo \cdot j * e^{-ln(b^p + ln(p))} \cdot x0 / (b^8 * (-1 + 8 * p) * (4 * p - 1)) : \]

\[ Phi0 := (phi0 + (int(integrands, s = 0..s1)) : \]

\[ Phimax := \left( subs(s1 = -ln(R0^2/a) / p * Bo, Phi0) : \right) \]

\[ Phix0y0z0 := \left( subs(piecewise(z > b, phix0, R1^2 < a^2 \text{ and } z < a^2, Phi0, R1^2 > a^2, Phimax), phi0, \right) : \]

\[ phixyz := \left( subs(y0 = \frac{2pye^{-Bo(p+1)s1} + ye^{-Bo(p+1)s1} + e^{Bo} \cdot jz - e^{-Bo(p+1)s1} \cdot jz) e^{Bo} \cdot s1}{(2 * p + 1)}, x0 = e^{-Bo} \cdot x, x02 = e^{-2Bo} \cdot x, s1 = \ln(z / z0) / Bo(p + 1), z0 = b, Phix0y0z0) : \]

\[ phineg := \left( subs(y = -y, z = -z, phi0) : \right) \]

\[ Phitot := piecewise(z >= 0, phix0, z < 0, phineg) : \]

\[ E := \left( grad(-Phitot, [x, y, z]) : \right) \]

\[ vperp := \left( crossprod(matadd(E, -etaJ), B/dotprod(B, B, orthogonal)) : \right) \]

\[ A.2 \quad \text{Input for maple worksheet} \]

The following gives the Maple commands used in the calculation described in Chapter 5, Section 5.2.
The magnetic field is

\[ B = \left[ \frac{2}{p+1} - \frac{1}{2} j y, \frac{2}{p} y / (p+1) + \frac{1}{2} j x, -2 z \right] \]

Here we have defined \( \eta \)

\[ \eta = \eta_0 \exp(-R^2/a^2) \times \exp(-z^2/b^2) \]

We have substituted the field line equation in integrand.

\[ \text{intagrand1} := \text{simplify} \left( \text{subs} \left( z = z_0 \exp(-2s) \times x = \frac{1}{2} (L x_0 - y_0 j p - y_0 j - 2 p x_0 + 2 x_0) \times \exp((p+1+(1/2)L) s / (p+1)) / L + (1/2) (y_0 j p + y_0 j + 2 p x_0 - 2 x_0 + L x_0) \times \exp((p+1-(1/2)L) s / (p+1)) / L, y = -(1/2) (2 p - 2 + L) (L x_0 - y_0 j p - y_0 j - 2 p x_0 + 2 x_0) \times \exp((p+1+(1/2)L) s / (p+1)) / (j (p+1) L) + (1/2) (-2 p + 2 + L) (y_0 j p + y_0 j + 2 p x_0 - 2 x_0 + L x_0) \times \exp((p+1-(1/2)L) s / (p+1)) / (j (p+1) L), \text{intagrand}) \right) \]

Here we calculate \( \phi \) as

\[ \phi := \text{simplify} \left( \text{ApproximateInt}(-\text{intagrand1}, s = 0..s1, \text{method} = \text{newtoncotes6}) \right) \]

\[ x := \text{xmin} - h1; \]

\[ \Phi := \text{Array}(1..nx,1..ny,1..nz); \]
Here we have substitute the inverse of magnetic field lines in $\phi$ and calculating $\phi$ numerically

```maple
> for i from 1 by 1 to nx do
  x:=x+h1; y:=ymin-h1;
  > for m from 1 by 1 to ny do
    > for k from 1 by 1 to nz do
      z:=z+h1(2):
      > Phi[i, m, k] := evalf(subs(x0 = (1/2)*(-2*exp((1/2)*(2*p+2+L)*s1/(p+1))*x
        +exp((1/2)*(2*p+2+L)*s1/(p+1))*x*L+exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*x*L
        -exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*y*j*p+exp((1/2)*(2*p+2+L)*s1/(p+1))*y*j
        +2*exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*x+exp((1/2)*(2*p+2+L)*s1/(p+1))*y*j*p
        -exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*y*j-2*exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*p*x
        +2*exp((1/2)*(2*p+2+L)*s1/(p+1))*p*x)*exp(-2*s1)/L, y0 = -(1/2)*(-4*exp(-(1/2)
          *(2*p-2+L)*s1/(p+1)))*x-4*p^2*exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*x
      +2*exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*y*j-2*p^2*exp(-(1/2)*(-2*p-2+L)
      )*s1/(p+1))*y*j-exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*y*j*L+exp(-(1/2)*(-2*p-2+L)
      )*s1/(p+1))*L^2*x+8*exp(-(1/2)*(-2*p-2+L)*s1/(p+1))*p*x-exp(-(1/2)*(-2*p-2+L)
      )*s1/(p+1))*y*j*L*p+4*exp((1/2)*(2*p+2+L)*s1/(p+1))*p^2*x-8*exp((1/2)*(2*p+2+L)
      )*s1/(p+1))*p*x-2*exp((1/2)*(2*p+2+L)*s1/(p+1))*y*j+4*exp((1/2)*(2*p+2+L)
      )*s1/(p+1))*x-exp((1/2)*(2*p+2+L)*s1/(p+1))*L^2*x+2*exp((1/2)*(2*p+2+L)*s1/(p+1))
      *j*p^2*y*exp((1/2)*(2*p+2+L)*s1/(p+1))*y*j*L-exp((1/2)*(2*p+2+L)*s1/(p+1))
      *y * j * L * p) * exp(-2 * s1)/(j * (p + 1) * L), s1 = (1/2) * ln(z0/(z), phi))
```
Here we are saving $\Phi$ as a array

```maple
> fd := fopen(spineJp1, WRITE, BINARY);
> for k from 1 to nz do
> for m from 1 to ny do
> for i from 1 to nx do
> fprintf(fd, "%.16f", Phi[i, m, k], "\n");
> end do: end do: end do;
>fclose(fd);
```

### A.3 Input for maple worksheet

The following gives the Maple commands used in the calculation described in Chapter 4, Section 5.4.

```maple
restart; with(plots); with(linalg):

> B := [R, j * R * (1 - (R/a)^6)^4 * (1 - (z/b)^4)^2, -2 * z]
> diverge(B, [R, theta, z], coords = cylindrical)
> J := curl(B, [R, theta, z], coords = cylindrical)
> eta2 := eta0 * piecewise(R^2 < a^2 and z^2 < b^2, 1, 0);
> JdotB := dotprod(J, B, orthogonal)
> integrand := expand(subs(z = z0 * exp(-2 * B0 * s), R = R0 * exp(B0 * s), JdotB * eta))
> phiout := 0
> Phi := -(int(integrand, s = 0..s1))
> Phimax := expand(subs(s1 = ln(a/R0), Phi))
```
\[ \Phi_R(0, z) := \text{piecewise}(z^2 > b^2, \phi_{\text{phiout}}, R^2 < a^2 \text{ and } z < \frac{a^2 b}{R^2}, \Phi_i, \text{and } (R^2 > a^2, z < \frac{a^2 b}{R^2}) \).

\[ \Phi_R(0, z) := \text{subs}(z = -z, \Phi_R(0, z)) :\]

\[ \Phi_R(z) := \text{expand}(\text{subs}(R_0 = R e^{-B_0 s_1}, z_0 = z e^{2B_0 s_1}, s_1 = -\frac{\ln(z/z_0)}{2B_0}, z_0 = b, \Phi_R(0, z))) \]

\[ \Phi_R(z) := \text{expand}(\text{subs}(R_0 = R e^{-B_0 s_1}, z_0 = z e^{2B_0 s_1}, s_1 = -\frac{\ln(z/z_0)}{2B_0}, z_0 = b, \Phi_R(0, z))) \]

\[ \phi_{\text{neg}} := \text{subs}(z = -z, \Phi_R(z)) :\]

\[ \Phi_{\text{tot}} := \text{piecewise}(z \geq 0, \Phi_R(z), z < 0, \phi_{\text{neg}}) :\]

\[ E := (\text{grad}(-\Phi_{\text{tot}}, [R, \theta, z], \text{coords = cylindrical})) :\]

\[ v_{\text{perp}} := (\text{crossprod}(\text{matadd}(E, -\eta^2 J), B / \text{dotprod}(B, B, \text{orthogonal}))) :\]

**A.4 Input for maple worksheet**

The following gives the Maple commands used in the calculation described in Chapter 5, Section 5.5

```
restart; with(plots); with(linalg):

> B := [R, j * R * z * (1 - (R/a)^6)^4 * (1 - (z/b)^4)^12, -2 * z]
> diverge(B, [R, theta, z], coords = cylindrical)
> J := curl(B, [R, theta, z], coords = cylindrical)
> eta2 := eta0 * piecewise(R^2 < a^2 \text{ and } z^2 < b^2, 1, 0);
> JdotB := dotprod(J, B, orthogonal)
> integrand := expand(subs(z = z0 * exp(-2 * B0 * s), R = R0 * exp(B0 * s), JdotB * eta))
> phiout := 0
> Phi := -(int(integrand, s = 0..s1))
```
> Phimax := expand(subs(s1 = -1/2*ln(b/z0), Phi))
> PhiR0z0 := piecewise(z^2 > b^2, phiout, R^2 < a^2 and -ln(z/z0) < a^2b/R^2, Phi, and (R^2 > a^2, z < a^2b/R^2, Phimax, phiout));
> PhiR0z0n := subs(z0 = -z0, PhiR0z0):
> PhiRz := expand(subs(R0 = Re^-B0s1, z0 = ze^2B0s1, s1 = ln(R/R0), z0 = b, PhiR0z0))
> PhiRzn := expand(subs(R0 = Re^-B0s1, z0 = ze^2B0s1, s1 = ln(R/R0), z0 = b, PhiR0z0n))
> phineg := subs(z = -z, PhiRzn):
> Phitot := piecewise(z >= 0, PhiRz, z < 0, phineg):
> E := (grad(-Phitot, [R, theta, z]), coords = cylindrical)):
> vperp := (crossprod(matadd(E, -eta2J), B/dotprod(B, B, orthogonal))):
Appendix B

Matlab commands

B.1 Input for Matlab worksheet

The following gives the Matlab commands used in the calculation described in Chapter 5, Section 5.2.

```matlab
clear all; close all;clc;
p=1;j=1;a=1;b=2;nx=101;ny=101;nz=101;xmax=2;
ymax=2;zmax=2;xmin=-2;ymin=-2;zmin=-2;h1=(xmax-xmin)/(nx-1);
input='spineJp1';
fid = fopen(input,'r','l');
Phi = fscanf(fid,'% g',[1,nx*ny*nz]);
fclose(fid);
Phi = reshape(Phi,101,101,101);
eta = zeros(nx,ny,nz);
Bx=zeros(nx,ny,nz);By=zeros(nx,ny,nz);Bz=zeros(nx,ny,nz);
Jx=zeros(nx,ny,nz);Jy=zeros(nx,ny,nz);Jz=zeros(nx,ny,nz);
wx=zeros(nx,ny,nz);wy=zeros(nx,ny,nz);wz=zeros(nx,ny,nz);
```
Ex=zeros(nx,ny,nz);Ey=zeros(nx,ny,nz);Ez=zeros(nx,ny,nz);
Ewx=zeros(nx,ny,nz);Ewy=zeros(nx,ny,nz);Ewz=zeros(nx,ny,nz);
vpx=zeros(nx,ny,nz);vpy=zeros(nx,ny,nz);vpz=zeros(nx,ny,nz);
vx=zeros(nx,ny,nz);vy=zeros(nx,ny,nz);vz=zeros(nx,ny,nz);

x = -1.960000000;
for i=3:1:99
  x = x+h1;
end

y = -1.960000000;
for g=3:1:99
  y = y+h1;
end

z =0.02000010000;
for k=3:1:99
  z = (z+h1/2);

  R = sqrt(x.^2 + p.*y.^2);
  eta(i-2,g-2,k-2) = eta0 * exp(-R.^2/a.^2) * exp(-z.^2/b.^2);

  if R^2 < 1 and z^2 < 4
    Bx(i-2,g-2,k-2) = 2*x/(p+1)-(1/2)*j*y;
    By(i-2,g-2,k-2) = 2*p*y/(p+1)+(1/2)*j*x ;
    Bz(i-2,g-2,k-2) =-2*z;
    Jx(i-2,g-2,k-2) = 0;
    Jy(i-2,g-2,k-2) = 0;
    Jz(i-2,g-2,k-2) = j;
  else
    Bx(i-2,g-2,k-2) = 2*x/(p+1);
    By(i-2,g-2,k-2) = 2*p*y/(p+1);
    Bz(i-2,g-2,k-2) =-2*z;
    Jx(i-2,g-2,k-2) = 0;
Jy(i-2,g-2,k-2) = 0;
Jz(i-2,g-2,k-2) = 0;
end

Ex(i-2,g-2,k-2)=-(Phi(i-2,g,k)-8.*Phi(i-1,g,k)+8.*Phi(i+1,g,k))/(12*h1);
Ey(i-2,g-2,k-2)=-(Phi(i,g-2,k)-8.*Phi(i,g-1,k)+8.*Phi(i,g+1,k))/(12*h1);
Ez(i-2,g-2,k-2)=-(Phi(i,g,k-2)-8.*Phi(i,g,k-1)+8.*Phi(i,g,k+1))/(12*h1/2);
wx(i-2,g-2,k-2)=Jx(i-2,g-2,k-2).*eta(i-2,g-2,k-2);
wy(i-2,g-2,k-2)=Jy(i-2,g-2,k-2).*eta(i-2,g-2,k-2);
wz(i-2,g-2,k-2)=Jz(i-2,g-2,k-2).*eta(i-2,g-2,k-2);
Ewx(i-2,g-2,k-2)=Ex(i-2,g-2,k-2)-wx(i-2,g-2,k-2);
Ewy(i-2,g-2,k-2)=Ey(i-2,g-2,k-2)-wy(i-2,g-2,k-2);
Ewz(i-2,g-2,k-2)=Ez(i-2,g-2,k-2)-wz(i-2,g-2,k-2);

vx(i-2,g-2,k-2)=Ewx(i-2,g-2,k-2).*Bz(i-2,g-2,k-2) - Ewz(i-2,g-2,k-2).*By(i-2,g-2,k-2);
vpy(i-2,g-2,k-2)=-(Ewx(i-2,g-2,k-2).*Bz(i-2,g-2,k-2))+(Ewz(i-2,g-2,k-2).*Bx(i-2,g-2,k-2));

vy(i-2,g-2,k-2)=vpy(i-2,g-2,k-2)-(vpz(i-2,g-2,k-2).*Bx(i-2,g-2,k-2)/Bz(i-2,g-2,k-2));

end

end

end
B.2 Input for Matlab worksheet

The following gives the Matlab commands used in the calculation described in Chapter 5, Section 5.4

function \( B = alialinew(s, y) \)

\[ j = 1; a = 1; b = 4; p = 1; q = 1; \]

\[ B = \left[ 2 \cdot y(1)/(p + 1) - j \cdot (1 - (y(1)^2 + q \cdot y(2)^2)^3/a^6)^4 \cdot (1 - y(3)^4/b^4)^2 \cdot y(2) \cdot q; \right. \]
\[ 2 \cdot p \cdot y(2)/(p + 1) + j \cdot (1 - (y(1)^2 + q \cdot y(2)^2)^3/a^6)^4 \cdot (1 - y(3)^4/b^4)^2 \cdot y(1); -2 \cdot y(3)]; \]

---------------------------------------------
clear all
close all
clc
j=1;a=1;b=4;nx=81;p=1;q=1;
s=0:1/(nx-1):1;
Here we are create array of all zeros.
X=zeros(nx,nx,nx);Y=zeros(nx,nx,nx);Z=zeros(nx,nx,nx);
Bx=zeros(nx,nx,nx);By=zeros(nx,nx,nx);Bz=zeros(nx,nx,nx);
Bx1=zeros(nx,nx,nx);By1=zeros(nx,nx,nx);Bz1=zeros(nx,nx,nx);
Jx=zeros(nx,nx,nx);Jy=zeros(nx,nx,nx);Jz=zeros(nx,nx,nx);
Jx1=zeros(nx,nx,nx);Jy1=zeros(nx,nx,nx);Jz1=zeros(nx,nx,nx);
eta=zeros(nx,nx,nx);etaJx=zeros(nx,nx,nx);etaJy=zeros(nx,nx,nx);etaJz=zeros(nx,nx,nx);
intagrand=zeros(nx,nx,nx);Phi=zeros(nx,nx,nx);Ex=zeros(nx-4,nx-4,nx-4);
Ey=zeros(nx-4,nx-4,nx-4);Ez=zeros(nx-4,nx-4,nx-4);wx=zeros(nx-4,nx-4,nx-4);
wy=zeros(nx-4,nx-4,nx-4);wz=zeros(nx-4,nx-4,nx-4);Ewx=zeros(nx-4,nx-4,nx-4);
Ewy=zeros(nx-4,nx-4,nx-4);Ewz=zeros(nx-4,nx-4,nx-4);
vpx=zeros(nx-4,nx-4,nx-4);vpy=zeros(nx-4,nx-4,nx-4);vpz=zeros(nx-4,nx-4,nx-4);
vx=zeros(nx-4,nx-4,nx-4);vy=zeros(nx-4,nx-4,nx-4);vz=zeros(nx-4,nx-4,nx-4);

Column vector of time points.
t=linspace(0,1,nx);

Here we are defining a vector of initial conditions for $x_0, y_0, z_0$.

for kk=1:nx
    k=2.1*((kk-1.0)/(nx-1.0)-0.5);
for mm=1:nx
    m=2.1*((mm-1.0)/(nx-1.0)-0.5);
    rr1 = k.^2 + q.*m.^2;
    if rr1 > 1
        x = k.*exp(t);
        y = m.*exp(t);
        z = 4.*exp(-2.*t);
    else
        Here we are find a field line equations.
        [s,B1] = ode23(@alialinew,t,[k m 4]);
        x = B1(:,1);
        y = B1(:,2);
        z = B1(:,3);
        rr = x.^2 + q*y.^2;
        pt = find(rr < 1,1,'last');
        if pt + 1 < nx
            S1 = s(pt + 1) : 1/(nx - 1) : 1;
            xx = x(pt).*exp(S1 - s(pt));
            yy = y(pt).*exp(S1 - s(pt));
            xx = xx'; yy = yy';
            x(pt + 1 : end) = xx;
\begin{align*}
y(pt + 1 : end) &= yy; \\
z &= z; \\
end \\
end \\
Here we are defined a regular mesh. \\
\begin{align*}
  xi &= (-40 : 40)/20; \\
yi &= (-40 : 40)/20; \\
zi &= 0.542 + (0 : 80)/23.2; \\
[x1, y1, z1] &= \text{meshgrid}(xi, yi, zi); \\
X(1: \text{length}(x), kk, mm) &= x; \\
Y(1: \text{length}(x), kk, mm) &= y; \\
Z(1: \text{length}(x), kk, mm) &= z; \\
Here we are define the magnetic field \\
Bx1 &= \frac{2}{p+1} \cdot X(1: \text{length}(x), kk, mm) - j \cdot (1 - (X(1: \text{length}(x), kk, mm)^2 + q \cdot Y(1: \text{length}(x), kk, mm)^2)^{3/a^6})^4 \cdot (1 - Z(1: \text{length}(x), kk, mm)^4/b^4)^2 \cdot Y(1: \text{length}(x), kk, mm)) * q; \\
By1 &= 2 * p/(p+1) \cdot Y(1: \text{length}(x), kk, mm) + j \cdot (1 - (X(1: \text{length}(x), kk, mm)^2 + q \cdot Y(1: \text{length}(x), kk, mm)^2)^{3/a^6})^4 \cdot (1 - (Z(1: \text{length}(x), kk, mm)^4/b^4)^2 \cdot X(1: \text{length}(x), kk, mm); \\
Bz1 &= -2 \cdot Z(1: \text{length}(x), kk, mm); \\
Here we are define the current components. \\
Jx1 &= 8 \cdot j \cdot (1 - (X(1: \text{length}(x), kk, mm)^2 + q \cdot Y(1: \text{length}(x), kk, mm)^2)^2)^3/a^6)^4 \cdot (1 - Z(1: \text{length}(x), kk, mm))^{4/b^4}) \cdot X(1: \text{length}(x), kk, mm); \\
Jy1 &= 8 \cdot j \cdot (1 - (X(1: \text{length}(x), kk, mm)^2 + q \cdot Y(1: \text{length}(x), kk, mm)^2)^2)^3/a^6)^4 \cdot (1 - Z(1: \text{length}(x), kk, mm))^{4/b^4}) \cdot Y(1: \text{length}(x), kk, mm)^3 * q/b^4; \\
\end{align*}
\[ Jz1(1 : \text{length}(x), kk, mm) = -24*j.*\left(1 - (X(1 : \text{length}(x), kk, mm)^2 + q*Y(1 : \text{length}(x), kk, mm))/b^4\right)^2 \cdot \Phi(1 : \text{length}(x), kk, mm) \cdot \Phi(i, kk, mm) \]
end
for $i = 1: length(x)$

$R = (X(i, kk, mm)).^2 + q. * (Y(i, kk, mm)).^2;$

if $R < 1$

$Bx(i, kk, mm) = 2/(p + 1). * X(i, kk, mm) - j. * (1 - (X(i, kk, mm))^2 + q. * Y(i, kk, mm))^2 / a^6).^4. * (1 - (Z(i, kk, mm))^4 / b^4).^2. * Y(i, kk, mm) * q;$

$By(i, kk, mm) = 2 * p/(p + 1). * Y(i, kk, mm) + j. * (1 - (X(i, kk, mm))^2 + q. * Y(i, kk, mm))^2 / a^6).^4. * (1 - (Z(i, kk, mm))^4 / b^4).^2. * X(i, kk, mm);$

$Bz(i, kk, mm) = -2. * Z(i, kk, mm);$

$Jx(i, kk, mm) = 8. * j. * (1 - (X(i, kk, mm))^2 + q. * (Y(i, kk, mm))^2).^3 / a^6).^4. * (1 - (Z(i, kk, mm))^4 / b^4). * (X(i, kk, mm)). * (Z(i, kk, mm))^3. * q/b^4;$

$Jy(i, kk, mm) = 8. * j. * (1 - (X(i, kk, mm))^2 + q. * (Y(i, kk, mm))^2).^3 / a^6).^4. * (1 - (Z(i, kk, mm))^4 / b^4). * (Y(i, kk, mm)). * (Z(i, kk, mm))^3. * q/b^4;$

$Jz(i, kk, mm) = -24 * j. * (1 - (X(i, kk, mm))^2 + q*Y(i, kk, mm))^2).^3 / a^6).^3. * (1 - Z(i, kk, mm))^4 / b^4).^2. * X(i, kk, mm)^2. * (X(i, kk, mm)^2 + q*Y(i, kk, mm))^2 / a^6 + j*(1 - ((X(i, kk, mm)^2 + q*Y(i, kk, mm)^2)^3) / (a^6))^4. * (1 - (Z(i, kk, mm))^4 / (b^4))^2 - 24 * j*(1 - ((X(i, kk, mm)^2 + q*Y(i, kk, mm)^2)^3) / (a^6))^3. * (1 - Z(i, kk, mm))^4 / b^4).^2. * Y(i, kk, mm)^2*q^2. * (X(i, kk, mm)^2 + q*Y(i, kk, mm)^2)^2 / a^6 + j*(1 - ((X(i, kk, mm)^2 + q*Y(i, kk, mm)^2)^3) / (a^6))^4. * (1 - (Z(i, kk, mm))^4 / (b^4))^2 * q;$

else

Here we are define the potential magnetic field out side diffusion region

$Bx(i, kk, mm) = 2/(p + 1). * X(i, kk, mm);$

$By(i, kk, mm) = 2 * p/(p + 1). * Y(i, kk, mm);$

$Bz(i, kk, mm) = -2. * Z(i, kk, mm);$

$Jx(i, kk, mm) = 0;$

$Jy(i, kk, mm) = 0;$

$Jz(i, kk, mm) = 0;$
we are substituting all calculation by regular mesh.

\[ \text{Phi}_1 = \text{griddata3}(X, Y, Z, \text{Phi}, x_1, y_1, z_1); \]
\[ \text{Phi}_1(\text{isnan}(	ext{Phi}_1)) = 0; \]
\[ \text{eta}_1 = \text{griddata3}(X, Y, Z, \text{eta}, x_1, y_1, z_1); \]
\[ \text{eta}_1(\text{isnan}(	ext{eta}_1)) = 0; \]
\[ \text{BB}_x = \text{griddata3}(X, Y, Z, \text{Bx}, x_1, y_1, z_1); \]
\[ \text{BB}_x(\text{isnan}(	ext{BB}_x)) = 0; \]
\[ \text{BB}_y = \text{griddata3}(X, Y, Z, \text{By}, x_1, y_1, z_1); \]
\[ \text{BB}_y(\text{isnan}(	ext{BB}_y)) = 0; \]
\[ \text{BB}_z = \text{griddata3}(X, Y, Z, \text{Bz}, x_1, y_1, z_1); \]
\[ \text{BB}_z(\text{isnan}(	ext{BB}_z)) = 0; \]
\[ \text{JJ}_x = \text{griddata3}(X, Y, Z, \text{Jx}, x_1, y_1, z_1); \]
\[ \text{JJ}_x(\text{isnan}(	ext{JJ}_x)) = 0; \]
\[ \text{JJ}_y = \text{griddata3}(X, Y, Z, \text{Jy}, x_1, y_1, z_1); \]
\[ \text{JJ}_y(\text{isnan}(	ext{JJ}_y)) = 0; \]
\[ \text{JJ}_z = \text{griddata3}(X, Y, Z, \text{Jz}, x_1, y_1, z_1); \]
\[ \text{JJ}_z(\text{isnan}(	ext{JJ}_z)) = 0; \]

Here we are calculating \( \mathbf{E} \) and \( \mathbf{v} \)

```plaintext
for ii = 3 : nx - 2 
for jj = 3 : nx - 2 
for ll = 3 : nx - 2 

\[ Ex(ii, jj, ll) = -(\text{Phi}_1(ii, jj - 2, ll) - 8 \cdot \text{Phi}_1(ii, jj - 1, ll) + 8 \cdot \text{Phi}_1(ii, jj + 1, ll) - \text{Phi}_1(ii, jj + 2, ll))/(12 \cdot 0.05); \]
```
\[ Ey(ii, jj, ll) = -(\Phi_1(ii - 2, jj, ll) - 8 \cdot \Phi_1(ii - 1, jj, ll) + 8 \cdot \Phi_1(ii + 1, jj, ll) - \Phi_1(ii + 2, jj, ll))/(12 \cdot 0.05); \]

\[ Ez(ii, jj, ll) = -(\Phi_1(ii, jj, ll - 2) - 8 \cdot \Phi_1(ii, jj, ll - 1) + 8 \cdot \Phi_1(ii, jj, ll + 1) - \Phi_1(ii, jj, ll + 2))/(12 \cdot 0.0431); \]

\[ wx(ii, jj, ll) = J J x(ii, jj, ll) \cdot \eta_1(ii, jj, ll); \]

\[ wy(ii, jj, ll) = J J y(ii, jj, ll) \cdot \eta_1(ii, jj, ll); \]

\[ wz(ii, jj, ll) = J J z(ii, jj, ll) \cdot \eta_1(ii, jj, ll); \]

\[ E w x(ii, jj, ll) = E x(ii, jj, ll) - J J x(ii, jj, ll); \]

\[ E w y(ii, jj, ll) = E y(ii, jj, ll) - J J y(ii, jj, ll); \]

\[ E w z(ii, jj, ll) = E z(ii, jj, ll) - J J z(ii, jj, ll); \]

\[ v p x(ii, jj, ll) = (((E w y(ii, jj, ll) \cdot B B z(ii, jj, ll)) - (E w z(ii, jj, ll) \cdot B B y(ii, jj, ll)))) /((B B x(ii, jj, ll)).^2 + (B B y(ii, jj, ll)).^2 + (B B z(ii, jj, ll)).^2); \]

\[ v p y(ii, jj, ll) = (((-E w x(ii, jj, ll) \cdot B B z(ii, jj, ll)) + (E w z(ii, jj, ll) \cdot B B x(ii, jj, ll)))) /((B B x(ii, jj, ll)).^2 + (B B y(ii, jj, ll)).^2 + (B B z(ii, jj, ll)).^2); \]

\[ v p z(ii, jj, ll) = (((E w x(ii, jj, ll) \cdot B B y(ii, jj, ll)) - (E w y(ii, jj, ll) \cdot B B x(ii, jj, ll)))) /((B B x(ii, jj, ll)).^2 + (B B y(ii, jj, ll)).^2 + (B B z(ii, jj, ll)).^2); \]

\[ v x(ii, jj, ll) = v p x(ii, jj, ll) - ((v p z(ii, jj, ll) \cdot B B z(ii, jj, ll)) \cdot B B x(ii, jj, ll)); \]

\[ v y(ii, jj, ll) = v p y(ii, jj, ll) - ((v p z(ii, jj, ll) \cdot B B z(ii, jj, ll)) \cdot B B y(ii, jj, ll)); \]

\[ v z(ii, jj, ll) = v p z(ii, jj, ll) - ((v p z(ii, jj, ll) \cdot B B z(ii, jj, ll)) \cdot B B z(ii, jj, ll)); \]

end

end
B.3 Input for Matlab worksheet

The following gives the Matlab commands used in the calculation described in Chapter 5, Section 5.5.

function \( B = G_{\text{opposite}}(s, y) \)

\[
j = 1; a = 4; b = 1; p = 1; q = 1/1;
B = \left[ -\left(2 \ast y(1)/(p + 1) - (1 - \text{heaviside}(y(3) - 1)) \ast j \ast y(3) \ast \left(1 - ((1/q) \ast y(1)^2 + y(2)^2) / a^6 \right)^4 \ast (1 - y(3)^4 / b^4)^{12} \ast y(2)); \right. \\
\left. -(2 \ast p \ast y(2)/(p + 1) + (1 - \text{heaviside}(y(3) - 1)) \ast j \ast y(3) \ast \left(1 - ((1/q) \ast y(1)^2 + y(2)^2) / a^6 \right)^4 \ast (1 - y(3)^4 / b^4)^{12} \ast y(1) \ast 1/q); 2 \ast y(3)); \right]
\]

clear all
close all
clc

\[
j = 1; a = 4; b = 1; nx = 81; p = 1; q = 1/1; h = 4;
\]
s1=0:h/(nx-1):h;

Here we are create an array of all zeros.

\[
X=\text{zeros}(nx,nx,nx); Y=\text{zeros}(nx,nx,nx); Z=\text{zeros}(nx,nx,nx);
Bx=\text{zeros}(nx,nx,nx); By=\text{zeros}(nx,nx,nx); Bz=\text{zeros}(nx,nx,nx);
Bx1=\text{zeros}(nx,nx,nx); By1=\text{zeros}(nx,nx,nx); Bz1=\text{zeros}(nx,nx,nx);
Jx=\text{zeros}(nx,nx,nx); Jy=\text{zeros}(nx,nx,nx); Jz=\text{zeros}(nx,nx,nx);
Jx1=\text{zeros}(nx,nx,nx); Jy1=\text{zeros}(nx,nx,nx); Jz1=\text{zeros}(nx,nx,nx);
eta=\text{zeros}(nx,nx,nx); etaJx=\text{zeros}(nx,nx,nx); etaJy=\text{zeros}(nx,nx,nx); etaJz=\text{zeros}(nx,nx,nx);
\text{intagrand}=\text{zeros}(nx,nx,nx); \text{Phi}=\text{zeros}(nx,nx,nx); \text{Ex}=\text{zeros}(nx-4,nx-4,nx-4);
Ey=\text{zeros}(nx-4,nx-4,nx-4); Ez=\text{zeros}(nx-4,nx-4,nx-4); wx=\text{zeros}(nx-4,nx-4,nx-4);
wz=\text{zeros}(nx-4,nx-4,nx-4); Ewx=\text{zeros}(nx-4,nx-4,nx-4);
Ewy=\text{zeros}(nx-4,nx-4,nx-4); Ewz=\text{zeros}(nx-4,nx-4,nx-4);
\]
vx=zeros(nx-4,nx-4,nx-4); vy=zeros(nx-4,nx-4,nx-4); vz=zeros(nx-4,nx-4,nx-4);
vpx=zeros(nx-4,nx-4,nx-4);

Column vector of time points.
t=linspace(0,h,nx);
Here we are defining a vector of initial conditions for $x_0, y_0, z_0$.

for $kk = 1 : nx$
  $k = 4.1 * ((q) * \cos((kk - 1.0) * 360/(nx - 1) * pi/180));$
  $m = 4.1 * (\sin((kk - 1.0) * 360/(nx - 1) * pi/180));$
for $mm = 1 : nx - 1$
  $l = 2 * ((mm - 1.0)/(nx - 1.99))^4;$
Here we are find a field line equations.

$[s, B1] = ode23(@Gopposite, t, [k m l]);$
$x = B1(:, 1);$
$y = B1(:, 2);$
$z = B1(:, 3);$

ZZ = z;

$pt = find(ZZ < 1, 1,'last');$

if $pt < nx$

  $S1 = s(pt + 1) : h/(nx - 1) : h;$
  $zz = z(pt) * exp(2 * (S1 - s(pt))));$
  $zz = zz';$
  $z(pt + 1 : end) = zz;$
end
Here we have defined a regular mesh.

$xi = (-4.1 : 8.2/81 : 4.1);$  
$yi = (-4.1 : 8.2/81 : 4.1);$  
$zi = (0 : 2.1/81 : 2);$
\([x_1, y_1, z_1] = \text{meshgrid}(x_i, y_i, z_i)\);

\[X(1 : \text{length}(x), kk, mm) = x;\]

\[Y(1 : \text{length}(x), kk, mm) = y;\]

\[Z(1 : \text{length}(x), kk, mm) = z;\]

\[Bx1(1 : \text{length}(x), kk, mm) = 2 \times X(1 : \text{length}(x), kk, mm) / (p + 1) - j. * Z(1 : \text{length}(x), kk, mm). * (1 - ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^3 / a_b^6).^4. * (1 - Z(1 : \text{length}(x), kk, mm))^4 / b^4).^12. * Y(1 : \text{length}(x), kk, mm);\]

\[By1(1 : \text{length}(x), kk, mm) = 2 \times p * Y(1 : \text{length}(x), kk, mm) / (p + 1) + j. * Z(1 : \text{length}(x), kk, mm). * (1 - ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^3 / a_b^6).^4. * (1 - Z(1 : \text{length}(x), kk, mm))^4 / b^4).^12. * X(1 : \text{length}(x), kk, mm) * 1/q;\]

\[Bz1(1 : \text{length}(x), kk, mm) = -2. * Z(1 : \text{length}(x), kk, mm);\]

\[Jx1(1 : \text{length}(x), kk, mm) = -j. * (1 - ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^3 / a_b^6).^4. * (1 - Z(1 : \text{length}(x), kk, mm))^4 / b^4).^12. * X(1 : \text{length}(x), kk, mm) + 1/q + 48 * j. * Z(1 : \text{length}(x), kk, mm)^4. * (1 - ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^3 / a_b^6).^4. * (1 - Z(1 : \text{length}(x), kk, mm))^4 / b^4).^11. * Y(1 : \text{length}(x), kk, mm) / (1/q) / b^4;\]

\[Jy1(1 : \text{length}(x), kk, mm) = -j. * (1 - ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^3 / a_b^6).^4. * (1 - Z(1 : \text{length}(x), kk, mm))^4 / b^4).^12. * Y(1 : \text{length}(x), kk, mm) + 48 * j. * Z(1 : \text{length}(x), kk, mm)^4. * (1 - ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^3 / a_b^6).^4. * (1 - Z(1 : \text{length}(x), kk, mm))^4 / b^4).^11. * Y(1 : \text{length}(x), kk, mm) / b^4;\]

\[Jz1(1 : \text{length}(x), kk, mm) = -24 * j. * Z(1 : \text{length}(x), kk, mm). * (1 - ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^3 / a_b^6).^3. * (1 - Z(1 : \text{length}(x), kk, mm))^4 / b^4).^12. * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^2 / a_b^6 + j * Z(1 : \text{length}(x), kk, mm). * ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^2 / a_b^6 + j * Z(1 : \text{length}(x), kk, mm). * (1 - ((1/q) * X(1 : \text{length}(x), kk, mm)^2 + Y(1 : \text{length}(x), kk, mm)^2)^3 / a_b^6).^4. *
\[(1 - Z(1 : length(x), kk, mm))^4 / b^4)^{12} \cdot (1/q) - 24 \cdot j \cdot Z(1 : length(x), kk, mm) \cdot (1 - (1/q) \cdot X(1 : length(x), kk, mm)^2 + Y(1 : length(x), kk, mm)^2)^3 / a^6)^3 \cdot (1 - Z(1 : length(x), kk, mm)^4 / b^4)^{12} \cdot Y(1 : length(x), kk, mm)^2 \cdot ((1/q) \cdot X(1 : length(x), kk, mm)^2 + Y(1 : length(x), kk, mm)^2)^2 / a^6 + j \cdot Z(1 : length(x), kk, mm) \cdot (1 - ((1/q) \cdot X(1 : length(x), kk, mm)^2 + Y(1 : length(x), kk, mm)^2)^3 / a^6)^4 \cdot (1 - Z(1 : length(x), kk, mm)^4 / b^4)^{12};
\]

\[\text{eta}(1 : length(x), kk, mm) = 1;}
\[\text{etaJx}(1 : length(x), kk, mm) = \text{eta}(1 : length(x), kk, mm) \cdot Jx1(1 : length(x), kk, mm);}
\[\text{etaJy}(1 : length(x), kk, mm) = \text{eta}(1 : length(x), kk, mm) \cdot Jy1(1 : length(x), kk, mm);}
\[\text{etaJz}(1 : length(x), kk, mm) = \text{eta}(1 : length(x), kk, mm) \cdot Jz1(1 : length(x), kk, mm);\]

\[\text{for } i = 1 : \text{length(x)}
\[R = ((1/q) \cdot X(i, kk, mm))^2 + (Y(i, kk, mm))^2;\]
\[ZZ = (Z(i, kk, mm));\]

\[\text{if } ZZ < 1
\[\text{intagrand} = \eta \text{J} \cdot \text{B};\]
\[\text{intagrand}(1 : \text{length(x), kk, mm}) = Bx1(1 : \text{length(x), kk, mm}) \cdot \text{etaJx}(1 : \text{length(x), kk, mm}) + By1(1 : \text{length(x), kk, mm}) \cdot \text{etaJy}(1 : \text{length(x), kk, mm}) + Bz1(1 : \text{length(x), kk, mm}) \cdot \text{etaJz}(1 : \text{length(x), kk, mm});\]
\[\text{else}
\[\text{intagrand}(i, kk, mm) = 0;\]
\[\text{end}
\[\text{end}\]

Here we are calculating \(\phi\)

\[\text{Phi}(1, kk, mm) = 0;\]

\[\text{for } i = 2 : \text{length(x)}\]
\[\text{Phi}(i, kk, mm) = \text{trapz}(s1(1 : i), \text{intagrand}(1 : i, kk, mm));\]
\[\text{end}\]
for $i = 1 : \text{length}(x)$

\[
R = (1/q) \times (X(i,kk,mm))^2 + (Y(i,kk,mm))^2;
\]

\[
ZZ = (Z(i,kk,mm));
\]

if $ZZ < 1$ and $R < 16$

\[
Bx(i,kk,mm) = 2 \times X(i,kk,mm)/(p + 1) - j \times Z(i,kk,mm). * (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).12 \times Y(i,kk,mm);
\]

\[
By(i,kk,mm) = 2 \times p \times Y(i,kk,mm)/(p + 1) + j \times Z(i,kk,mm). * (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).12 \times X(i,kk,mm); \]

\[
Bz(i,kk,mm) = -2 \times Z(i,kk,mm);
\]

\[
Jx(i,kk,mm) = -j \times (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).12 \times X(i,kk,mm) + 48 \times j \times Z(i,kk,mm).4 \times (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).11 \times X(i,kk,mm)/(q/b^4);
\]

\[
Jy(i,kk,mm) = -j \times (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).12 \times Y(i,kk,mm) + 48 \times j \times Z(i,kk,mm).4 \times (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).11 \times Y(i,kk,mm)/(q/b^4);
\]

\[
Jz(i,kk,mm) = -24 \times j \times Z(i,kk,mm). * (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).3 \times (1 - Z(i,kk,mm).4/b^4).12 \times X(i,kk,mm)^2 \times (1/q^2) \times ((1/q) \times X(i,kk,mm)^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).12 \times Y(i,kk,mm)^2/a^6 + j \times Z(i,kk,mm). * (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).12 \times Y(i,kk,mm)^3/a^6).3 \times (1 - Z(i,kk,mm).4/b^4).12 \times Y(i,kk,mm)^2/a^6 + j \times Z(i,kk,mm). * (1 - ((1/q) \times X(i,kk,mm))^2 + Y(i,kk,mm)^2)^3/a^6).4 \times (1 - Z(i,kk,mm).4/b^4).12; \]

else

Here we are define the potential magnetic field out side diffusion region

\[
Bx(i,kk,mm) = 2 \times X(i,kk,mm)/(p + 1);
\]
\[ By(i, kk, mm) = 2 \ast p \ast Y(i, kk, mm)/(p + 1); \]
\[ Bz(i, kk, mm) = -2 \ast Z(i, kk, mm); Jx(i, kk, mm) = 0; \]
\[ Jy(i, kk, mm) = 0; \]
\[ Jz(i, kk, mm) = 0; \]
\end{verbatim}

Here we are substituting all calculation by regular mesh.

\[ Phi1 = \text{griddata3}(X, Y, Z, Phi, x1, y1, z1); \]
\[ Phi1(\text{isnan}(Phi1)) = 0; \]
\[ eta1 = \text{griddata3}(X, Y, Z, eta, x1, y1, z1); \]
\[ eta1(\text{isnan}(eta1)) = 0; \]
\[ BBx = \text{griddata3}(X, Y, Z, Bx, x1, y1, z1); \]
\[ BBx(\text{isnan}(BBx)) = 0; \]
\[ BBy = \text{griddata3}(X, Y, Z, By, x1, y1, z1); \]
\[ BBy(\text{isnan}(BBy)) = 0; \]
\[ BBz = \text{griddata3}(X, Y, Z, Bz, x1, y1, z1); \]
\[ BBz(\text{isnan}(BBz)) = 0; \]
\[ JJx = \text{griddata3}(X, Y, Z, Jx, x1, y1, z1); \]
\[ JJx(\text{isnan}(JJx)) = 0; \]
\[ JJy = \text{griddata3}(X, Y, Z, Jy, x1, y1, z1); \]
\[ JJy(\text{isnan}(JJy)) = 0; \]
\[ JJz = \text{griddata3}(X, Y, Z, Jz, x1, y1, z1); \]
\[ JJz(\text{isnan}(JJz)) = 0; \]
\[ x2 = x1; y2 = y1; \]
\[ \text{for } ii = 1 : nx \]
for \( jj = 1 : nx \)
\[
x2(ii, jj,:) = x1(jj, ii,:);
\]
\[
y2(ii, jj,:) = y1(jj, ii,:);
\]
end
end

Here we are calculating \( \mathbf{E} \) and \( \mathbf{v} \)
\[
x1 = x2; y1 = y2;
\]
for \( ii = 3 : nx - 2 \)
for \( jj = 3 : nx - 2 \)
for \( ll = 3 : nx - 2 \)
\[
Ex(ii, jj, ll) = -(Phi1(ii, jj - 2, ll) - 8 * Phi1(ii, jj - 1, ll) + 8 * Phi1(ii, jj + 1, ll) - Phi1(ii, jj + 2, ll))/(12 * 0.1012);
\]
\[
Ey(ii, jj, ll) = -(Phi1(ii - 2, jj, ll) - 8 * Phi1(ii - 1, jj, ll) + 8 * Phi1(ii + 1, jj, ll) - Phi1(ii + 2, jj, ll))/(12 * 0.1012);
\]
\[
Ez(ii, jj, ll) = -(Phi1(ii, jj, ll - 2) - 8 * Phi1(ii, jj, ll - 1) + 8 * Phi1(ii, jj, ll + 1) - Phi1(ii, jj, ll + 2))/(12 * 0.0247);
\]
\[
xw(ii, jj, ll) = JJx(ii, jj, ll) * eta1(ii, jj, ll);
\]
\[
wz(ii, jj, ll) = JJz(ii, jj, ll) * eta1(ii, jj, ll);
\]
\[
Ewx(ii, jj, ll) = Ex(ii, jj, ll) - JJx(ii, jj, ll);
\]
\[
Ewy(ii, jj, ll) = Ey(ii, jj, ll) - JJy(ii, jj, ll);
\]
\[
Ewz(ii, jj, ll) = Ez(ii, jj, ll) - JJz(ii, jj, ll);
\]
\[
vpx(ii, jj, ll) = (((Ewy(ii, jj, ll) * BBz(ii, jj, ll)) - (Ewz(ii, jj, ll) * BBy(ii, jj, ll)))) / ((BBx(ii, jj, ll))^2 + (BBy(ii, jj, ll))^2 + (BBz(ii, jj, ll))^2);
\]
\[
vpy(ii, jj, ll) = ((-(Ewx(ii, jj, ll) * BBz(ii, jj, ll)) + (Ewz(ii, jj, ll) * BBx(ii, jj, ll)))) / ((BBx(ii, jj, ll))^2 + (BBy(ii, jj, ll))^2 + (BBz(ii, jj, ll))^2);
\]
\[
vpz(ii, jj, ll) = (((Ewx(ii, jj, ll) * BBy(ii, jj, ll)) - (Ewy(ii, jj, ll) * BBx(ii, jj, ll))))
\[ ((BBx(ii, jj, ll))^2 + (BBy(ii, jj, ll))^2 + (BBz(ii, jj, ll))^2); \]

\[
vx(ii, jj, ll) = vpx(ii, jj, ll) - ((vpz(ii, jj, ll)./BBz(ii, jj, ll)). * BBx(ii, jj, ll));
\]

\[
vy(ii, jj, ll) = vpy(ii, jj, ll) - ((vpz(ii, jj, ll)./BBz(ii, jj, ll)). * BBy(ii, jj, ll));
\]

\[
vz(ii, jj, ll) = vpz(ii, jj, ll) - ((vpz(ii, jj, ll)./BBz(ii, jj, ll)). * BBz(ii, jj, ll));
\]

end

end

end
Bibliography


