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A numerical study for multiple solutions of a singular boundary value problem arising from laminar flow in a porous pipe with moving wall

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Abstract

This paper is concerned with multiple solutions of a singular nonlinear boundary value problem (BVP) on the interval $[0, 1]$, which arises in a study of laminar flow in a porous pipe with an expanding or contracting wall. For the singular nonlinear BVP, the correct boundary conditions are derived to guarantee that its linearization has a unique continuous solution, and the smooth results on its solutions are extended. Furthermore, a numerical technique is proposed to find all possible multiple solutions. The computed results are compared with those obtained by AUTO. For the suction driven pipe flow with the expanding wall (e.g. $\alpha = 2$), we find a new solution numerically and classify it as a type VI solution. In addition, we also construct asymptotic solutions for a few cases of parameters, which agree well with numerical solutions. This serves as a validation of our numerical results. Thus we believe that the numerical technique designed in Section 3 is reliable, and may be further applied to solve a variety of nonlinear equations that arise from other flow problems.

Keywords: singular boundary value problem; multiple solutions; singular perturbation method; expanding porous circular pipe

1. Introduction

The laminar flow in a porous circular pipe or channel with an expanding or contracting wall has received considerable attention in recent years due to their relevance to a number of biological and engineering models, such as the transport of biological fluids through contracting or expanding vessels and the air circulation in the respiratory system. The earliest workers on the unsteady flow across an expanding wall can probably be traced back to Uchida and Aoki [1], in which the flow equations in a pipe are reduced to a single fourth-order nonlinear ordinary differential equation with the wall expansion ratio as a parameter. In order to simulate the laminar flow field in cylindrical solid rocket motors, Goto and Uchida [2] analyzed the laminar incompressible flow in a semi-infinite porous pipe whose radius varies with time. Following the route of this investigation, a variety of methods have been used to study this problem. For example, Boutros et al. [3, 4] applied a Lie-group method to the equations of motion to determine symmetry reductions of partial differential equations. The resulting fourth-order nonlinear differential equation is then solved using small-parameter perturbations, and the results are compared with numerical solutions using shooting method. Asghar et al. [5] and Dinarvand and Rashidi [6] also discussed the flow in

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a slowly deforming channel with weak permeability using homotopy analysis method (HAM) and adomian decomposition method (AMD), respectively.

Besides the above mentioned results, multiple solutions are usually found for governing equations in the porous pipe or channel with stationary walls. For example, Robinson [7] considered the inclusion of exponentially small terms in an asymptotic series to find two of the solutions analytically for the flow in a porous channel. Using the HAM, Xu et al. [8] recently investigated the multiple solutions of the flow in a porous channel with expanding or contracting walls and explored some ranges of the control parameters. However, little work is found in literature on multiple solutions of the laminar flow in a pipe with an expanding or contracting wall in a full range of Reynolds numbers. The main purpose of this paper is to find multiple solutions corresponding to governing equations. Since the transformation of governing equations to a singular nonlinear boundary value problem (BVP) can be found in previous literature (e.g. [9]), for the sake of simplicity, we only present the resulting BVP of the form

$$\eta f''' + f'' + \frac{\alpha}{2}(\eta f'' + f') + \frac{Re}{2}(f'' f - f'^2) = k, \tag{1}$$

where $'$ denotes the derivative with respect to η, α and Re are wall expansion ratio and cross-flow Reynolds number, respectively, and k is an integration constant. Another form of the Eq.(1) is often used to study its solution. It is obtained through a simple differentiation:

$$\eta f'''' + 2f''' + \frac{\alpha}{2}(\eta f''' + 2f'') + \frac{Re}{2}(f''' f - f'' f') = 0. \tag{2}$$

The corresponding boundary conditions are

$$f'(1) = 0, \quad f(1) = 1, \quad f(0) = 0, \quad \lim_{\eta\to 0} \eta^{\frac{1}{2}} f''(\eta) = 0. \tag{3}$$

Noting that when the wall is stationary (i.e. $\alpha = 0$), Terrill and Thomas [10] have presented the multiple solutions using a numerical technique, in which the boundary value problem is rewritten as an initial value problem at the left endpoint, and the initial values are updated to meet the boundary conditions at the other endpoint. This process is similar to that of shooting method. To overcome the singularity at $\eta = 0$, a Taylor expansion is used in the neighborhood of $\eta = 0$ and a Runge-Kutta method is then applied thereafter. Using the technique, Shankararaman and Liu [11] also considered the effect of the slip on existence and uniqueness of similarity solutions in a porous pipe with stationary walls. However for the laminar flow in a porous pipe with an expanding or contracting wall (i.e. $\alpha \neq 0$), if we continue to use their technique, the BVP (i.e. (2) and (3)) may not be easy to solve due to the singularity at $\eta = 0$ and multiple parameters (i.e. α and Re). Furthermore, we aim to find all possible multiple solutions for the full range of parameters. This increases the difficulty of the computation dramatically. In other words, to obtain multiple solutions of (2) and (3), it is necessary to design a new numerical technique.

In the current paper, we mainly focus on how to solve the problem (2) and (3) to obtain multiple solutions. Since Eq.(2) has the singularity at $\eta = 0$, a solution to its linearization may blow up (see Section 2 for more detail on it). Therefore, before solving it, we first analyze the problem from the aspects of the singularity. There also exist plenty of papers dealing with the smooth properties of solutions for singular BVPs. Here, by combining the problem (2) and (3), the studies on such singular BVPs will be simply introduced. In fact, the problem (2) and (3) is a typical BVP with a singularity of the first kind [12]. Such BVPs often arise in numerous applications in natural sciences and engineering, e.g. when a partial differential equation (PDE) is reduced to an ordinary differential equation (ODE) by cylindrical or spherical symmetry. Since the singular BVP is not evaluated easily at the singular point, the studies on it have become a recurring topic in the field of numerical calculation (e.g. [12-18]), where their attentions mainly focus on the existence, uniqueness and smoothness of solutions. In particular, the structure of the boundary conditions which are necessary and sufficient for the linearization of a singular nonlinear BVP to have a reasonable smoothness of the solution on a closed interval including the singular point is of special interest. Other studies on the applications and convergence properties of several finite difference, collocation and Galerkin schemes for singular BVPs may also be found in e.g. [12][16][19-22] and a nice bibliography on solving singular BVP numerically [23].

The rest of the paper is organised as follow. In Section 2, we will focus first on the following question: What boundary conditions are appropriate so that the linearization of the problem (2) and (3) may have a reasonably smooth solution. To obtain all possible multiple solutions, we propose a technique in Section 3.1, where translating the problem (2) and (3) into a initial value problem (IVP) will be needed. The resulting IVP also has the singularity at $\eta = 0$. Therefore, in Section 3.2, according to the existing literature [24-28], we analyse the smoothness of the solution of the singular IVP near the singular point, and thus ode45 given in Matlab can be used to solve it. Numerical results and multiple solutions are presented in Section 4. In order to further verify the numerical results, asymptotic solutions for some ranges of parameters are constructed by a few suitable perturbation methods and asymptotic results are compared with the numerical ones in Section 5. Finally, Section 6 concludes the paper.

2. The BVP with a singularity of the first kind

The problem we consider is singular and formulated as a two-point BVP, where correct boundary conditions can result in a well-posed BVP whose lineralization has a unique continuous solution. This property is crucial when solving a nonlinear BVP numerically using the Newton method. So the following question arises: What boundary conditions may be derived to guarantee that the linearization of the problem (i.e. (2) and (3)) is to have a unique continuous solution on the interval [0, 1]?

Before we do such an analysis, the notation $C_q^p[0,1]$ is introduced. It is the space whose elements have the form

$$\mathbf{x}(t) = (\mathbf{x}_1(t),\ \mathbf{x}_2(t), \cdots,\ \mathbf{x}_q(t))^T, \quad 0 \le t \le 1, \tag{4}$$

where $\mathbf{x}_i(t),\ i = 1, 2, \cdots, q$, are p times continuously differentiable functions on $[0, 1]$, and $C_q^p(0, 1]$ is defined in a similar way. For simplicity, we will delete the subscripts q in the subsequent analysis and simply call $C[0, 1] = C^0[0, 1],\ C(0, 1] = C^0(0, 1]$.

In general, for a BVP with a singularity of the first kind, its standard form is formulated as:

$$\mathbf{y}' = S\mathbf{y}/\eta + \mathbf{g}(\eta, \mathbf{y}), \quad 0 < \eta \le 1, \quad \mathbf{y} \in C[0, 1] \cap C^1(0, 1], \tag{5}$$

$$\mathbf{b}(\mathbf{y}(0), \mathbf{y}(1)) = 0. \tag{6}$$

Here, $\mathbf{y}$ and $\mathbf{g}$ are vector-valued functions of dimension n, $\mathbf{b}$ is a vector-valued function of dimension $m \le n$. S is a constant $n \times n$ matrix. To write (2) and (3) into the standard form (5) and (6), we introduce the following specific new variables:

$$y_1 = \frac{f}{\eta}, \quad y_2 = f', \quad y_3 = f'', \quad y_4 = \eta f''', \tag{7}$$

and collect them as a vector variable $\mathbf{y} = (y_1 \quad y_2 \quad y_3 \quad y_4)^T$. Note that $f(0) = 0$, thus y_1 is well defined. Where

$$S = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{8}$$

and

$$\mathbf{g}(\eta, \mathbf{y}) = (0 \quad y_3 \quad 0 \quad -\frac{\alpha}{2}(y_4 + 2y_3) - \frac{Re}{2}(y_1 y_4 - y_2 y_3))^T. \tag{9}$$

In fact, the problem (5) and (6) has been investigated by de Hoog and Weiss [16]. They have not only developed a canonical form for such BVPs, but also established a Fredholm theory for linear problems in this canonical form. Further, its analysis on more general problems can be found in the literature [12], and these problems cover very general nonlinear systems. Therefore, we will follow their line in the subsequent analysis. It is assumed that the nonlinear two-point BVP (i.e. (2) and (3)) has an isolated solution $\mathbf{y}(\eta)$. This means that the linearized problem

$$\phi'(\eta) = S\phi(\eta)/\eta + \mathbf{A}(\eta)\phi(\eta), \qquad 0 < \eta \leq 1, \tag{10}$$

$$\mathbf{B}_0\phi(0) + \mathbf{B}_1\phi(1) = 0, \tag{11}$$

where

$$\mathbf{A}(\eta) = \frac{\partial \mathbf{g}(\eta, \mathbf{y}(\eta))}{\partial \mathbf{y}}; \qquad \mathbf{B}_i = \frac{\partial \mathbf{b}(\mathbf{y}(0), \mathbf{y}(1))}{\partial \mathbf{y}(i)}, \qquad i = 0, 1; \tag{12}$$

has only the trivial solution. We note that $\frac{\partial \mathbf{g}}{\partial \mathbf{y}}$ is smooth, which indicates that the eigenvalues of S play a major role in posing the correct boundary conditions [16]. Before deriving the correct boundary conditions, we introduce some notations that will be used in the subsequent analysis.

Let X_0 and X_+ be the eigenspaces of S corresponding to the eigenvalue zero and the eigenvalues with positive real part, respectively, R and M be the projection matrices onto X_0 and X_+, respectively, and define

$$Q = I - R - M, \tag{13}$$

where I is an identity matrix. According to lemmas 3.6, 3.7 and theorem 3.1 in [16], the correct boundary conditions which are necessary and sufficient for the problem (10) and (11) to have a unique solution $\phi \equiv 0$ are $Q\phi(0) = 0$, $M\phi(0) = 0$ and rank$[\mathbf{B_0}R, \mathbf{B_1}]$=2. Here the projection matrices Q, M and R are as follows:

$$Q = \begin{pmatrix} \frac{1}{2} & -\frac{1}{2} & 0 & 0 \\ -\frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad M = \mathbf{0}, \qquad R = \begin{pmatrix} \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{14}$$

It is easily verified that the additional boundary conditions $Q\phi(0) = 0$ and $M\phi(0) = 0$ required for the correct boundary conditions are

$$y_1(0) - y_2(0) = 0, \qquad y_4(0) = 0. \tag{15}$$

On the other hand, according to (12), we can easily obtain

$$\mathbf{B_0} = \begin{pmatrix} 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \mathbf{B_1} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \tag{16}$$

which verifies rank$[\mathbf{B_0}R, \mathbf{B_1}]$=2.

As an aside, since the problem is a result of the actual model in a cylindrical pipe, on physical grounds the flow variables at the center of the pipe should be smooth (even if the BVP (2) and (3) has the singularity at $\eta = 0$ due to the cylindrical symmetry [9]). Thus, we assume that the solutions of the BVP (2) and (3) are well-behaved or analytic at the origin (i.e. $\eta = 0$). Moreover, the condition $S\mathbf{y}(0) = 0$ is necessary for $\mathbf{y}$ to have a bounded limit for $\eta \to 0$. In fact, it is not difficult to calculate

$$S\mathbf{y}(0) = \begin{pmatrix} -1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}_{\eta=0} = \begin{pmatrix} y_2 - y_1 \\ 0 \\ y_4 \\ -y_4 \end{pmatrix}_{\eta=0} = \begin{pmatrix} f' - \frac{1}{\eta} f \\ 0 \\ \eta f''' \\ -\eta f''' \end{pmatrix}_{\eta=0}. \tag{17}$$

Since we look only for analytic solutions as mentioned earlier, we may have $\eta f'''(\eta) = 0$ at $\eta = 0$ (i.e. $y_4(0) = 0$) and additionally from $f(0) = 0$ and L'Hopital's rule we have $f'(0) = \lim_{\eta \to 0} \dfrac{f(\eta)}{\eta}$. So we do have $S\mathbf{y}(0) = 0$ and the right hand side of the ODE system (5) is well defined at $\eta = 0$ and can be evaluated without any problem. On the other hand, the unconventional boundary condition (i.e. the last condition of (3)) is automatically satisfied due to the smoothness of the solution at $\eta = 0$. The condition

$$y_4(0) = 0 \tag{18}$$

is also automatically satisfied and arises naturally from $S\mathbf{y}(0) = 0$ or (15). Therefore, we will replace the unconventional boundary condition by (18). $y_1(0) = y_2(0)$ from (15) is equivalent to the condition $f(0) = 0$ since $f(0) = f(\eta) - \eta f'(\eta) + \frac{\eta^2}{2} f''(\theta\eta)$, $0 < \theta < 1$. Thus a correct set of boundary conditions may be as follows:

$$y_1(0) = y_2(0), \qquad y_1(1) = 1, \qquad y_2(1) = 0, \qquad y_4(0) = 0. \tag{19}$$

Next, we obtain the smoothness result for the solution of the problem (2) and (3). For the reader's convenience, we first state a theorem in [16] for the BVP (5) and (6):

Let $\mathbf{g}(\eta, \mathbf{y}(\eta)) \in C^p[T_\rho]$, where $T_\rho = \{(\eta, \mathbf{x}) | 0 \leq \eta \leq 1, \mathbf{x} \in S_\rho(\mathbf{y}(\eta))\}$ and $S_\rho(\mathbf{y}(\eta)) = \{\mathbf{x} | \|\mathbf{y} - \mathbf{x}\| \leq \rho, \rho > 0\}$, $\quad p \geq 0$. Then

(i) $\mathbf{y} \in C^{p+1}(0, 1]$.

(ii) $\mathbf{y} \in C^{p+1}[0, 1]$ if all eigenvalues of S have nonpositive real parts.

Note that $\mathbf{g}(\eta, \mathbf{y}(\eta))$, given by (9), satisfies the condition of the theorem above, and that all eigenvalues of S in (8) have nonpositive real parts. A straightforward application of the theorem above yields that the solution $\mathbf{y}$ for the problem (2) and (3) has

$$\mathbf{y} \in C^p[0,1], \quad p \geq 0. \tag{20}$$

3. The computational technique

The main aim of this section is to present an idea on finding multiple solutions and to solve a relevant singular IVP (see (26) and (27) in Section 3.2). In order to state them more clearly, the contents on them will be separated in sections 3.1 and 3.2, respectively.

3.1. The technique for finding multiple solutions

We now explain how we find multiple solutions of the singular BVP (5), (7)-(9) and (19). From the earlier analysis the solver bvp4c in MATLAB can be naturally applied to solve the BVP (5), (7)-(9) and (19). However, in bvp4c, an initial guess of the solution need be provided (this is also crucial in finding multiple solutions). According to our computational experience for this singular BVP (5) and (19), the solution could be very sensitive to the initial guess of the solution. The numerical continuation technique will always be crucial in obtaining a good initial guess of the solution. For example, we will discretise the Reynolds number to a number of equally spaced discrete points (e.g. Re = -3, -2, -1, 0, 1, 2, 3). Once we obtain solutions at one discrete point of the Reynolds number (e.g. $Re = 1$), each of these solutions will be used as the initial guess of the solution of the BVP at the next neighbour discrete point of the Reynolds number (e.g. $Re = 0$ or 2). Similarly, we can apply numerical continuation for the expansion ratio as well.

As we see above the numerical continuation will start from an obtained solution at certain Reynolds number. So we need to find a solution at a Reynolds number first. Also we cannot guarantee that the numerical continuation would not stop at some discrete point of the Reynolds number* (see also Fig.1 for $\alpha = 0$). So we sometimes need to restart the continuation process from another solution at another Reynolds number. To achieve this, let $F = Ref$, and Eq.(1) becomes

$$8\eta F''' + 4FF'' + 8F'' + \alpha(4\eta F'' + 4F') - 4F'^2 = K, \tag{21}$$

where $K = 8Rek$. The corresponding (3) (or equivalently (19) since the solution is smooth at $\eta = 0$) becomes

$$F'(1) = 0, \quad F(1) = Re, \quad F(0) = 0, \quad \lim_{\eta\to 0} \eta^{\frac{1}{2}} F''(\eta) = 0. \tag{22}$$

Then Eq.(21) will be the formulation to make use of an initial value method (to be explained in Section 3.2) for this high order ODE problem. We will use the following initial conditions:

$$F(0) = 0, \quad F'(0) = A, \quad F''(0) = B. \tag{23}$$

Then let $\eta = 0$ we have

$$K = 8B + 4\alpha A - 4A^2. \tag{24}$$

We will solve (21) and (23) for different A and B until finding a solution to satisfy the first condition

*In fact the earlier studies for the case of $\alpha = 0$ revealed that there possibly be no solution for certain interval of Re. We expect that the numerical continuation will stop at the end points of the no-solution interval.

of (22), i.e. $F'(1) = 0$ and then obtaining corresponding Reynolds number $Re = F(1)$ (here the scheme of updating A and B is based on one step of the Newton-Raphson method). The last condition of (22) is automatically satisfied for any given values of A and B due to the smoothness of the solution (see **Remark 1** in Section 3.2). Thus, we can find a solution of (21) satisfying (22) for the corresponding Re. Similar ideas have been seen in literature [10-11] for different boundary conditions. But there they test all possible values of A and B until obtaining solutions for entire range of Re. We only use the idea to find one or a few solutions and their corresponding Re to start the BVP method and the numerical continuation.

3.2. The singular IVP

To achieve the above idea, Eq.(21) is written into a first order system by introducing new variables:

$$\mathbf{z} = (z_1, z_2, z_3)^T, \quad \text{where } z_1 = F,\ z_2 = F',\ z_3 = F'', \tag{25}$$

and the Eq.(21) becomes

$$\begin{pmatrix} z_1' \\ z_2' \\ z_3' \end{pmatrix} = \begin{pmatrix} z_2 \\ z_3 \\ \frac{w(\eta,\mathbf{z})}{\eta} \end{pmatrix}, \tag{26}$$

where $w(\eta, \mathbf{z}) = \frac{K}{8} + \frac{1}{2}z_2^2 - \frac{1}{2}z_1 z_3 - z_3 - \frac{\alpha}{2}(\eta z_3 + z_2)$. The corresponding initial conditions become

$$z_1(0) = 0, \quad z_2(0) = A, \quad z_3(0) = B. \tag{27}$$

Here, we denote $\mathbf{z}_0 = (0, A, B)$. In fact, (26) and (27) obtained from (21) and (22) are formed as a singular IVP, where the singularity occurs at $\eta = 0$ (see $\frac{w(\eta,\mathbf{z})}{\eta}$ in (26)). It is necessary to examine whether the solution near the singular point is smooth for the singular IVP.

Before proceeding, we first introduce a result in [27]: For a nonlinear system of singular ODEs of the form

$$\eta \mathbf{u}'(\eta) = \mathbf{r}(\eta, \mathbf{u}(\eta)), \quad 0 < \eta \le T_1, \tag{28}$$

where $\mathbf{r}$: $[0, T_1] \times \wp \mapsto \mathbf{R}^n$ is a vector function, $\frac{\partial \mathbf{r}}{\partial \mathbf{u}} : [0, T_1] \times \wp \mapsto \mathbf{R}^{n \times n}$ is a matrix function, $\wp \subset \mathbf{R}^n$ is an open domain. The following theorem was obtained in [27].

Theorem 1. Assume that $\mathbf{r} \in C^m([0, T_1] \times \wp)$, $\frac{\partial \mathbf{r}}{\partial \mathbf{u}} \in C^m([0, T_1] \times \wp)$, and that equation $\mathbf{r}(0, \mathbf{v}) = \mathbf{0}$ has a solution $\mathbf{v}_0 \in \wp$. Then system (28) has for a sufficiently small $T_2 \in (0, T_1]$ a unique solution $\mathbf{u} \in C^m[0, T_2]$ such that $\mathbf{u}(0) = \mathbf{v}_0$, where

$$m \ge \max_{\lambda_j \in \delta(A_0)} Re\lambda_j, \quad A_0 = \frac{\partial \mathbf{r}(0, \mathbf{u})}{\partial \mathbf{u}}|_{\mathbf{u}=\mathbf{v}_0}, \tag{29}$$

and $\delta(A_0)$ is the set of eigenvalues of the matrix A_0.

The system of first order ODEs (26) is written into the following form

$$\eta \mathbf{z}'(\eta) = \tilde{w}(\eta, \mathbf{z}) = (\eta z_2, \eta z_3, w(\eta, \mathbf{z}))^T. \tag{30}$$

Since $\mathbf{z}_0$ is always satisfied with the system $\tilde{w}(0, \mathbf{z}_0) = 0$ for any values of A and B, and $\tilde{w}(\eta, \mathbf{z}) \in C^p([0, 1] \times \wp)$, $\frac{\partial \tilde{w}(\eta,\mathbf{z})}{\partial \mathbf{z}} \in C^p([0, 1] \times \wp)$ for a $p \ge 0$. According to the above theorem 1, the solvability of system (30) in $C^p[0,T_2]$ can be guaranteed. Note that the Jacobi matrix $A_0 = \frac{\partial \tilde{w}(0,\mathbf{z})}{\partial \mathbf{z}}|_{\mathbf{z}=\mathbf{z}_0}$ is

$$A_0 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\frac{1}{2}B & A-\frac{\alpha}{2} & -1 \end{pmatrix}, \tag{31}$$

and the eigenvalues of A_0 are 0 and -1, so $p \geq 0$ satisfies (29).

Remark 1. The fact that eigenvalues of A_0 are independent of α, A and B, indicates that the request for the smoothness of $\tilde{w}(\eta,\mathbf{z})$ is also independent of these parameters. In other words, for any given values of A and B, the resulting solution $\mathbf{z}$ near the singular point is always smooth (or analytical) since the right hand side function $\tilde{w}$ is of polynomial and thus always smooth.

We can now evaluate the right hand side of (26) at the singular point $\eta = 0$. According to the result of previous analysis (see **Remark 1**), we have known that the solution near the singular point is analytical. Therefore, an analytical approximation at the singular point is considered, namely, if we let $\eta \to 0$ in (26), we find that

$$\begin{pmatrix} z_1'(0) \\ z_2'(0) \\ z_3'(0) \end{pmatrix} = \begin{pmatrix} z_2(0) \\ z_3(0) \\ \hat{w}(0,\mathbf{z}(0)) \end{pmatrix}, \tag{32}$$

where $\hat{w}(0,\mathbf{z}(0)) = \lim_{\eta\to 0} \dfrac{w(\eta,\mathbf{z})}{\eta}$. According to L'Hopital's rule and analytical property at $\eta = 0$, we can further improve

$$\begin{aligned} \hat{w}(0,\mathbf{z}(0)) &= \lim_{\eta\to 0} \frac{w(\eta,\mathbf{z})}{\eta} \\ &= w'(0,\mathbf{z}(0)) \\ &= \frac{1}{2}(\frac{1}{2}z_2(0) - \alpha)z_3(0) \\ &= \frac{1}{2}(\frac{1}{2}A - \alpha)B. \end{aligned} \tag{33}$$

So the right hand side function of (26) is evaluated by (32) and (33) at the singular point. Finally, multiple solutions obtained by the technique in this section will be presented and interpreted in the next section.

4. Multiple solutions

The main aim of this section is to present multiple solutions, namely, the BVP (i.e. (5), (7)-(9) and (19)) and the IVP (i.e. (26)-(27)) are solved. For the BVP, so far there exist many freely available softwares to solve it. These softwares include Matlab codes bvp4c [29] and bvp5c [30], sbvp [31], bvpsuite [32-33], and Fortran codes, BVP-solver specified in [34] and COLNEW described in [35] and COLSYS [36]. In this paper we simply use bvp4c given in Matlab to solve the BVP. When concerning bvp4c solver, it is requested in the code to provide a guess for the solution desired. Further, a sufficiently good guess is very important for obtaining a good numerical solution. To provide a good guess, the numerical continuation technique is used (see Section 3.1 for more detail on it). As for the IVP, here we simply use the solver ode45 given in Matlab. In addition, unless stated otherwise, for all our computations, we use the relative error tolerance 10^{-3} and the absolute error tolerance 10^{-6}.

Next, our aim is to present multiple solutions of the BVP. Before presenting multiple solutions, the quantity $[-f''(1)]$ is introduced. It is proportional to the wall skin friction [10] and is usually used to show multiple solutions [7, 10-11]. Here we present multiple solutions in the same way, that is, plotting $[-f''(1)]$ against Re. In Fig.1, multiple solutions are presented for some given values of expansion ratio taken over a full range of cross-flow Reynolds number. Let us briefly explain how the computational technique presented in Section 3 is used to obtain multiple solutions in Fig.1. Firstly, we start from $\alpha = 0$ since at this α multiple solutions information is mostly known from literature [10-11], where it is also shown that Sec.I contains all the well behaved solutions and no solution exists in the range $-9.1 < Re < -2.3$. So we start from a relatively large injection Reynolds number (e.g. $Re = 50$). According to our computational experience, in this case, the solution can be easily obtained by bvp4c for an initial guess (e.g. $[0, 0, 0, 0]$). Then, the numerical continuation is used to obtain the solution at the next neighbour Reynolds number (e.g. $Re = 49$) until near $Re = -2.3$. At this juncture, we begin to find Sec.II solution instead of Sec.I solution. However, we find that Sec.II solution can not be easily obtained by bvp4c for arbitrarily chosen initial guesses. Under this situation IVP method given in Section 3.2 is used to provide an initial guess of the solution for bvp4c. When a solution at certain Reynolds number (denoted as $\tilde{Re}$) is obtained, the numerical continuation is again used to find the solution near $\tilde{Re}$ until near $Re = -2.3$. Finally, the continuous deformation of the velocity profiles (i.e. as $Re \to -2.3$ from above and below the limiting profiles are identical) is used to stop the numerical continuation. Next, the computational process above is extended to find other solutions and multiple solutions corresponding to $\alpha = -2, 2$.

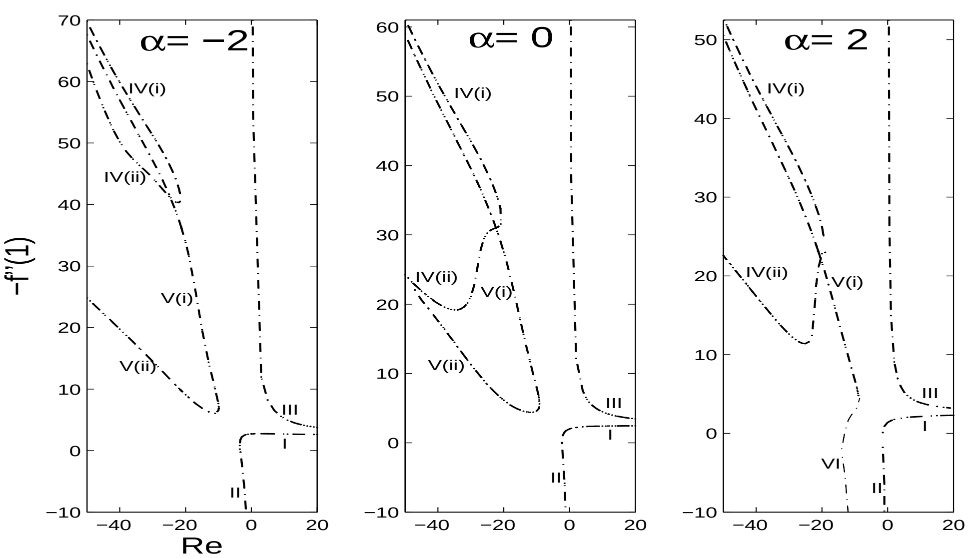

Figure 1: Branches of $-f''(1)$ vs Re at $\alpha = -2, 0, 2$.

As observed in Fig.1, we note that the numerical results at $\alpha = 0$ are the same as what Terrill and Thomas [10] or Shankararamann and Liu [11] obtained. In a sense, this may illustrate the reliability of our numerical technique. On the other hand, the well-known bifurcation software package AUTO [37] is used to further validate our computational results. In order to use AUTO, the BVP (i.e. (5), (7)-(9) and (19)) is transited into an autonomous system by $y_5 = \eta$, and the condition $y_5(0) = 0$ (or $y_5(1) = 1$) is added to (19). Then, the constant H is defined to measure the bifurcation, i.e.

$$H = \sqrt{\int_0^1 \sum_{i=1}^{5} y_i^2(\eta)\, d\eta}. \tag{34}$$

From the respect of bifurcation analysis, it is in fact a process of seeking fold point (or limit point (LP)). Due to the high sensitivity to initial guess and the difficulty in changing the tangential direction for the problem with singularity, we have difficulty to obtain all limit points using AUTO, but we still managed to obtain a few, which are consistent to the results obtained by our method. The results are listed in the following figures and table 1. In Fig.2 and Fig.3a, the computed results on LP are showed (Noting that a black dot represents a LP). Further, Table 1 presents the comparison of the computation on LP for different methods. Obviously, these results are found to be in very well agreement, which indicates that our numerical technique is solid and effective in computing LP. In Fig.4a, we once again show multiple solutions in the range $-13.33 < Re < -8.8$ for the sake of clarity of explanation. With AUTO, we start from near $Re = -12$, and the bifurcation results are showed in Fig.4b. During the calculation of the bifurcation, we not only find that the point F is a bifurcation point, but also detect that the point E is both a bifurcation point and limit point (LP). These results illustrate the correctness of multiple solutions presented in Fig.4a. To be specific, the point E is a bifurcation point, which indicates the existence of multiple solutions (see the points C and D in Fig.4a). On the other hand, the point E is a LP, which indicates the change of the solution near the point D. In other words, Sec.V(ii) solution will not exist near $Re = -40$, or there are only three solutions near $Re = -40$. The sharp decrease of

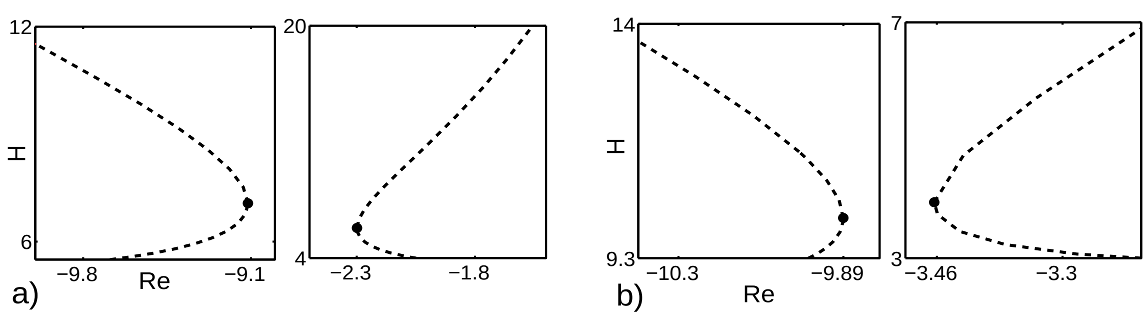

Figure 2: Bifurcation diagrams for H at a) $\alpha = 0$, and b) $\alpha = -2$.

the value of $-f''(1)$ corresponding to Sec.VI solution* in Fig.4a agrees well with the solid line in Fig.4b due to $H \propto |-f''(1)|$ (see (34)). In Fig.4, the bifurcation point F corresponds to the points A and B, and the closed curve formed by the points E and F corresponds to the curve $ABCD$. Since the Sec.VI solution is found for the first time, Fig.3b shows the behavior of the axial velocity profiles $[f'(\eta)]$ (its physical meaning can be found in [10]) for different cross-flow Reynolds numbers. As for axial velocity profiles corresponding to other solutions (e.g. solution Sec.I or solution Sec.II in Fig.1), the reader can find them in literature [10-11], where the characteristic features of those solutions are summarized.

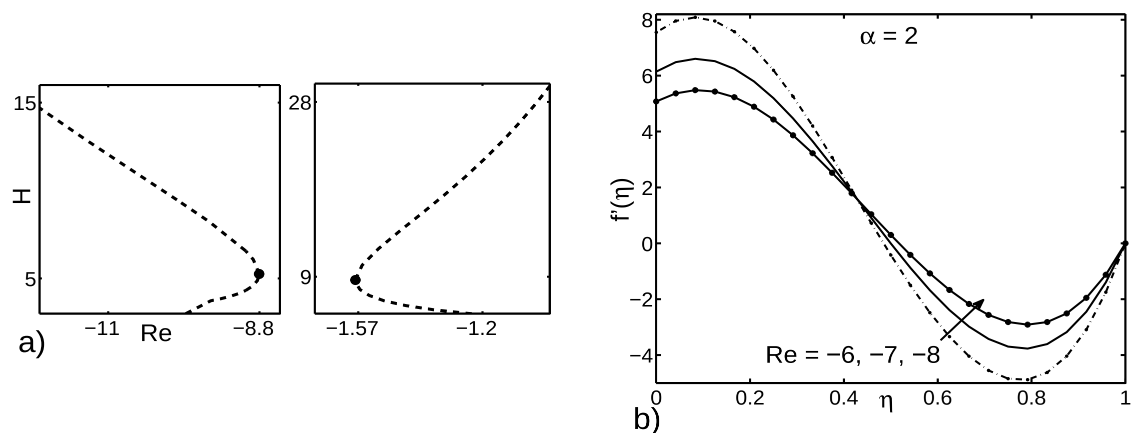

Figure 3: Bifurcation diagrams (left) and solution Sec.VI (right) for $\alpha = 2$.

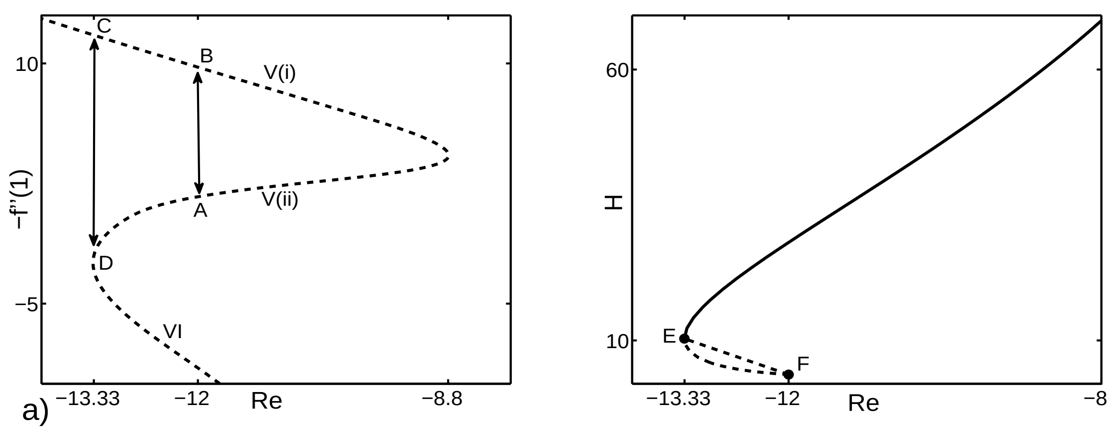

Figure 4: Multiple solutions (left) and bifurcation diagram (right) for $\alpha = 2$.

*In Fig.1, values of $-f''(1)$ for Sec.VI have not been completely included for the sake of clarity of presentation because they are much high in magnitude compared to Sec.II. For example, the value of $-f''(1)$ is -106.8324 as $Re = -3$.

Table 1: The comparison of the computation on LP for different methods

α	Our numerical technique	AUTO	Terrill and Thomas [10]
-2	(-9.8872, -3.4711)	(-9.8901, -3.4634)	—
0	(-9.1147, -2.2997)	(-9.1125, -2.2990)	(-9.1, -2.3)
2	(-8.8493, -1.5764)	(-8.8033, -1.5785)	—

5. Asymptotic solutions as a validation of our numerical technique

In this section we construct asymptotic solutions for certain range of Reynolds number Re and expansion ratio α. However, our goal is not to construct all possible asymptotic solutions in each category of parameters, but to obtain some of these solutions from another perspective in order to partially validate numerical solutions we obtained in the previous section.

5.1 Solution for large injection Reynolds numbers

The asymptotic solution of Eq.(1) for the large injection Reynolds number can be obtained by the Lighthill method. As indicated in [38], this is the available method which can construct a sufficiently smooth asymptotic solution for all $\eta \in [0, 1]$. Eq.(1) can be written as

$$\varepsilon(\eta f''' + f'') + \varepsilon\frac{\alpha}{2}(\eta f'' + f') + ff'' - f'^2 = \lambda, \tag{35}$$

where

$$\varepsilon = \frac{2}{Re}, \quad \lambda = \frac{2k}{Re}. \tag{36}$$

By introducing a new variable ξ, let

$$\eta = \xi + \varepsilon X_1(\xi) + o(\varepsilon). \tag{37}$$

$X_1(\xi)$ is unknown to be determined next. Assuming the expansion of the solution to be

$$f(\eta) = g(\xi) = \sum_{i=0}^{\infty} g_i(\xi)\varepsilon^i, \ \lambda = \sum_{i=0}^{\infty} \lambda_i \varepsilon^i. \tag{38}$$

Substituting Eq.(38) into Eq.(35), and equating coefficients of $\varepsilon^i (i = 0, 1, 2, ...)$ yields the equations

$$g_0\ddot{g}_0 - (\dot{g}_0)^2 = \lambda_0, \tag{39}$$

$$g_0\ddot{g}_1 - 2g_0\dot{g}_1 + \ddot{g}_0 g_1 = 2\lambda_0\dot{X}_1(\xi) + \lambda_1 - [\xi\dddot{g}_0 + \ddot{g}_0 + \frac{\alpha}{2}(\xi\ddot{g}_0 + \dot{g}_0)], \tag{40}$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots,$$

where $\cdot$ denotes the derivative with respect to ξ. Assuming $\bar{\xi}$ is the root of Eq.(37) when $\eta = 1$, namely

$$1 = \bar{\xi} + \varepsilon X_1(\bar{\xi}) + \cdots. \tag{41}$$

From Eq.(41) we can obtain

$$\bar{\xi} = 1 - \varepsilon X_1(1) + \varepsilon^2[X_1(1)\dot{X}_1(1) - X_2(1)] + \cdots. \tag{42}$$

The first two conditions of Eq.(3) become

$$g_0(1) = 1, \quad g_1(1) = \dot{g}_0(1)X_1(1), \tag{43}$$

$$\dot{g}_0(1) = 0, \quad \dot{g}_1(1) = X_1(1)\ddot{g}_0(1). \tag{44}$$

Similar to the above process, when $\eta = 0$, the third condition of Eq.(3) becomes

$$g_0(0) = 0, \quad g_1(0) = X_1(0)\dot{g}_0(0). \tag{45}$$

Combining Eqs.(39) and (40) with Eqs.(43), (44) and (45), the results are as follow:

$$\lambda_0 = -\frac{\pi^2}{4}, \quad \lambda_1 = -\frac{\pi^2}{4} + \frac{\pi}{2} - 1, \tag{46}$$

$$X_1(\frac{\pi}{2}\xi) = \frac{\xi}{2}\sin(\frac{\pi}{2}\xi) - (1+\frac{\alpha}{2})\frac{\xi}{\pi}\cos(\frac{\pi}{2}\xi) + \frac{2}{\pi^2}\sin(\frac{\pi}{2}\xi) + \frac{1}{\pi}\cos(\frac{\pi}{2}\xi) + (\frac{1}{\pi} - \frac{2}{\pi^2} - \frac{1}{2})\xi, \tag{47}$$

$$g_0(\xi) = \sin(\frac{\pi}{2}\xi),\ g_1(\xi) = \frac{1}{2}\cos(\frac{\pi}{2}\xi). \tag{48}$$

So $f(\eta) = g(\xi) = g_0(\xi) + \varepsilon g_1(\xi) + \cdots$ and an asymptotic approximation of $f''(1)$ can be computed. The numerical and asymptotic values of $-f''(1)$ are listed in Table 2 with expansion ratios $\alpha = 0$ and ± 2. The results agree well.

Table 2: The numerical and asymptotic values of $-f''(1)$ for Section I

	$\alpha = 0$		$\alpha = -2$		$\alpha = 2$	
Re	Numerical	Asymptotic	Numerical	Asymptotic	Numerical	Asymptotic
60	2.46836251	2.51726594	2.53323720	2.46493193	2.40705480	2.56959995
70	2.46878514	2.50978489	2.52479941	2.46492725	2.41547402	2.55464253
80	2.46899697	2.50417413	2.51825890	2.46492375	2.42184869	2.54342451
90	2.46908870	2.49981021	2.51305787	2.46491028	2.42683540	2.53469940
100	2.46913215	2.49631908	2.50881277	2.46491884	2.43083795	2.52771932

5.2 Solution for small Reynolds number and small expansion ratio

The asymptotic solution for small Reynolds number *Re* and small expansion ratio α can be obtained by a regular perturbation method, which is corresponding to small suction and injection for Solution Sec.I.

Let $\varepsilon = \frac{Re}{2}$ be a small perturbation parameter. Assuming the solution of Eq.(2) to be

$$f = f_0(\eta) + \varepsilon f_1(\eta) + O(\varepsilon^2), \tag{49}$$

substituting Eq.(49) into Eq.(2) and equating the coefficient of like powers of ε on both sides, we get the leading order and the first order equations as follows:

$$\eta f_0'''' + 2f_0''' + \frac{\alpha}{2}(\eta f_0''' + 2f_0'') = 0, \tag{50}$$

$$\eta f_1'''' + 2f_1''' + \frac{\alpha}{2}(\eta f_1''' + 2f_1'') + f_0 f_0''' - f_0' f_0'' = 0. \tag{51}$$

The corresponding boundary conditions of Eqs.(50) and (51) are

$$f_0(1)=1,\quad f_0'(1)=0,\quad f_0(0)=0,\quad \lim_{\eta\to 0}\eta^{\frac{1}{2}}f_0''(\eta)=0, \tag{52}$$

and

$$f_1(1)=0,\quad f_1'(1)=0,\quad f_1(0)=0,\quad \lim_{\eta\to 0}\eta^{\frac{1}{2}}f_1''(\eta)=0, \tag{53}$$

respectively. Secondly, because α is also small, we can use α as a secondary parameter and expand f_0, f_1 in the following forms.

$$f_i=f_{i0}+\alpha f_{i1}+O(\alpha^2),\ i=0,1. \tag{54}$$

Substituting (54) into Eq.(50), we can obtain the equations of the leading and first order in α

$$\eta f_{00}''''+2f_{00}'''=0, \tag{55}$$

$$f_{00}(1)=1,\quad f_{00}'(1)=0,\quad f_{00}(0)=0,\quad \lim_{\eta\to 0}\eta^{\frac{1}{2}}f_{00}''(\eta)=0, \tag{56}$$

and

$$\eta f_{01}''''+2f_{01}'''+\frac{1}{2}(\eta f_{00}'''+2f_{00}'')=0, \tag{57}$$

$$f_{01}(1)=0,\quad f_{01}'(1)=0,\quad f_{01}(0)=0,\quad \lim_{\eta\to 0}\eta^{\frac{1}{2}}f_{01}''(\eta)=0. \tag{58}$$

The solutions of f_{00}, f_{01} are

$$f_{00}=-\eta^2+2\eta, \tag{59}$$

$$f_{01}=\frac{1}{6}\eta^3-\frac{1}{3}\eta^2+\frac{1}{6}\eta. \tag{60}$$

In the similar process, using α as the secondary parameter , substituting (54) into Eq.(51), and collecting terms of the same order in α, we can obtain

$$\eta f_{10}''''+2f_{10}'''+f_{00}f_{00}'''-f_{00}'f_{00}''=0, \tag{61}$$

$$f_{10}(1)=0,\quad f_{10}'(1)=0,\quad f_{10}(0)=0,\quad \lim_{\eta\to 0}\eta^{\frac{1}{2}}f_{10}''(\eta)=0, \tag{62}$$

and

$$\eta f_{11}''''+2f_{11}'''+\frac{1}{2}(\eta f_{10}'''+2f_{10}'')+f_{00}f_{01}'''+f_{01}f_{00}'''-f_{00}'f_{01}''-f_{01}'f_{00}''=0, \tag{63}$$

$$f_{11}(1)=0,\quad f_{11}'(1)=0,\quad f_{11}(0)=0,\quad \lim_{\eta\to 0}\eta^{\frac{1}{2}}f_{11}''(\eta)=0, \tag{64}$$

where $f_0=f_{00}+\alpha f_{01}$. The solutions of (61) and (62) and (63) and (64) can be obtained.

$$f_{10}=\frac{1}{18}\eta^4-\frac{1}{3}\eta^3+\frac{1}{2}\eta^2-\frac{2}{9}\eta, \tag{65}$$

$$f_{11}=-\frac{1}{72}\eta^5+\frac{17}{216}\eta^4-\frac{2}{9}\eta^3+\frac{19}{72}\eta^2-\frac{23}{216}\eta. \tag{66}$$

Substituting $f_{00}, f_{01}, f_{10}, f_{11}$ into $f=f_{00}+\alpha f_{01}+\varepsilon(f_{10}+\alpha f_{11})+O(\varepsilon^2)$, we can obtain the

Table 3: The numerical and asymptotic values of $-f''(1)$ for section I

	$\alpha = 0$		$\alpha = -0.05$		$\alpha = 0.05$	
Re	Numerical	Asymptotic	Numerical	Asymptotic	Numerical	Asymptotic
0.592	2.0843	2.0987	2.0992	2.1133	2.0694	2.0841
0.297	2.0456	2.0495	2.0614	2.0651	2.0299	2.0339
0.041	2.0068	2.0068	2.0233	2.0234	1.9903	1.9903
-0.052	1.9912	1.9913	2.0081	2.0082	1.9744	1.9745
-0.141	1.9755	1.9765	1.9927	1.9937	1.9584	1.9593
-0.422	1.9199	1.9297	1.9383	1.9478	1.9015	1.9115

expression of f as the Reynolds number and expansion ratio are both small. The numerical and asymptotic values of $-f''(1)$ are compared for some values of α and Re in Table 3. The smaller the Reynolds number Re and expansion ratio α, the closer the numerical and asymptotic solutions are, indicating that our numerical computations are reliable.

5.3 Solution for large suction Reynolds numbers

In general, there will be a boundary layer at the wall when there is large suction. The numerical results will confirm it. As the viscous terms become dominant, the perturbation solution of Eq.(1) for large suction valid outside the boundary layer would break down inside the layer and the inner solution should satisfy the conditions at the wall. In order to obtain the perturbation expansion of Eq.(1) corresponding to the large suction Reynolds number, Eq.(1) can be written as

$$\varepsilon(\eta f''' + f'') + \varepsilon\frac{\alpha}{2}(\eta f'' + f') + f'^2 - ff'' = \beta^2 + (\frac{\alpha}{2}\beta + \gamma)\varepsilon, \tag{67}$$

where

$$\varepsilon = -\frac{2}{Re}, \quad \beta^2 + (\frac{\alpha}{2}\beta + \gamma)\varepsilon = -\frac{2k}{Re}, \tag{68}$$

and

$$\beta = f'(0) = \beta_0 + \varepsilon\beta_1 + \varepsilon^2\beta_2 + \cdots, \gamma = f''(0) = \gamma_0 + \varepsilon\gamma_1 + \varepsilon^2\gamma_2 + \cdots. \tag{69}$$

From the Eq.(67), the reduced problem can be obtained.

$$f_0'^2 - f_0 f_0'' = \beta_0^2, \tag{70}$$

the corresponding boundary conditions are

$$f_0(0) = 0, \quad f_0(1) = 1, \quad f_0'(1) = 0, \quad \lim_{\eta \to 0} \eta^{\frac{1}{2}} f_0''(\eta) = 0. \tag{71}$$

One solution of Eq.(70) is

$$f_0 = \beta_0 \eta. \tag{72}$$

According to the condition $f(1) = 1$, one obtains

$$\beta_0 = 1, \quad \gamma_0 = f_0''(0) = 0. \tag{73}$$

Letting

$$f = \eta + \bar{f}, \tag{74}$$

and substituting it into Eq.(67) yields

$$\varepsilon(\eta \bar{f}''' + \bar{f}'') + \varepsilon\frac{\alpha}{2}(\eta \bar{f}'' + \bar{f}') + \bar{f}'^2 + 2\bar{f}' - (\bar{f}+\eta)\bar{f}'' = 2\beta_1\varepsilon + (\beta_1^2 + \frac{\alpha}{2}\beta_1 + 2\beta_2 + \gamma_1)\varepsilon^2 + \cdots . \tag{75}$$

The asymptotic solution of Eq.(75) may be written in the form (boundary layer correction method [39-41]):

$$\bar{f} = \varepsilon(f_1(\eta) + g_1(\tau)) + \varepsilon^2(f_2(\eta) + g_2(\tau)) + \varepsilon^3(f_3(\eta) + g_3(\tau)) + \cdots , \tag{76}$$

where $\tau = \frac{1-\eta}{\varepsilon}$ is the stretching transformation near the wall and $g_i(\tau)$, $i = 1, 2, \ldots$, are boundary layer functions (rapidly decay when η is away from the wall). The boundary conditions satisfied by $\bar{f}(\eta)$ are

$$\bar{f}(0) = 0, \quad \bar{f}(1) = 0, \quad \bar{f}'(1) = -1, \quad \lim_{\eta\to 0} \eta^{\frac{1}{2}} \bar{f}''(\eta) = 0. \tag{77}$$

Substituting (76) into Eq.(75) and equating coefficients of ε^r yields

$$\varepsilon : 2f_1' - \eta f_1'' = 2\beta_1, \tag{78}$$

$$\varepsilon^2 : 2f_2' - \eta f_2'' = \frac{\alpha}{2}\beta_1 + \gamma_1 + \beta_1^2 + 2\beta_2 - \eta f_1''' - f_1'' - \frac{\alpha}{2}\eta f_1'' - \frac{\alpha}{2}f_1' + f_1'' f_1 - f_1'^2, \tag{79}$$

$$\varepsilon^3 : 2f_3' - \eta f_3'' = \frac{\alpha}{2}\beta_2 + \gamma_2 + 2\beta_1\beta_2 + 2\beta_0\beta_3 - \eta f_2''' - f_2'' - \frac{\alpha}{2}\eta f_2'' - \frac{\alpha}{2} - f_2' - 2f_1' f_2' + f_1'' f_2 + f_2'' f_1, \tag{80}$$

$$\varepsilon^4 : 2f_4' - \eta f_4'' = \frac{\alpha}{2}\beta_3 + \gamma_3 + 2\beta_1\beta_3 + 2\beta_0\beta_4 + \beta_2^2 - \eta f_3''' - f_3'' - \frac{\alpha}{2}\eta f_3'' - \frac{\alpha}{2}f_3' - 2f_1' f_3' - f_2'^2 + f_1'' f_3 + f_3'' f_1 + f_2'' f_2, \tag{81}$$

$$\varepsilon^{-1} : \dddot{g}_1 + \ddot{g}_1 = 0, \tag{82}$$

$$\varepsilon^0 : \dddot{g}_2 + \ddot{g}_2 = \ddot{g}_1 + \tau\dddot{g}_1 + \frac{\alpha}{2}\ddot{g}_1 - 2\dot{g}_1 + \tau\ddot{g}_1 + \dot{g}_1^2 - \ddot{g}_1 f_1(1) - \ddot{g}_1 g_1, \tag{83}$$

$$\begin{aligned} \varepsilon : \dddot{g}_3 + \ddot{g}_3 = \quad & \ddot{g}_2 + \tau\dddot{g}_2 + \tfrac{\alpha}{2}(\ddot{g}_2 - \dot{g}_1 - \tau\ddot{g}_1) - 2\dot{g}_2 + \tau\ddot{g}_2 - 2\dot{g}_1 f_1'(1) \\ & + 2\dot{g}_1\dot{g}_2 - \ddot{g}_1(f_2(1) - \tau f_1'(1)) - \ddot{g}_1 g_2 - \ddot{g}_2 f_1(1) - \ddot{g}_2 g_1, \end{aligned} \tag{84}$$

$$\cdots\cdots$$

where $\cdot$ and $'$ denote the derivative with respect to τ and η, respectively, and we have used $f_1(\eta) = f_1(1-\varepsilon\tau) = f_1(1) - \varepsilon\tau f_1'(1) + \frac{1}{2}\varepsilon^2\tau^2 f_1''(1) + \cdots$ and $f_2(\eta) = f_2(1-\varepsilon\tau) = f_2(1) - \varepsilon\tau f_2'(1) + \cdots$. The boundary conditions to be satisfied by $f_i(\eta)$ and $g_i(\tau)$ at $\eta = 1$ or $\tau = 0$ are

$$f_1(0) = 0, \quad \varepsilon g_1'(\tau)|_{\eta=1} (= -\dot{g}_1(\tau)|_{\tau=0}) = -1, \quad f_1(\eta)|_{\eta=1} + g_1(\tau)|_{\tau=0} = 0, \tag{85}$$

$$f_i(0) = 0, \quad f_{i-1}'(\eta)|_{\eta=1} - \dot{g}_i(\tau)|_{\tau=0} = 0, \quad f_i(\eta)|_{\eta=1} + g_i(\tau)|_{\tau=0} = 0, (i = 2, 3, 4, \cdots). \tag{86}$$

The boundary layer solution of Eq.(82) is

$$g_1(\tau) = C_1 e^{-\tau}. \tag{87}$$

From the condition $\dot{g}_1(\tau)|_{\tau=0} = 1$, one obtain $C_1 = -1$. The solution of the Eq.(78) satisfying (85) is

$$f_1(\eta) = \beta_1\eta + (1 - \beta_1)\eta^3. \tag{88}$$

From (88), one can obtain $\gamma_1 = f_1''(0) = 0$. The parameter β_1 is still unknown to be determined next. Substituting (88) into Eq.(79) and noticing $f_2(0) = 0$ yields

$$f_2(\eta) = (1-\beta_1)\frac{3\alpha}{2}\eta^3 \ln\eta - \eta^5(-\frac{3\beta_1}{5} + \frac{3}{10} + \frac{3\beta_1^2}{10}) + \eta^3(\frac{A_2}{3} + \frac{\beta_1\alpha}{2} - \frac{\alpha}{2}) + 6\eta^2(\beta_1 - 1) + \beta_2\eta, \quad (89)$$

where A_2 is an integration constant. Since we look for the analytic solution we thus set $\beta_1 = 1$ (otherwise, $f_2'''(\eta) \to \infty$ as $\eta \to 0$). Hence,

$$f_2(\eta) = \frac{A_2}{3}\eta^3 + \beta_2\eta. \quad (90)$$

Then Eq.(83) becomes

$$\dddot{g}_2 + \ddot{g}_2 = \tau\dddot{g}_1 + \frac{\alpha}{2}\ddot{g}_1 - 2\dot{g}_1 + \tau\ddot{g}_1 + \dot{g}_1^2 - \ddot{g}_1 g_1. \quad (91)$$

Subject to the condition $f_1'(\eta)|_{\eta=1} - \dot{g}_2(\tau)|_{\tau=0} = 0$, the boundary layer solution of Eq.(91) is then

$$g_2 = -\frac{1}{2}(6 + \alpha + 4\tau + \tau\alpha)e^{-\tau}. \quad (92)$$

From the condition $f_2(\eta)|_{\eta=1} + g_2(\tau)|_{\tau=0} = 0$, one also obtains

$$f_2 = (3 + \frac{\alpha}{2} - \beta_2)\eta^3 + \beta_2\eta, \quad (93)$$

from (93),

$$\gamma_2 = f_2''(0) = 0. \quad (94)$$

The parameter β_2 will be determined later from the solution $f_3(\eta)$ of Eq.(80). We neglect tedious calculation and simply summarize the results below. The coefficients $\beta_2, \beta_3, \beta_4$ are:

$$\beta_2 = 3 + \frac{\alpha}{2}, \quad \beta_3 = 18 + 4\alpha + \frac{1}{4}\alpha^2, \quad \beta_4 = \frac{591}{4} + \frac{79}{2}\alpha + \frac{15}{4}\alpha^2 + \frac{1}{8}\alpha^3, \quad \gamma_3 = \gamma_4 = 0, \quad (95)$$

and $f_2, f_3, f_4, g_2, g_3, g_4$ are:

$$f_2 = (3 + \frac{\alpha}{2})\eta, \quad (96)$$

$$f_3 = (18 + 4\alpha + \frac{\alpha^2}{4})\eta, \quad (97)$$

$$f_4 = (\frac{591}{4} + \frac{15\alpha^2}{4} + \frac{\alpha^3}{8} + \frac{79\alpha}{2})\eta, \quad (98)$$

$$g_2 = -\frac{1}{2}(6 + \alpha + 4\tau + \tau\alpha)e^{-\tau}, \quad (99)$$

$$g_3 = -[\tau^2(\alpha + \frac{7}{2} + \frac{\alpha^2}{8}) + \tau(15 + \frac{7\alpha}{2} + \frac{\alpha^2}{4}) + 18 + 4\alpha + \frac{\alpha^2}{4}]e^{-\tau}, \quad (100)$$

$$\begin{aligned} g_4 = \; & \tfrac{3}{4}e^{-2\tau} - [(\tfrac{1}{4}\alpha^2 + \tfrac{1}{48}\alpha^3 + \tfrac{7}{4}\alpha + \tfrac{16}{3})\tau^3 + (\tfrac{11}{8}\alpha^2 + \tfrac{1}{16}\alpha^3 + 12\alpha + \tfrac{81}{2})\tau^2 \\ & +(\tfrac{7}{2}\alpha^2 + \tfrac{31}{8}\alpha^3 + \tfrac{71}{2}\alpha + 129)\tau + \tfrac{15}{4}\alpha^2 + \tfrac{1}{8}\alpha^3 + \tfrac{79}{2}\alpha + \tfrac{297}{2})]e^{-\tau}. \end{aligned} \quad (101)$$

Then the asymptotic solution for all $\eta \in [0,1]$ is

$$f(\eta) = \eta + \varepsilon(f_1(\eta) + g_1(\tau)) + \varepsilon^2(f_2(\eta) + g_2(\tau)) + \varepsilon^3(f_3(\eta) + g_3(\tau)) + \varepsilon^4(f_4(\eta) + g_4(\tau)) + \cdots. \quad (102)$$

Then we can calculate $f''(1)$ based on the asymptotic solution above:

$$\frac{d^2 f}{d\eta^2}|_{\eta=1} = \frac{Re}{2} + 1 + \frac{\alpha}{2} - \frac{10+2\alpha}{Re} + \frac{126 + 2\alpha^2 + 30\alpha}{Re^2}. \quad (103)$$

The values of (103) are compared with numerical results in Table 4, which shows that the smaller the Reynolds number Re, the closer the numerical and asymptotic values of $-f''(1)$ are.

Table 4: The numerical and asymptotic values of $-f''(1)$

	$\alpha = 0$		$\alpha = -2$		$\alpha = 2$	
Re	Numerical	Asymptotic	Numerical	Asymptotic	Numerical	Asymptotic
-30.978	11.9219	14.0369	15.0322	15.2182	13.0232	12.8349
-37.952	16.3119	17.6250	18.7029	18.7665	16.3830	16.4724
-41.993	18.8162	19.6869	20.7739	20.8117	18.4633	18.5531
-51.475	24.6565	24.4957	25.5782	25.5930	23.3437	23.3923
-72.439	36.1969	35.0574	36.1180	36.1226	33.9753	33.9893
-92.48	45.1126	45.1171	46.1634	46.1665	44.0590	44.0659
-103.05	50.4133	50.4161	51.4566	51.4598	49.3652	49.3709
-122.81	60.3127	60.3152	61.3467	61.3512	59.2746	59.2781

6. Conclusions

We have investigated multiple solutions of a singular nonlinear BVP arising from a study of laminar flow in a porous pipe with an expanding or contracting wall. We propose a numerical technique for the singular nonlinear BVP and multiple solutions are presented for some typical values of the expansion ratio and a full range of cross-flow Reynolds number. Based on the comparison of calculation results, the numerical technique proposed in this paper is robust and efficient in solving this nonlinear singular BVP for the entire range of Reynolds number Re. We believe that it can be used to some similar problems from fluid mechanics and other scientific fields.

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References

[1] S. Uchida and H. Aoki, Unsteady flows in a semi-infinite contracting or expanding pipe, J. Fluid. Mech. 82(1977) 371-387.

[2] M. Goto and S. Uchida, Unsteady flow in a semi-infinite expanding pipe with injection through wall, Japan. Soc. Aeronaut. Space. SCI. 33(1990) 14-27.

[3] Z.B. Youssef, B.A. Mina, A.B. Nagwa and S.H. Hossam, Lie-group method for unsteady flows in a semi-infinite expanding or contracting pipe with injection or suction through a porous wall, J. Comput. Appl. Math. 197(2006) 465-494.

[4] Z.B. Youssef, B.A. Mina, A.B. Nagwa and S.H. Hossam, Lie-group method solution for two-dimensional viscous flow between slowly expanding or contracting walls with weak permeability, Appl. comp. Model. 31(2007) 1092-1108.

[5] S. Asghar, M. Mushtaq and T. Hayat, Flow in a slowly deforming channel with weak permeability: an analytical approach, Nonliear. Anal-Real. 11(2010) 555-561.

[6] S. Dinarvand and M.M. Rashidi, A reliable treatment of a homotopy analysis method for two-dimensional viscous flow in a rectangular domain bounded by two moving porous walls, Nonlinear. Anal-Real. 11(2010) 1502-1512.

[7] W.A. Robinson, The existence of multiple solutions for the laminar flow in a uniformly porous channel with suction at both walls, J. Eng. Math. 10(1976) 23-40.

[8] H. Xu, Z.L. Lin, S.J. Liao, J.Z. Wu and J. Majdalani, Homotopy based solutions of the Navier-Stokes equations for a porous channel with orthogonally moving walls, Phys. fluids 22(2010) 1-18.

[9] X.H. Si, L.C. Zheng, X.X. Zhang, M. Li, J.H. Yang and Y. Chao, Multiple solutions for the laminar flow in a porous pipe with suction at slowly expanding or contracting wall, Appl. Math. Comp. 218(2011) 3515-3521.

[10] R.M. Terrill and P.W. Thomas, On laminar flow through a uniformly porous pipe, Appl. Sci. Res. 21(1969) 37-67.

[11] C. Shankararaman, M. Liu, Effect of slip on existence,uniqueness, and behavier of similarity solutions for steady incompressible laminar flow in porous tubes and channels, Phys. Fluids 18(2006) 1-10.

[12] F.R. De Hoog and R. Weiss, The numerical solution of boundary value problems with an essential singularity, SIAM J. Numer. Anal. 16(1979) 637-669.

[13] U. Ascher and R. Russell, Reformulation of boundary value problems into standard form, SIAM Review. 23(1981) 238-254.

[14] F. Natterer, A generalized spline method for singular boundary value problems of ordinary differential equations, Linear Algebra Appl. 7(1973) 189-216.

[15] D.C. Brabston and H.B. Keller, A numerical method for singular two point boundary value problems, SIAM J. Numer. Anal. 14(1977) 779-791.

[16] F.R. De Hoog and R. Weiss, Difference methods for boundary value problems with a singularity of first kind, SIAM J. Numer. Anal. 13(1976) 775-813.

[17] L.F. Shampine, Singular boundary value problems for ODEs, Appl. Math. Comput. 138(1977) 99-112.

[18] W. Auzinger, O. Koch and E. Weinmuller, Analysis of a New Error Estimate for Collocation Methods Applied to Singular Boundary Value Problem, SIAM J. Numer. Anal. 42(2005) 2366-2386.

[19] D. Jespersen, Ritz-Galerkin methods for singular boundary value problems, SIAM J. Numer. Anal. 15(1978) 813-834.

[20] M. Lentini, Boundary value problems over semi-infinite intervals, Ph. D. thesis, California Institute of Technology, Pasadena, 1978.

[21] M. Lentini, Boundary value problems on semi-infinite intervals and their numerical solution, SIAM J. Numer. Anal. 17(1980) 577-604.

[22] S.C. Eisenstat, R.S. Schreiber and M.H. Schultz, Finite element methods for spherially symmetric elliptic equation, Research Report 109, Comput Science, Yale University, New Haven CN, 1977.

[23] http://www.math.tuwien.ac.at/ ewa/fwfbibli.html.

[24] O. Koch and E. Weinmuller, Analytical and numerical treatment of a singular initial value problem in avalanche modeling, Appl. Math. Comput. 148(2004) 561-570.

[25] G. Vainikko, A smooth solution to a linear systems of singular ODEs, Z. Anal. Anwend. 32(2013) 349-370.

[26] O. Koch, P. Kofler and E. Weinmuller, Initial value problems for systems of ordinary first and second order differential equations with a singularity of the first kind, Analysis. 21(2001) 373-389.

[27] G. Vainikko, A smooth solution to a nonlinear systems of singular ODEs, AIP Conf. Proc. 1558(2013) 758-761.

[28] I. Rachunkova, S. Stanek, J. Vampolova and E. Weinmuller, On linear ODEs with a time singularity of the first kind and unsmooth inhomogeneity, ASC Report 07/2014, Institute for Analysis and Scientific Computing, Vienna University of Technology, Vienna(2014).

[29] L.F. Shampine, J. Kierzenka and M. Reichelt, Solving boundary value problems for ordinary differential equations in MATLAB with bvp4c, 2000.

[30] J. Kierzenka and L.F. Shampine, A BVP solver that controls residual and error, J. Numer. Anal. Indust. Appl. Math. 3(2008) 27-41.

[31] W. Auzinger, G. Kneisl, O. Koch and E. Weinmuller, A collocation code for boundary value problems in ordinary differential equations, Numer. Algorithms. 33(2003) 27-39.

[32] G. Kitzhofer, G. Pulverer, O. Koch, Ch. Simon and E. Weinmuller, BVPSUITE- A New MATLAB code for singular implicit boundary value problems, 2009.

[33] G. Kitzhofer, O. Koch, G. Pulverer, Ch. Simon and E. Weinmuller, The new MATLAB code BVPSUITE for the solution of singular implicit boundary value problem, J. Numer. Anal. Indust. Appl. Math. 5(2010) 113-134.

[34] L.F. Shampine, P. Muir and H. Xu, A user-friendly fortran BVP solver, J. Numer. Anal. Indust. Appl. Math. 1(2006) 201-217.

[35] U. Ascher and U. Bader, A new basis implementation for a mixed order boundary value ODE solver, SIAM J. Scient. Stat. Comput. 8(1987) 483-500.

[36] U. Ascher, J. Christiansen and R.D. Russell, A collocation solver for mixed order systems of boundary values problems, Math. Comp. 33(1978) 659-679.

[37] E.J. Doedel, A.R. Champneys, T.F. Fairgrieve, Y.A. Kuznetsov, B. Sandstede, X.J. Wang, AUTO97-00: Continuation and bifurcation software for ordinary differential equations (with HomCont), User's Guide, Concordia University, Montreal, Cannada (1997-2000).

[38] L. Li, H. Zhang, P. Lin, X.H. Si, Asymptotic solutions for the laminar flow through a channel with porous retractable walls in the presence of a transverse magnetic field, Submitted.

[39] Y. CH. SU, Boundary layer correction method for singular perturbation problems, ShangHai Scinece and Technology Press(In Chinese) 1983.

[40] M.I. Vishik and L.A. Lyusternik, Regular degeneration and boundary layer for linear differential equations with small parameter, Uspekhi. Matematicheskikh. Nauk. 1957.

[41] M.I. Vishik and L.A. Lyusternik, The solution of some perturbation problems for matrices and selfadjoint or non-selfadjoint differential equations, Russ. Math. Surv. 1960.

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