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HOMOGENIZATION OF BIOMECHANICAL MODELS FOR PLANT TISSUES*

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Abstract. In this paper homogenization of a mathematical model for plant tissue biomechanics is presented. The microscopic model constitutes a strongly coupled system of reaction-diffusion-convection equations for chemical processes in plant cells, the equations of poroelasticity for elastic deformations of plant cell walls and middle lamella, and Stokes equations for fluid flow inside the cells. The chemical process in cells and the elastic properties of cell walls and middle lamella are coupled because elastic moduli depend on densities involved in chemical reactions, whereas chemical reactions depend on mechanical stresses. Using homogenization techniques, we derive rigorously a macroscopic model for plant biomechanics. To pass to the limit in the nonlinear reaction terms, which depend on elastic strain, we prove the strong two-scale convergence of the displacement gradient and velocity field.

Key words. homogenization, two-scale convergence, periodic unfolding method, poroelasticity, Stokes system, biomechanics of plant tissues

AMS subject classification. 35B27

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1. Introduction.

Analysis of interactions between mechanical properties and chemical processes, which influence the elasticity and extensibility of plant cell tissues, is important for better understanding of plant growth and development, as well as their response to environmental changes. Plant tissues are composed of cells surrounded by cell walls and connected by a cross-linked pectin network of middle lamella. Plant cell walls must be very strong to resist high internal hydrostatic pressure and at the same time flexible to permit growth. It is supposed that calcium-pectin cross-linking chemistry is one of the main regulators of plant cell wall elasticity and extension [51]. Pectin is deposited to cell walls in a methylesterified form. In cell walls and middle lamella, pectin can be modified by the enzyme pectin methylesterase (PME), which removes methyl groups by breaking ester bonds. The de-esterified pectin is able to form calcium-pectin cross-links, and thus stiffen the cell wall and reduce its expansion; see, e.g., [50]. On the other hand, mechanical stresses can break calcium-pectin cross-links and hence increase the extensibility of plant cell walls and middle lamella. It has been shown that chemical properties of pectin and the control of the density of calcium-pectin cross-links greatly influence the mechanical deformations of plant cell walls [34], and the interference with PME activity causes dramatic changes in growth behavior of plant tissues [50].

To analyze the interactions between calcium-pectin dynamics and deformations of a plant tissue, we derive a mathematical model for plant tissue biomechanics at the

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length scale of plant cells. In the microscopic model we consider a system of reaction-diffusion-convection equations describing the dynamics of the methylesterified pectin, demethylesterified pectin, calcium ions, and calcium-pectin cross-links. Elastic deformations and water flow are modelled by the equations of poroelasticity for cell walls and middle lamella coupled with the Stokes system for the flow velocity inside cells. The interplay between the mechanics and the chemistry comes in by assuming that the elastic properties of cell walls and middle lamella depend on the density of the calcium-pectin cross-links and that the stress within cell walls and middle lamella can break the cross-links. Thus the microscopic problem is a strongly coupled system of the Stokes equations, reaction-diffusion-convection equations, with reaction terms depending on the displacement gradient, and equations of poroelasticity, with elastic moduli depending on the density of cross-links. To address the situations when a plant tissue is completely and not completely saturated by water, we consider both evolutional and quasi-stationary equations of poroelasticity.

To show the existence of a weak solution of the microscopic equations, we use a classical approach and apply the Banach fixed-point theorem. However, due to quadratic nonlinearities of reaction terms, the proof of the contraction inequality is not standard and relies on delicate a priori estimates for the L^∞-norm of a solution of the reaction-diffusion-convection system in terms of the L^2-norm of displacement gradient and flow velocity. The Alikakos iteration technique [2] is applied to show the uniform boundedness of some components of solutions of the microscopic equations.

To analyze effective mechanical properties of plant tissues, we derive rigorously a macroscopic model for plant biomechanics using homogenization techniques. The two-scale convergence, e.g., [3, 31], and the periodic unfolding method, e.g., [15], are applied to obtain the macroscopic equations. The main mathematical difficulty in the derivation of the macroscopic problem arises from the strong coupling between the equations of poroelasticity and the system of reaction-diffusion-convection equations. In order to pass to the limit in the nonlinear reaction terms, we prove the strong two-scale convergence for the displacement gradient and fluid flow velocity, essential for the homogenization of the coupled problem considered here. Due to the dependence of the elasticity tensor on the time variable, in the proof of the strong two-scale convergence a specific form of the energy functional is considered.

Similar to the microscopic problem, to prove uniqueness of a solution of the macroscopic equations, we derive a contraction inequality involving the L^∞-norm of the difference of two solutions of the reaction-diffusion-convection equations. This contraction inequality also ensures the well-posedness of the limit system.

The poroelasticity equations, modelling interactions between fluid flow and elastic stresses in porous media, were first obtained by Biot using a phenomenological approach [10, 9, 8] and subsequently derived by applying techniques of homogenization theory. Formal asymptotic expansion was undertaken by the authors of [5, 13, 23, 42] to derive Biot equations from microscopic description of elastic deformations of a solid matrix and fluid flow in porous space. The rigorous homogenization of the coupled system of equations of linear elasticity for a solid matrix combined with the Stokes or Navier–Stokes equations for the fluid part was conducted in [17, 19, 24, 32] by using the two-scale convergence method. Depending on the ratios between the physical parameters, different macroscopic equations were obtained, e.g., Biot's equations of poroelasticity, the system consisting of the anisotropic Lamé equations for the solid component, and the acoustic equations for the fluid component, the equations of viscoelasticity. The homogenization of a mathematical model for elastic deformations, fluid flow, and chemical processes in a cell tissue was considered in [20]. In contrast to the problem considered in the present paper, in [20] the coupling between the equations of linear elasticity and reaction-diffusion-convection equations for a concentration was given only through the dependence of the elasticity tensor on the chemical concentration. The existence and uniqueness of a solution for equations of poroelasticity were studied in [45, 53].

Compared to the many results for the equations of poroelasticity, there exist only a few studies of interactions between a free fluid and a deformable porous medium. In [46] a nonlinear semigroup method was used for mathematical analysis of a system of poroelastic equations coupled with the Stokes equations for free fluid flow. A rigorous derivation of interface conditions between a poroelastic medium and an elastic body was considered in [26]. Numerical methods for a coupled Biot poroelastic system and Navier-Stokes equations were derived in [6]. The Nitsche method for enforcing interface conditions was applied in [12] for numerical simulation of the Stokes–Biot coupled system.

Several results on homogenization of equations of linear elasticity can be found in [7, 21, 33, 42] (and the references therein). Homogenization of the microscopic model for plant cell wall biomechanics, composed of equations of linear elasticity and reaction-diffusion equations for chemical processes, has been studied in [39].

This paper is organized as follows. In section 2 we derive the microscopic model for plant tissue biomechanics. A priori estimates as well as the existence and uniqueness of a weak solution of the microscopic problem are obtained in section 3. In section 4 we prove the convergence results for solutions of the microscopic problem. The multiscale analysis of the coupled poroelastic and Stokes problem is conducted in section 5. In section 6 we show strong two-scale convergence of the displacement gradient and flow velocity. The macroscopic equations for the system of reaction-diffusion-convection equations are derived in section 7. The well-posedness and uniqueness of a solution of the macroscopic problem are proved in section 8. In section 9 we consider the incompressible and quasi-stationary cases for the equations of poroelasticity.

2. Microscopic model.

In the mathematical model for plant tissue biomechanics we consider interactions between the mechanical properties of a plant tissue and the chemical processes in plant cells. A plant tissue is composed of the cell interior (intracellular space), the plasma membrane, plant cell walls, and the cross-linked pectin network of the middle lamella joining individual cells together. Primary plant cell walls consist mainly of oriented cellulose microfibrils (that strongly influence the cell wall stiffness), pectin, hemicellulose, proteins, and water. It is supposed that calcium-pectin chemistry, given by the de-esterification of pectin and creation and breakage of calcium-pectin cross-links, is one of the main regulators of cell wall elasticity; see, e.g., [51]. Hence in our mathematical model we consider the interactions and two-way coupling between calcium-pectin chemistry and elastic deformations of a plant tissue. To describe the coupling between the mechanics and chemistry, we consider the dynamics of pectins, calcium, and calcium-pectin cross-links, water flow in a plant tissue, and the poroelastic nature of cell walls and middle lamella.

To derive a mathematical model for plant tissue biomechanics, we denote a domain occupied by a plant tissue by $\Omega \subset \mathbb{R}^3$, where Ω is a bounded domain with $C^{1,\alpha}$ boundary for some $\alpha > 0$. Notice that all results also hold for a two-dimensional domain. Then the time-independent domains $\Omega_f \subset \Omega$ and $\Omega_e \subset \Omega$, with $\overline{\Omega} = \overline{\Omega}_e \cup \overline{\Omega}_f$ and $\Omega_e \cap \Omega_f = \emptyset$, represent the reference (Lagrangian) configurations of the intracellular (cell interior) and intercellular (cell walls and middle lamella) spaces, respectively, and Γ denotes the boundaries between the cell interior and cell walls and corresponds to the plasma membrane. Since Γ represents the interface between elastic material and fluid in the Lagrangian configuration, it is also independent of time.

Pectin is deposited into the cell wall in a highly methylesterified state and is modified by the wall enzyme PME, which removes methyl groups [50]. It was observed experimentally that pectins can diffuse in a plant cell wall matrix; see, e.g., [18, 35, 48]. Thus in the balance equation for the density of the methylesterified pectin $b_{e,1}$ and demethylesterified pectin $b_{e,2}$,

$$\partial_t b_{e,j} + \mathrm{div} J_{b,j} = g_{b,j} \quad \text{in } \Omega_e, \quad j = 1, 2,$$

we assume the flux to be determined by Fick's law, i.e., $J_{b,j} = -D_{b_e,j} \nabla b_{e,j}$, with $j = 1, 2$ and $D_{b_e,j} > 0$. The term $g_{b,j}$ models chemical reactions that correspond to the demethylesterification processes and creation and breakage of calcium-pectin cross-links. In general, diffusion coefficients for pectins and calcium depend on the microscopic structure of the cell wall given by the cell wall microfibrils and hemicellulose network, which is assumed to be given and not to change in time, as well as on the density of pectins and calcium-pectin cross-links. For presentation simplicity we assume here that the diffusion coefficient does not depend on the dynamics of pectin and calcium-pectin cross-link densities. However, the analysis can be conducted in the same way for the generalized model in which the diffusion of pectin, calcium, and cross-links depends on pectin and cross-link densities, assuming that diffusion coefficients are uniformly bounded from below and above, which is biologically sensible. The modification of methylesterified pectin by PME is modelled by the reaction term $g_{b,1} = -\mu_1 b_{e,1}$ with some $\mu_1 > 0$. For simplicity we assume that there is a constant concentration of PME enzyme in the cell wall. By simple modifications of the analysis considered here, the same results can be obtained for a generalized model including the dynamics of PME and chemical reactions between PME and pectin; see [39] for the derivation of the corresponding system of equations.

The deposition of the methylesterified pectin is described by the inflow boundary condition on the cell plasma membrane. We also assume that the demethylesterified pectin cannot move back into the cell interior:

$$D_{b_e,1} \nabla b_{e,1} \cdot n = P_1(b_{e,1}, b_{e,2}, b_{e,3}), \quad D_{b_e,2} \nabla b_{e,2} \cdot n = 0 \quad \text{on } \Gamma.$$

To account for mechanisms controlling the amount of pectin in the cell wall, we assume that the inflow of new methylesterified pectin depends on the density of methylesterified and demethylesterified pectin, i.e., $b_{e,1}$ and $b_{e,2}$, and calcium-pectin cross-links $b_{e,3}$.

We consider the diffusion and transport by water flow of calcium molecules in the symplast (in the cell interior) and diffusion of calcium in the apoplast (cell walls and middle lamella); see, e.g., [49]. Then the balance equations for calcium densities c_f and c_e in Ω_f and Ω_e, respectively, are given by

$$\begin{aligned} &\partial_t c_f - \mathrm{div}(D_f \nabla c_f - \mathcal{G}(\partial_t u_f) c_f) = g_f && \text{in } \Omega_f, \\ &\partial_t c_e - \mathrm{div}(D_e \nabla c_e) = g_e && \text{in } \Omega_e, \end{aligned}$$

where the chemical reaction term $g_f = g_f(c_f)$ in Ω_f describes the decay and/or buffering of calcium inside the plant cells (see, e.g., [52]), g_e models the interactions between calcium and demethylesterified pectin in cell walls and middle lamella and the creation and breakage of calcium-pectin cross-links, and $\mathcal{G}$ is a bounded function of the intracellular flow velocity $\partial_t u_f$. The condition that $\mathcal{G}$ is bounded is natural from the biological and physical point of view, because the flow velocity in plant tissues is bounded. This condition is also essential for a rigorous mathematical analysis of the model. We assume that as the result of the breakage of a calcium-pectin cross-link by mechanical stresses we obtain one calcium molecule and two galacturonic acid monomers of demethylesterified pectin. A detailed derivation of the chemical reaction term g_e is given in [39]. See also Remark 2.3 for the detailed form of the reaction terms. We assume a passive flow of calcium between cell walls and cell interior and assume that the exchange of calcium between cell interior and cell walls is facilitated only on parts of the cell membrane $\Gamma \setminus \widetilde{\Gamma}$, i.e.,

$$\begin{aligned} &c_f = c_e, && (D_f \nabla c_f - \mathcal{G}(\partial_t u_f) c_f) \cdot n = D_e \nabla c_e \cdot n && \text{on} \quad \Gamma \setminus \widetilde{\Gamma}, \\ &D_e \nabla c_e \cdot n = 0, && (D_f \nabla c_f - \mathcal{G}(\partial_t u_f) c_f) \cdot n = 0 && \text{on} \quad \widetilde{\Gamma}. \end{aligned}$$

The regulation of calcium flow by mechanical properties of the cell wall will be considered in future studies.

Calcium-pectin cross-links $b_{e,3}$ are created by electrostatic and ionic binding between two galacturonic acid monomers of pectin chains and calcium molecules. It is also known that these cross-links are very stable and can be disturbed mainly by thermal treatments and/or mechanical forces; see, e.g., [38, 37]. Thus assuming a constant temperature, the calcium-pectin chemistry can be described as a reaction between calcium molecules and pectins, where the breakage of cross-links depends on the deformation gradient of the cell walls. Hence we assume that the cross-links are disturbed by the mechanical stresses in cell walls and middle lamella; see [39] for a detailed description of the modelling of the calcium-pectin chemistry. A similar approach was used in [41] to model a chemically mediated mechanical expansion of the cell wall of a pollen tube. There are no experimental observations of diffusion of calcium-pectin cross-links $b_{e,3}$; however, since most calcium-pectin cross-links are not attached to cell wall microfibrils [18], it is possible that cross-links can move inside the cell wall matrix by a very slow diffusion

$$\partial_t b_{e,3} - \operatorname{div}(D_{b_e,3} \nabla b_{e,3}) = g_{b,3} \quad \text{in } \Omega_e,$$

where $D_{b_e,3} > 0$ and the reaction term $g_{b,3}$ models the creation and breakage by mechanical stresses of calcium-pectin cross-links (see Remark 2.3 for a detailed form of $g_{b,3}$). For the analysis presented here the diffusion term in the equations for calcium-pectin cross-link density is important. However, the same results can be obtained if one assumes that calcium-pectin cross-links do not diffuse and that the reaction terms in equations for pectin, calcium, and calcium-pectin cross-links depend on a local average of the deformation gradient, reflecting the fact that in a dense pectin network mechanical forces have a nonlocal effect on the calcium-pectin chemistry; see [39].

To describe elastic deformations of plant cell walls and middle lamella, we consider the equations of poroelasticity reflecting the microscopic structure of cell walls and middle lamella permeable to fluid flow:

$$\begin{aligned} &\rho_e \partial_t^2 u_e - \operatorname{div}(\mathbf{E}(b_{e,3}) \mathbf{e}(u_e)) + \alpha \nabla p_e = 0 && \text{in } \Omega_e, \\ &\rho_p \partial_t p_e - \operatorname{div}(K_p \nabla p_e - \alpha\, \partial_t u_e) = 0 && \text{in } \Omega_e. \end{aligned}$$

Here u_e denotes the displacement from the equilibrium position, $\mathbf{e}(u_e)$ stands for the symmetrized gradient of u_e, and ρ_e denotes the poroelastic wall density. Since we consider the equations of poroelasticity, one more unknown function that should be determined is the pressure, denoted by p_e. The mass storativity coefficient is denoted by ρ_p, and K_p denotes the hydraulic conductivity of cell walls and middle lamella. In what follows, we assume that the Biot–Willis constant is $\alpha = 1$.

It is observed experimentally that the load-bearing calcium-pectin cross-links reduce cell wall expansion; see, e.g., [51]. Hence elastic properties of cell walls and middle lamella depend on the chemical configuration of pectin and density of calcium-pectin cross-links; see, e.g., [55]. This is reflected in the dependence of the elasticity tensor $\mathbf{E}$ of the cell wall and middle lamella on the density of calcium-pectin cross-links $b_{e,3}$. The differences in the elastic properties of cell walls and middle lamella result in the dependence of the elasticity tensor $\mathbf{E}$ on the spatial variables. Since the properties of calcium-pectin cross-links are changing during the deformation and the stretched cross-links have different impact (stress drive hardening) on the elastic properties of the cell wall matrix from that of newly created cross-links (see, e.g., [11, 36, 43]), we consider a nonlocal-in-time dependence of the Young modulus of the cell wall matrix on the density of calcium-pectin cross-links; see Assumption **A1**. A similar approach was used in [20] to model the dependence of cell deformations on the concentration of a chemical substance. We assume that the hydraulic conductivity tensor varies between cell wall and middle lamella and, hence, K_p depends on the spacial variables.

In the cell interior, that is, in Ω_f, the water flow is modelled by the Stokes system

$$\rho_f \partial_t^2 u_f - \mu \operatorname{div}(\mathbf{e}(\partial_t u_f)) + \nabla p_f = 0, \qquad \operatorname{div}\partial_t u_f = 0 \qquad \text{in } \Omega_f,$$

where $\partial_t u_f$ denotes the fluid velocity, p_f the fluid pressure, μ the fluid viscosity, and ρ_f the fluid density.

As transmission conditions between free fluid and poroelastic domains we consider the continuity of normal flux, which corresponds to mass conservation, and the continuity of the normal component of total stress on the interface Γ; i.e., the total stress of the poroelastic medium is balanced by the total stress of the fluid, representing the conservation of momentum,

$$\begin{aligned} &(-K_p \nabla p_e + \partial_t u_e) \cdot n = \partial_t u_f \cdot n && \text{on } \Gamma, \\ &(\mathbf{E}(b_{e,3})\, \mathbf{e}(u_e) - p_e I)\, n = (\mu\, \mathbf{e}(\partial_t u_f) - p_f I)\, n && \text{on } \Gamma. \end{aligned} \tag{1}$$

Also taking into account the channel structure of a cell membrane separating cell interior and cell wall, given by the presence of aquaporins (see, e.g., [14]), we assume that the water flow between the poroelastic cell wall and cell interior is in the direction normal to the interface between the free fluid and the poroelastic medium. Hence we assume the no-slip interface condition, which is appropriate for problems where at the interface the fluid flow in the tangential direction is not allowed (see, e.g., [12]),

$$\Pi_\tau \partial_t u_e = \Pi_\tau \partial_t u_f \qquad \text{on } \Gamma.$$

By $\Pi_\tau w$ we define the tangential projection of a vector w, i.e., $\Pi_\tau w = w - (w \cdot n)n$, where n is a normal vector and τ indicates the tangential subspace to the boundary. The balance of the normal components of the stress in the fluid phase across the interphase is given by

$$n \cdot (\mu\, \mathbf{e}(\partial_t u_f) - p_f I)\, n = -p_e \qquad \text{on } \Gamma. \tag{2}$$

Notice that the transmission conditions (1) and (2) imply $\mathbf{E}(b_{e,3})\, \mathbf{e}(u_e)\, n \cdot n = 0$ on Γ. The transmission conditions are motivated by the models describing coupling between

Biot and Navier–Stokes or Stokes equations considered in, e.g., [6, 12, 27, 28, 46]. The coupling between elastic deformations and fluid flow is described in the Lagrangian configuration, and hence Γ is a fixed interface between the fluid domain and elastic material. Since in our model we consider only the linear elastic nature of the solid skeleton of the cell walls, the transmission conditions (1) and (2) are the corresponding linearizations of the fluid-solid interface conditions; i.e., $|\det(I+\nabla u_e)|(\mu\, \mathbf{e}(\partial_t u_f(t, x+u_e)) - p_f(t, x+u_e)I)(I+\nabla u_e)^{-T} n$ is approximated by $(\mu\, \mathbf{e}(\partial_t u_f(t,x)) - p_f(t,x)I)n$ on Γ, and the first Piola–Kirchhoff stress tensor is equal to the Cauchy stress tensor in the first order approximation.

Then the model for the densities of calcium, pectins, and calcium-pectin cross-links reads as

$$
(3)\qquad
\begin{aligned}
&\partial_t b_e = \operatorname{div}(D_b \nabla b_e) + g_b(c_e, b_e, \mathbf{e}(u_e)) && \text{in } \Omega_e,\ t>0\\
&\partial_t c_e = \operatorname{div}(D_e \nabla c_e) + g_e(c_e, b_e, \mathbf{e}(u_e)) && \text{in } \Omega_e,\ t>0,\\
&\partial_t c_f = \operatorname{div}(D_f \nabla c_f - \mathcal{G}(\partial_t u_f)c_f) + g_f(c_f) && \text{in } \Omega_f,\ t>0,\\
&D_b \nabla b_e \cdot n = P(b_e) && \text{on } \Gamma,\quad t>0,\\
&c_e = c_f, \qquad D_e \nabla c_e \cdot n = (D_f \nabla c_f - \mathcal{G}(\partial_t u_f)c_f)\cdot n && \text{on } \Gamma\setminus\widetilde{\Gamma},\ t>0,\\
&D_e \nabla c_e \cdot n = 0, \qquad (D_f \nabla c_f - \mathcal{G}(\partial_t u_f)c_f)\cdot n = 0 && \text{on } \widetilde{\Gamma},\quad t>0,\\
&b_e(0,x) = b_{e0}(x), \quad c_e(0,x) = c_0(x) && \text{in } \Omega_e,\\
&c_f(0,x) = c_0(x) && \text{in } \Omega_f,
\end{aligned}
$$

where $b_e = (b_{e,1}, b_{e,2}, b_{e,3})$, $D_f > 0$, $D_e > 0$, and $D_b = \operatorname{diag}(D_{b_{e,1}}, D_{b_{e,2}}, D_{b_{e,3}})$ with $D_{b_{e,j}} > 0$, $j = 1,2,3$, stands for the diagonal matrix of diffusion coefficients for $b_{e,1}$, $b_{e,2}$, and $b_{e,3}$.

For elastic deformations of cell walls and middle lamella and fluid flow inside the cells we have a coupled system of Stokes equations and poroelastic (Biot) equations:

$$
(4)\qquad
\begin{aligned}
&\rho_e \partial_t^2 u_e - \operatorname{div}(\mathbf{E}(b_{e,3})\mathbf{e}(u_e)) + \nabla p_e = 0 && \text{in } \Omega_e,\ t>0,\\
&\rho_p \partial_t p_e - \operatorname{div}(K_p \nabla p_e - \partial_t u_e) = 0 && \text{in } \Omega_e,\ t>0,\\
&\rho_f \partial_t^2 u_f - \mu \operatorname{div}(\mathbf{e}(\partial_t u_f)) + \nabla p_f = 0 && \text{in } \Omega_f,\ t>0,\\
&\operatorname{div} \partial_t u_f = 0 && \text{in } \Omega_f,\ t>0,\\
&(\mathbf{E}(b_{e,3})\,\mathbf{e}(u_e) - p_e I)\, n = (\mu\,\mathbf{e}(\partial_t u_f) - p_f I)\, n && \text{on } \Gamma,\ t>0,\\
&\Pi_\tau \partial_t u_e = \Pi_\tau \partial_t u_f, \qquad n\cdot(\mu\,\mathbf{e}(\partial_t u_f) - p_f I)\, n = -p_e && \text{on } \Gamma,\ t>0,\\
&(-K_p \nabla p_e + \partial_t u_e)\cdot n = \partial_t u_f \cdot n && \text{on } \Gamma,\ t>0,\\
&u_e(0,x) = u_{e0}(x), \quad \partial_t u_e(0,x) = u^1_{e0}(x), \quad p_e(0,x) = p_{e0}(x) && \text{in } \Omega_e,\\
&\partial_t u_f(0,x) = u^1_{f0}(x) && \text{in } \Omega_f.
\end{aligned}
$$

For multiscale analysis of the mathematical model (3)–(4) we derive the nondimensionalized equations from the dimensional model by considering $t = \hat{t}t^*$, $x = \hat{x}x^*$, $b_e = \hat{b}b_e^*$, $c_j = \hat{c}c_j^*$, $u_j = \hat{u}u_j^*$, $p_j = \hat{p}p_j^*$, with $j = e, f$, $\mathbf{E} = \hat{E}\mathbf{E}^*$, $K_p = \hat{K}K_p^*$, $\mu = \hat{\mu}\mu^*$, $\rho_p = \hat{\rho}_p\rho_p^*$, $\rho_j = \hat{\rho}\rho_j^*$, with $j = e, f$, $D_j = \hat{D}D_j^*$ for $j = b, e, f$, $P(b_e) = \hat{R}\hat{b}P^*(b_e^*)$, $g_j(c_e, b_e, \mathbf{e}(u_e)) = \hat{g}\hat{b}g_j^*(c_e^*, b_e^*, \mathbf{e}(u_e^*))$ for $j = b, e$, and $g_f(c_f) = \hat{g}_f\hat{b}g_f^*(c_f^*)$. The dimensionless small parameter $\varepsilon = l/L$ represents the ratio between the representative size of a plant cell l and the considered size of a plant tissue L and reflects the size of the microstructure. For a plant root cell we can consider $l = 10\mu$m and $L = 1$m, and, hence, ε is of order 10^{-5}. We consider $\hat{x} = L$, $\hat{p} = \Lambda\varepsilon$, with $\Lambda = 1$MPa, and

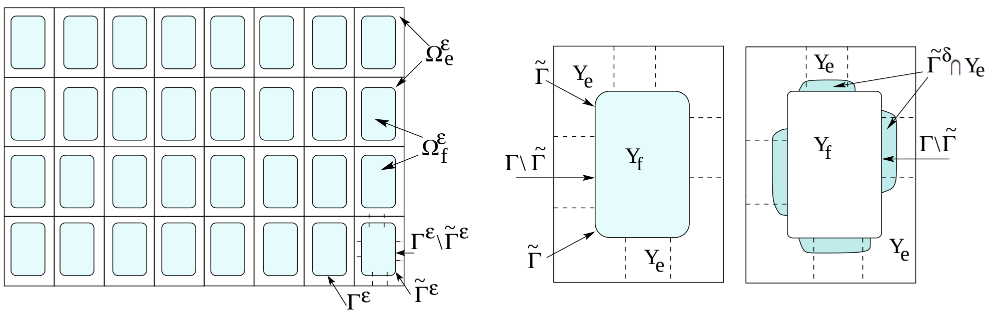

FIG. 1. *Schematic diagram of the geometry of a plant tissue and unit cell.*

$\hat{u} = l$. For the time scale we take $\hat{t} = \hat{\mu}/(\Lambda\varepsilon^2)$, which together with $\hat{\mu} = 10^{-2}$Pa·s corresponds approximately to 1.7min. We also consider $\hat{E} = \Lambda$, $\hat{K} = \hat{x}^2\varepsilon/(\hat{p}\hat{t}) = l^2/\hat{\mu}$, $\hat{\rho} = (\Lambda\hat{t}^2)/\hat{x}^2 = \hat{\mu}^2/(\Lambda\varepsilon^4 L^2)$, $\hat{\rho}_p = 1/\Lambda$, $\hat{D} = \hat{x}^2/\hat{t} = l^2\Lambda/\hat{\mu}$, and $\hat{R} = \hat{x}\varepsilon/\hat{t} = \varepsilon^3 L\Lambda/\hat{\mu}$. Hydraulic conductivity K_p is of order 10^{-9}–10^{-8} m^2· s^{-1}· Pa^{-1}, and the minimal value of the elasticity tensor is of order 10MPa [55]. Hence the minimal value of the nondimensionalized elasticity tensor $\mathbf{E}^*$ is approximately 10, and $K_p^* \sim 0.01 - 0.1$. The parameters in the inflow boundary condition, i.e., in $P(b_e)$, are of order 10^{-7}m/s, and with $\hat{R} = 10^{-7}$m/s we obtain the nondimensional parameters in the boundary condition for b_e to be of order 1. Here we assume that $\rho_j > 0$, with $j = e, p, f$, are fixed. The case when the density ρ_e and/or ρ_p is of order ε^2 can be analyzed in the same way as the case when $\rho_e = 0$ and $\rho_p = 0$, considered in section 9.

To describe the microscopic structure of a plant tissue, we assume that cells in the tissue are distributed periodically and have a diameter of order ε. The stochastic case will be analyzed in a future paper. We consider a unit cell $\overline{Y} = \overline{Y}_e \cup \overline{Y}_f$, with $\overline{Y} = [0, a_1] \times [0, a_2] \times [0, a_3]$, for $a_j > 0$ with $j = 1, 2, 3$, where Y_e represents the cell wall and a part of the middle lamella, and Y_f, with $\overline{Y}_f \subset Y$, defines the inner part of a cell. We denote $\partial Y_f = \Gamma$ and let $\widetilde{\Gamma}$ be an open on Γ regular subset of Γ.

Then the time-independent domains Ω_f^ε and Ω_e^ε, representing the reference (Lagrangian) configuration of the intracellular (cell interior) and intercellular (cell walls and middle lamella) spaces, are defined by

$$\Omega_f^\varepsilon = \mathrm{Int}\Big(\bigcup_{\xi\in\Xi^\varepsilon} \varepsilon(\overline{Y}_f + \xi)\Big) \quad \text{and} \quad \Omega_e^\varepsilon = \Omega \setminus \overline{\Omega}_f^\varepsilon, \tag{5}$$

respectively, where $\Xi^\varepsilon = \{\xi = (a_1\eta_1, a_2\eta_2, a_3\eta_3),\ \eta = (\eta_1, \eta_2, \eta_3) \in \mathbb{Z}^3 :\ \varepsilon(\overline{Y}_f + \xi) \subset \Omega\}$, and $\Gamma^\varepsilon = \bigcup_{\xi\in\Xi^\varepsilon} \varepsilon(\Gamma + \xi)$ defines the boundaries between cell interior and cell walls, $\widetilde{\Gamma}^\varepsilon = \bigcup_{\xi\in\Xi^\varepsilon} \varepsilon(\widetilde{\Gamma} + \xi)$; see Figure 1.

We shall use the following notation for time-space domains: $\Omega_s = (0, s) \times \Omega$, $(\partial\Omega)_s = (0, s) \times \partial\Omega$, $\Omega_{j,s}^\varepsilon = (0, s) \times \Omega_j^\varepsilon$ for $j = e, f$, $\Gamma_s^\varepsilon = (0, s) \times \Gamma^\varepsilon$, and $\widetilde{\Gamma}_s^\varepsilon = (0, s) \times \widetilde{\Gamma}^\varepsilon$ for $s \in (0, T]$.

Neglecting *, we obtain the nondimensionalized microscopic model for plant tissue biomechanics

$$
(6)\qquad
\begin{aligned}
&\partial_t b_e^\varepsilon = \operatorname{div}(D_b \nabla b_e^\varepsilon) + g_b(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)) && \text{in } \Omega_{e,T}^\varepsilon,\\
&\partial_t c_e^\varepsilon = \operatorname{div}(D_e \nabla c_e^\varepsilon) + g_e(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)) && \text{in } \Omega_{e,T}^\varepsilon,\\
&\partial_t c_f^\varepsilon = \operatorname{div}(D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t u_f^\varepsilon) c_f^\varepsilon) + g_f(c_f^\varepsilon) && \text{in } \Omega_{f,T}^\varepsilon,\\
&D_b \nabla b_e^\varepsilon \cdot n = \varepsilon\, P(b_e^\varepsilon) && \text{on } \Gamma_T^\varepsilon,\\
&c_e^\varepsilon = c_f^\varepsilon, \qquad D_e \nabla c_e^\varepsilon \cdot n = (D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t u_f^\varepsilon) c_f^\varepsilon) \cdot n && \text{on } \Gamma_T^\varepsilon \setminus \widetilde{\Gamma}_T^\varepsilon,\\
&D_e \nabla c_e^\varepsilon \cdot n = 0, \qquad (D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t u_f^\varepsilon) c_f^\varepsilon) \cdot n = 0 && \text{on } \widetilde{\Gamma}_T^\varepsilon,\\
&b_e^\varepsilon(0,x) = b_{e0}(x), \quad c_e^\varepsilon(0,x) = c_0(x) && \text{in } \Omega_e^\varepsilon,\\
&c_f^\varepsilon(0,x) = c_0(x) && \text{in } \Omega_f^\varepsilon
\end{aligned}
$$

and

$$
(7)\qquad
\begin{aligned}
&\rho_e \partial_t^2 u_e^\varepsilon - \operatorname{div}(\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon)) + \nabla p_e^\varepsilon = 0 && \text{in } \Omega_{e,T}^\varepsilon,\\
&\rho_p \partial_t p_e^\varepsilon - \operatorname{div}(K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t u_e^\varepsilon) = 0 && \text{in } \Omega_{e,T}^\varepsilon,\\
&\rho_f \partial_t^2 u_f^\varepsilon - \varepsilon^2 \mu \operatorname{div}(\mathbf{e}(\partial_t u_f^\varepsilon)) + \nabla p_f^\varepsilon = 0 && \text{in } \Omega_{f,T}^\varepsilon,\\
&\operatorname{div} \partial_t u_f^\varepsilon = 0 && \text{in } \Omega_{f,T}^\varepsilon,\\
&(\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\, \mathbf{e}(u_e^\varepsilon) - p_e^\varepsilon I)\, n = (\varepsilon^2 \mu\, \mathbf{e}(\partial_t u_f^\varepsilon) - p_f^\varepsilon I)\, n && \text{on } \Gamma_T^\varepsilon,\\
&\Pi_\tau \partial_t u_e^\varepsilon = \Pi_\tau \partial_t u_f^\varepsilon, \qquad n \cdot (\varepsilon^2 \mu\, \mathbf{e}(\partial_t u_f^\varepsilon) - p_f^\varepsilon I)\, n = -p_e^\varepsilon && \text{on } \Gamma_T^\varepsilon,\\
&(-K_p^\varepsilon \nabla p_e^\varepsilon + \partial_t u_e^\varepsilon) \cdot n = \partial_t u_f^\varepsilon \cdot n && \text{on } \Gamma_T^\varepsilon,\\
&u_e^\varepsilon(0,x) = u_{e0}^\varepsilon(x), \quad \partial_t u_e^\varepsilon(0,x) = u_{e0}^1(x), \quad p_e^\varepsilon(0,x) = p_{e0}^\varepsilon(x) && \text{in } \Omega_e^\varepsilon,\\
&\partial_t u_f^\varepsilon(0,x) = u_{f0}^1(x) && \text{in } \Omega_f^\varepsilon.
\end{aligned}
$$

On the external boundaries we prescribe the following force and flux conditions:

$$
(8)\qquad
\begin{aligned}
&D_b \nabla b_e^\varepsilon \cdot n = F_b(b_e^\varepsilon), \qquad D_e \nabla c_e^\varepsilon \cdot n = F_c(c_e^\varepsilon) && \text{on } (\partial\Omega)_T,\\
&\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon)\, n = F_u && \text{on } (\partial\Omega)_T,\\
&(K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t u_e^\varepsilon) \cdot n = F_p && \text{on } (\partial\Omega)_T.
\end{aligned}
$$

The elasticity and permeability tensors are defined by Y-periodic functions

$$
\mathbf{E}^\varepsilon(x,\xi) = \mathbf{E}(x/\varepsilon, \xi) \quad \text{and} \quad K_p^\varepsilon(x) = K_p(x, x/\varepsilon),
$$

where $\mathbf{E}(\cdot,\xi)$ and $K_p(x,\cdot)$ are Y-periodic for a.a. $\xi \in \mathbb{R}$ and $x \in \Omega$.

We emphasize that the diffusion coefficients D_b, D_e, and D_f in (6) are supposed to be constant just for presentation simplicity. The method developed in this paper also applies to the case of nonconstant uniformly elliptic diffusion coefficients.

We suppose the following conditions hold:

A1. Elasticity tensor $\mathbf{E}(y,\zeta) = (E_{ijkl}(y,\zeta))_{1\le i,j,k,l\le 3}$ satisfies $E_{ijkl} = E_{klij} = E_{jikl} = E_{ijlk}$ and $\alpha_1|A|^2 \le \mathbf{E}(y,\zeta)A \cdot A \le \alpha_2|A|^2$ for all symmetric matrices $A \in \mathbb{R}^{3\times 3}$, $\zeta \in \mathbb{R}_+$, and $y \in Y$, and for some α_1 and α_2 such that $0 < \alpha_1 \le \alpha_2 < \infty$.

$\mathbf{E}(y,\zeta) = \mathbf{E}_1(y, \mathcal{F}(\zeta))$, where

$$
\mathbf{E}_1 \in C_{\rm per}(Y; C_b^2(\mathbb{R})) \quad \text{and} \quad \mathcal{F}(\zeta) = \int_0^t \kappa(t-\tau)\zeta(\tau,x)d\tau,
$$

with a smooth function $\kappa : \mathbb{R}_+ \to \mathbb{R}_+$ such that $\kappa(0) = 0$, and $x \in \Omega$.

A2. $K_p \in C(\overline{\Omega}; L^\infty_{\rm per}(Y))$ and $K_p(x,y)\eta\cdot\eta \geq k_1|\eta|^2$ for $\eta\in\mathbb{R}^3$, a.a. $y\in Y$ and $x\in\Omega$, and $k_1>0$.

A3. $\mathcal{G}$ is a Lipschitz continuous function on $\mathbb{R}^3$ such that $|\mathcal{G}(r)|\leq R$ for some $R>0$ and all $r\in\mathbb{R}^3$.

A4. For functions g_b, g_e, g_f, P, F_b, and F_c we assume that

$$g_b \in C(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^6;\mathbb{R}^3), \quad g_e\in C(\mathbb{R}\times\mathbb{R}^3\times\mathbb{R}^6), \quad F_b,\, P\in C(\mathbb{R}^3;\mathbb{R}^3),$$

and F_c and g_f are Lipschitz continuous. Moreover, the following estimates hold:

$$\begin{aligned}
&|g_b(s,r,\xi)| \leq C_1(1+|s|+|r|)+C_2|r||\xi|,\\
&|g_e(s,r,\xi)| \leq C_3(1+|s|+|r|)+C_4(|s|+|r|)|\xi|,\\
&|F_b(r)|+|P(r)|\leq C(1+|r|),\\
&|F_c(s)|+|g_f(s)|\leq C(1+|s|),
\end{aligned}$$

where $s\in\mathbb{R}_+$, $r\in\mathbb{R}^3_+$, and ξ is a symmetric 3×3 matrix. Here and in what follows we identify the space of symmetric 3×3 matrices with $\mathbb{R}^6$.

It is also assumed that for any symmetric 3×3 matrix ξ we have that $g_{b,j}(s,r,\xi)$, $F_{b,j}(r)$, and $P_j(r)$ are nonnegative for $r_j=0$, $s\geq 0$, and $r_i\geq 0$, with $i=1,2,3$ and $j\neq i$, and $g_e(s,r,\xi)$, $g_f(s)$, and $F_c(s)$ are nonnegative for $s=0$ and $r_j\geq 0$, with $j=1,2,3$.

We assume also that $g_b(\cdot,\cdot,\xi)$, $g_e(\cdot,\cdot,\xi)$, F_b, and P are locally Lipschitz continuous and

$$\begin{aligned}
&|g_b(s_1,r_1,\xi_1)-g_b(s_2,r_2,\xi_2)| \leq C_1(|r_1|+|r_2|)|s_1-s_2|\\
&\quad + C_2(|s_1|+|s_2|+|\xi_1|+|\xi_2|)|r_1-r_2| + C_3(|r_1|+|r_2|)|\xi_1-\xi_2|,
\end{aligned}$$

$$\begin{aligned}
&|g_e(s_1,r_1,\xi_1)-g_e(s_2,r_2,\xi_2)| \leq C_1(|r_1|+|r_2|+|\xi_1|+|\xi_2|)|s_1-s_2|\\
&\quad + C_2(|s_1|+|s_2|+|\xi_1|+|\xi_2|)|r_1-r_2| + C_3(|r_1|+|r_2|+|s_1|+|s_2|)|\xi_1-\xi_2|
\end{aligned}$$

for $s_1,s_2\in\mathbb{R}_+$, $r_1,r_2\in\mathbb{R}^3_+$, and ξ,ξ_1,ξ_2 are symmetric 3×3 matrices.

A5. $b_{e0}\in L^\infty(\Omega)^3$, $c_0\in L^\infty(\Omega)$, and $b_{e0,j}\geq 0$, $c_0\geq 0$ a.e. in Ω, where $j=1,2,3$. $u^1_{e0}\in H^1(\Omega)^3$, $u^1_{f0}\in H^2(\Omega)^3$, and $\operatorname{div} u^1_{f0}=0$ in Ω^ε_f.
$u^\varepsilon_{e0}\in H^1(\Omega^\varepsilon_e)^3$, $p^\varepsilon_{e0}\in H^1(\Omega)$ are defined as solutions of

$$\begin{aligned}
&\operatorname{div}(\mathbf{E}^\varepsilon(b_{e0,3})\mathbf{e}(u^\varepsilon_{e0})) = f_u && \text{in } \Omega^\varepsilon_e,\\
&\Pi_\tau(\mathbf{E}^\varepsilon(b_{e0,3})\mathbf{e}(u^\varepsilon_{e0})\, n) = \varepsilon^2\mu\,\Pi_\tau(\mathbf{e}(u^1_{f0})n) && \text{on } \Gamma^\varepsilon,\\
&n\cdot\mathbf{E}^\varepsilon(b_{e0,3})\mathbf{e}(u^\varepsilon_{e0})\, n = 0 && \text{on } \Gamma^\varepsilon, \quad u^\varepsilon_{e0}=0 \quad \text{on } \partial\Omega,\\
&\operatorname{div}(K^\varepsilon_p\nabla p^\varepsilon_{e0}) = f_p && \text{in } \Omega, \quad p^\varepsilon_{e0}=0 \quad \text{on } \partial\Omega,
\end{aligned}$$

for given $f_u\in L^2(\Omega)^3$ and $f_p\in L^2(\Omega)$.
$F_p\in H^1(0,T;L^2(\partial\Omega))$, $F_u\in H^2(0,T;L^2(\partial\Omega))^3$.

Remark 2.1. Under the assumptions on u^ε_{e0} and p^ε_{e0} by the standard homogenization results, we obtain

$$\begin{aligned}
&\tilde u^\varepsilon_{e0}\to u_{e0}, \qquad p^\varepsilon_{e0}\to p_{e0} && \text{strongly in } L^2(\Omega),\\
&\mathbf{e}(u^\varepsilon_{e0})\to \mathbf{e}(u_{e0})+\mathbf{e}_y(\hat u_{e0}) && \text{strongly two-scale, } \hat u_{e0}\in L^2(\Omega;H^1(Y_e)/\mathbb{R})^3,
\end{aligned}$$

where $\tilde u^\varepsilon_{e0}$ is an extension of u^ε_{e0}, and $u_{e0}\in H^1(\Omega)^3$ and $p_{e0}\in H^1(\Omega)$ are solutions of the corresponding macroscopic (homogenized) equations.

Remark 2.2. Our approach also applies to the case when the initial velocity u^1_{f0} has the form $u^{1,\varepsilon}_{f0}(x) = U^1_{f0}(x, x/\varepsilon)$, where the vector function $U^1_{f0}(x, y)$ is periodic in y, sufficiently regular, and such that $\operatorname{div}_x U^1_{f0}(x, y) = 0$, $\operatorname{div}_y U^1_{f0}(x, y) = 0$.

Remark 2.3. The reaction terms for c^ε_f, $b^\varepsilon_{e,1}$, $b^\varepsilon_{e,2}$, $b^\varepsilon_{e,3}$, and c^ε_e can be considered in the following form:

$$
\begin{aligned}
&g_f(c^\varepsilon_f) = -\mu_2 c^\varepsilon_f, \qquad g_{b,1}(b^\varepsilon_e, c^\varepsilon_e, \mathbf{e}(u^\varepsilon_e)) = -\mu_1 b^\varepsilon_{e,1},\\
&g_{b,2}(c^\varepsilon_e, b^\varepsilon_e, \mathbf{e}(u^\varepsilon_e)) = \mu_1 b^\varepsilon_{e,1} - 2r_{dc}\frac{b^\varepsilon_{e,2}c^\varepsilon_e}{\kappa + c^\varepsilon_e} + 2R_b(b^\varepsilon_{e,3})(\mathrm{tr}\mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\mathbf{e}(u^\varepsilon_e))^+ - r_d b^\varepsilon_{e,2},\\
&g_{b,3}(c^\varepsilon_e, b^\varepsilon_e, \mathbf{e}(u^\varepsilon_e)) = r_{dc}\frac{b^\varepsilon_{e,2}c^\varepsilon_e}{\kappa + c^\varepsilon_e} - R_b(b^\varepsilon_{e,3})(\mathrm{tr}\mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\mathbf{e}(u^\varepsilon_e))^+,\\
&g_e(c^\varepsilon_e, b^\varepsilon_e, \mathbf{e}(\mathbf{u}^\varepsilon_e)) = -r_{dc}\frac{b^\varepsilon_{e,2}c^\varepsilon_e}{\kappa + c^\varepsilon_e} + R_b(b^\varepsilon_{e,3})(\mathrm{tr}\mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\mathbf{e}(u^\varepsilon_e))^+,
\end{aligned}
$$

where $\mu_1, \mu_2, r_{dc}, r_d, \kappa > 0$, and $R_b(b^\varepsilon_{e,3})$ is a Lipschitz continuous function of calcium-pectin cross-links density, e.g., $R_b(b^\varepsilon_{e,3}) = r_b b^\varepsilon_{e,3}$ with some constant $r_b > 0$. We assume that the concentration of the enzyme PME is constant, and hence methylesterified pectin is de-esterified at a constant rate. The demethylesterified pectin is produced through the de-esterification of methylesterified pectin by PME, demethylesterified pectin can decay, and through the interaction between two galacturonic acid groups of pectin chains and a calcium molecule a calcium-pectin cross-link is produced. If a cross-link breaks due to mechanical forces, we regain two acid groups of demethylesterified pectin and one calcium molecule. We consider the decay of calcium inside the cells. The positive part of the trace of the elastic stress reflects the fact that extension rather than compression causes the breakage of calcium-pectin cross-links. See [39] for more details on the derivation of a microscopic model for the biomechanics of a plant cell wall.

In what follows we use the notation $\langle \cdot, \cdot \rangle_{H^1(A)', H^1}$ for the duality product between $L^2(0, s; (H^1(A))')$ and $L^2(0, s; H^1(A))$, and

$$
\langle \phi, \psi \rangle_{A_s} = \int_0^s \int_A \phi\, \psi\, dxdt \quad \text{ for } \quad \phi \in L^q(0, s; L^p(A)) \text{ and } \psi \in L^{q'}(0, s; L^{p'}(A)),
$$

where $1/q + 1/q' = 1$ and $1/p + 1/p' = 1$ for any $s > 0$ and domain $A \subset \mathbb{R}^3$.

We also use the notation

$$
c^\varepsilon = \begin{cases} c^\varepsilon_e & \text{in } \Omega^\varepsilon_{e,T}, \\ c^\varepsilon_f & \text{in } \Omega^\varepsilon_{f,T}. \end{cases}
$$

Next we define a weak solution of the coupled system (6)–(8).

Definition 2.4. *Functions*

$$
\begin{aligned}
&u^\varepsilon_e \in \left[L^2(0, T; H^1(\Omega^\varepsilon_e)) \cap H^2(0, T; L^2(\Omega^\varepsilon_e))\right]^3,\\
&p^\varepsilon_e \in L^2(0, T; H^1(\Omega^\varepsilon_e)) \cap H^1(0, T; L^2(\Omega^\varepsilon_e)),\\
&\partial_t u^\varepsilon_f \in \left[L^2(0, T; H^1(\Omega^\varepsilon_f)) \cap H^1(0, T; L^2(\Omega^\varepsilon_f))\right]^3, \qquad p^\varepsilon_f \in L^2((0, T) \times \Omega^\varepsilon_f),\\
&\Pi_\tau \partial_t u^\varepsilon_e = \Pi_\tau \partial_t u^\varepsilon_f \quad \text{ on } \Gamma^\varepsilon_T, \qquad \operatorname{div} \partial_t u^\varepsilon_f = 0 \quad \text{in } \Omega^\varepsilon_{f,T},
\end{aligned}
$$

and

$$
\begin{aligned}
&b^\varepsilon_e \in \left[L^2(0, T; H^1(\Omega^\varepsilon_e)) \cap L^\infty(0, T; L^2(\Omega^\varepsilon_e))\right]^3,\\
&c^\varepsilon \in L^2(0, T; H^1(\Omega \setminus \widetilde{\Gamma}^\varepsilon)) \cap L^\infty(0, T; L^2(\Omega))
\end{aligned}
$$

are a weak solution of (6)–(8) *if*

(i) $(u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon, p_f^\varepsilon)$ *satisfy the integral relation*

$$\begin{aligned}
&\langle \rho_e\, \partial_t^2 u_e^\varepsilon, \phi\rangle_{\Omega^\varepsilon_{e,T}} + \big\langle \mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\mathbf{e}(u_e^\varepsilon), \mathbf{e}(\phi)\big\rangle_{\Omega^\varepsilon_{e,T}} + \langle \nabla p_e^\varepsilon, \phi\rangle_{\Omega^\varepsilon_{e,T}} \\
&+ \langle \rho_p\, \partial_t p_e^\varepsilon, \psi\rangle_{\Omega^\varepsilon_{e,T}} + \big\langle K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t u_e^\varepsilon, \nabla\psi\big\rangle_{\Omega^\varepsilon_{e,T}} + \langle \partial_t u_f^\varepsilon \cdot n, \psi\rangle_{\Gamma^\varepsilon_T} - \langle p_e^\varepsilon, \eta\cdot n\rangle_{\Gamma^\varepsilon_T} \\
&+ \langle \rho_f\, \partial_t^2 u_f^\varepsilon, \eta\rangle_{\Omega^\varepsilon_{f,T}} + \varepsilon^2 \mu \, \big\langle \mathbf{e}(\partial_t u_f^\varepsilon), \mathbf{e}(\eta)\big\rangle_{\Omega^\varepsilon_{f,T}} = \langle F_u, \phi\rangle_{(\partial\Omega)_T} + \langle F_p, \psi\rangle_{(\partial\Omega)_T}
\end{aligned} \tag{9}$$

for all $\psi \in L^2(0,T; H^1(\Omega_e^\varepsilon))$, $\phi \in L^2(0,T;H^1(\Omega_e^\varepsilon))^3$, *and* $\eta \in L^2(0,T;H^1(\Omega_f^\varepsilon))^3$, *with* $\Pi_\tau \phi = \Pi_\tau \eta$ *on* Γ^ε_T *and* $\mathrm{div}\eta = 0$ *in* $(0,T)\times\Omega_f^\varepsilon$,

(ii) $(b_e^\varepsilon, c^\varepsilon)$ *satisfy the integral relations*

$$\begin{aligned}
&\langle \partial_t b_e^\varepsilon, \varphi_1\rangle_{H^1(\Omega_e^\varepsilon)', H^1} + \langle D_b \nabla b_e^\varepsilon, \nabla \varphi_1\rangle_{\Omega^\varepsilon_{e,T}} - \langle g_b(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)), \varphi_1\rangle_{\Omega^\varepsilon_{e,T}} \\
&\qquad = \varepsilon \langle P(b_e^\varepsilon), \varphi_1\rangle_{\Gamma^\varepsilon_T} + \langle F_b(b_e^\varepsilon), \varphi_1\rangle_{(\partial\Omega)_T}
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
&\langle \partial_t c_e^\varepsilon, \varphi_2\rangle_{H^1(\Omega_e^\varepsilon)', H^1} + \langle D_e \nabla c_e^\varepsilon, \nabla\varphi_2\rangle_{\Omega^\varepsilon_{e,T}} - \langle g_e(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)), \varphi_2\rangle_{\Omega^\varepsilon_{e,T}} \\
&+ \langle \partial_t c_f^\varepsilon, \varphi_2\rangle_{H^1(\Omega_f^\varepsilon)', H^1} + \langle D_f \nabla c_f^\varepsilon - \mathcal{G}(\partial_t u_f^\varepsilon) c_f^\varepsilon, \nabla \varphi_2\rangle_{\Omega^\varepsilon_{f,T}} - \langle g_f(c_f^\varepsilon), \varphi_2\rangle_{\Omega^\varepsilon_{f,T}} \\
&\qquad = \langle F_c(c_e^\varepsilon), \varphi_2\rangle_{(\partial\Omega)_T}
\end{aligned} \tag{11}$$

for all $\varphi_1 \in L^2(0,T;H^1(\Omega_e^\varepsilon))^3$ *and* $\varphi_2 \in L^2(0,T;H^1(\Omega\setminus\widetilde{\Gamma}^\varepsilon))$,

(iii) *the corresponding initial conditions are satisfied. Namely, as* $t \to 0$, $u_e^\varepsilon(t,\cdot) \to u_{e0}^\varepsilon(\cdot)$ *and* $\partial_t u_e^\varepsilon(t,\cdot) \to u^1_{e0}(\cdot)$ *in* $L^2(\Omega_e^\varepsilon)^3$, $p_e^\varepsilon(t,\cdot)\to p_{e0}^\varepsilon(\cdot)$ *in* $L^2(\Omega_e^\varepsilon)$, $\partial_t u_f^\varepsilon(t,\cdot) \to u^1_{f0}(\cdot)$ *in* $L^2(\Omega_f^\varepsilon)^3$, $b_e^\varepsilon(t,\cdot) \to b_{e0}(\cdot)$ *in* $L^2(\Omega_e^\varepsilon)^3$, *and* $c^\varepsilon(t,\cdot)\to c_0(\cdot)$ *in* $L^2(\Omega)$.

3. A priori estimates, existence and uniqueness of a solution of the microscopic problem. We begin by proving the existence of a weak solution of the microscopic model (6)–(8) and uniform in ε a priori estimates. In order to obtain uniform in ε estimates, we shall extend H^1-functions from a perforated domain into the whole domain.

LEMMA 3.1.

- *There exist extensions* $\overline{b}_e^\varepsilon$ *and* $\overline{c}_e^\varepsilon$ *of* b_e^ε *and* c_e^ε, *respectively, from* $L^2(0,T;H^1(\Omega_e^\varepsilon))$ *to* $L^2(0,T;H^1(\Omega))$ *such that*

$$\|\overline{b}_e^\varepsilon\|_{L^2(\Omega_T)} \le C\|b_e^\varepsilon\|_{L^2(\Omega^\varepsilon_{e,T})}, \quad \|\nabla \overline{b}_e^\varepsilon\|_{L^2(\Omega_T)} \le C\|\nabla b_e^\varepsilon\|_{L^2(\Omega^\varepsilon_{e,T})}, \tag{12}$$

$$\|\overline{c}_e^\varepsilon\|_{L^2(\Omega_T)} \le C\|c_e^\varepsilon\|_{L^2(\Omega^\varepsilon_{e,T})}, \quad \|\nabla \overline{c}_e^\varepsilon\|_{L^2(\Omega_T)} \le C\|\nabla c_e^\varepsilon\|_{L^2(\Omega^\varepsilon_{e,T})}. \tag{13}$$

- *There exists an extension* $\overline{c}^\varepsilon$ *of* c^ε *from* $L^2(0,T;H^1(\widetilde{\Omega}^\varepsilon_{ef}))$ *to* $L^2(0,T;H^1(\Omega))$ *such that*

$$\|\overline{c}^\varepsilon\|_{L^2(\Omega_T)} \le C\|c^\varepsilon\|_{L^2(\widetilde{\Omega}^\varepsilon_{ef,T})}, \quad \|\nabla\overline{c}^\varepsilon\|_{L^2(\Omega_T)} \le C\|\nabla c^\varepsilon\|_{L^2(\widetilde{\Omega}^\varepsilon_{ef,T})}. \tag{14}$$

Here the constant C *is independent of* ε, *and* $\widetilde{\Omega}^\varepsilon_{ef} = \Omega\setminus\widetilde{\Omega}^\varepsilon$, *with* $\widetilde{\Omega}^\varepsilon = \bigcup_{\xi\in\Xi^\varepsilon} \varepsilon(\widetilde{\Gamma}^\delta \cap Y_e + \xi)$, *where* $\widetilde{\Gamma}^\delta$ *is a* δ*-neighborhood of* $\widetilde{\Gamma}$ *such that* $\widetilde{\Gamma}^\delta \cap \partial Y = \emptyset$ *and* $Y\setminus\overline{\widetilde{\Gamma}^\delta\cap Y_e}$ *is a connected set.*

Sketch of proof. The assumptions on the geometry of Ω_e^ε and $\widetilde{\Omega}_{ef}^\varepsilon$ and a standard extension operator (see, e.g., [1, 16]) ensure the existence of extensions of b_e^ε, c_e^ε, and c^ε satisfying estimates (12), (13), and (14), respectively. ∎

Remark. Notice that we have a jump in c^ε across $\widetilde{\Gamma}$. Thus in order to construct an extension of c^ε in $H^1(\Omega)$ we have to consider c^ε outside a δ-neighborhood of $\widetilde{\Gamma}$. Also since we would like to have an extension of c_f^ε from Ω_f^ε to Ω, we have to consider $\widetilde{\Gamma}^\delta \cap Y_e$; see Figure 1.

Notice that, since $Y_f \subset Y$ with $\partial Y_f \cap \partial Y = \emptyset$ and $\Gamma = \partial Y_f$, for $\delta > 0$ sufficiently small $\widetilde{\Gamma}^\delta$ will satisfy the assumption of the lemma.

LEMMA 3.2. *Under assumptions* **A1**–**A5**, *solutions of the microscopic problem* (6)–(8) *satisfy the following a priori estimates:*

For elastic deformation, pressures, and flow velocity we have

$$
\begin{aligned}
&\|u_e^\varepsilon\|_{L^\infty(0,T;H^1(\Omega_e^\varepsilon))} + \|\partial_t u_e^\varepsilon\|_{L^\infty(0,T;H^1(\Omega_e^\varepsilon))} + \|\partial_t^2 u_e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} \le C,\\
&\|p_e^\varepsilon\|_{L^2(0,T;H^1(\Omega_e^\varepsilon))} + \|\partial_t p_e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|\partial_t p_e^\varepsilon\|_{L^2(0,T;H^1(\Omega_e^\varepsilon))} \le C,\\
&\|\partial_t u_f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_f^\varepsilon))} + \|\partial_t^2 u_f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_f^\varepsilon))} + \varepsilon\|\nabla\partial_t u_f^\varepsilon\|_{H^1(0,T;L^2(\Omega_f^\varepsilon))}\\
&\qquad\qquad + \|p_f^\varepsilon\|_{L^2(\Omega_{f,T}^\varepsilon)} \le C.
\end{aligned}
\tag{15}
$$

For the densities we have

$$
\begin{aligned}
&b_{e,i}^\varepsilon \ge 0, \quad c_e^\varepsilon \ge 0 \quad \text{a.e. in } \Omega_{e,T}^\varepsilon, \quad c_f^\varepsilon \ge 0 \quad \text{a.e. in } \Omega_{f,T}^\varepsilon, \qquad i = 1,2,3,\\
&\|b_e^\varepsilon\|_{L^2(0,T;H^1(\Omega_e^\varepsilon))} + \varepsilon^{1/2}\|b_e^\varepsilon\|_{L^2(\Gamma_T^\varepsilon)} + \|b_e^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_e^\varepsilon))} \le C,\\
&\|c_j^\varepsilon\|_{L^2(0,T;H^1(\Omega_j^\varepsilon))} + \|c_j^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_j^\varepsilon))} + \|c_j^\varepsilon\|_{L^\infty(0,T;L^4(\Omega_j^\varepsilon))} \le C, \quad j = e,f,
\end{aligned}
\tag{16}
$$

and

$$
\|\theta_h b_e^\varepsilon - b_e^\varepsilon\|_{L^2((0,\tilde{T})\times\Omega_e^\varepsilon)} + \|\theta_h c_j^\varepsilon - c_j^\varepsilon\|_{L^2((0,\tilde{T})\times\Omega_j^\varepsilon)} \le Ch^{1/4}, \qquad j = e,f,
\tag{17}
$$

for $\tilde{T} \in (0, T-h]$, *where* $\theta_h v(t,x) = v(t+h,x)$ *for* $(t,x) \in (0,T-h] \times \Omega_j^\varepsilon$, *with* $j = e,f$, *and the constant* C *is independent of* ε.

Proof. The nonnegativity of c_e^ε, c_f^ε, and b_e^ε is justified in the proof of Theorem 3.3 on the existence and uniqueness of a weak solution of the microscopic problem (6)–(8).

To derive the estimates in (15), we first take $(\partial_t u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon)$ as test functions in (9) and obtain

$$
\begin{aligned}
&\rho_e\|\partial_t u_e^\varepsilon(s)\|_{L^2(\Omega_e^\varepsilon)}^2 + \langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon(s)), \mathbf{e}(u_e^\varepsilon(s))\rangle_{\Omega_e^\varepsilon} - \langle \partial_t\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon), \mathbf{e}(u_e^\varepsilon)\rangle_{\Omega_{e,s}^\varepsilon}\\
&+ 2\langle\nabla p_e^\varepsilon, \partial_t u_e^\varepsilon\rangle_{\Omega_{e,s}^\varepsilon} + \rho_p\|p_e^\varepsilon(s)\|_{L^2(\Omega_e^\varepsilon)}^2 + 2\langle K_p^\varepsilon\nabla p_e^\varepsilon, \nabla p_e^\varepsilon\rangle_{\Omega_{e,s}^\varepsilon} - 2\langle\partial_t u_e^\varepsilon, \nabla p_e^\varepsilon\rangle_{\Omega_{e,s}^\varepsilon}\\
&+ \rho_f\|\partial_t u_f^\varepsilon(s)\|_{L^2(\Omega_f^\varepsilon)}^2 + 2\varepsilon^2\mu\|\mathbf{e}(\partial_t u_f^\varepsilon)\|_{L^2(\Omega_{f,s}^\varepsilon)}^2\\
&= 2\langle F_u, \partial_t u_e^\varepsilon\rangle_{(\partial\Omega)_s} + 2\langle F_p, p_e^\varepsilon\rangle_{(\partial\Omega)_s} + \rho_e\|\partial_t u_e^\varepsilon(0)\|_{L^2(\Omega_e^\varepsilon)}^2\\
&+ \rho_p\|p_e^\varepsilon(0)\|_{L^2(\Omega_e^\varepsilon)}^2 + \rho_f\|\partial_t u_f^\varepsilon(0)\|_{L^2(\Omega_f^\varepsilon)}^2 + \langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon(0)), \mathbf{e}(u_e^\varepsilon(0))\rangle_{\Omega_e^\varepsilon}
\end{aligned}
$$

for $s \in (0,T]$. As was defined just after formula (5), $\Omega_{j,s}^\varepsilon := (0,s) \times \Omega_j^\varepsilon$ for $j = e,f$.

Using assumptions **A1**, **A2**, and **A5** yields

$$
\begin{aligned}
&\|\partial_t u_e^\varepsilon(s)\|^2_{L^2(\Omega_e^\varepsilon)} + \|\mathbf{e}(u_e^\varepsilon(s))\|^2_{L^2(\Omega_e^\varepsilon)} + \|\partial_t u_f^\varepsilon(s)\|^2_{L^2(\Omega_f^\varepsilon)} + \varepsilon^2\|\mathbf{e}(\partial_t u_f^\varepsilon)\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \\
&\qquad + \|p_e^\varepsilon(s)\|^2_{L^2(\Omega_f^\varepsilon)} + \|\nabla p_e^\varepsilon\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \\
&\le \delta\big[\|u_e^\varepsilon(s)\|^2_{L^2(\partial\Omega)} + \|p_e^\varepsilon\|^2_{L^2((0,s)\times\partial\Omega)}\big] + C_1\langle|\partial_t \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)|\mathbf{e}(u_e^\varepsilon), \mathbf{e}(u_e^\varepsilon)\rangle_{\Omega_{e,s}^\varepsilon} \\
&\quad + C_\delta\big[\|F_u\|^2_{L^\infty(0,s;L^2(\partial\Omega))} + \|\partial_t F_u\|^2_{L^2((0,s)\times\partial\Omega)} + \|F_p\|^2_{L^2((0,s)\times\partial\Omega)}\big] + C_2
\end{aligned}
\tag{18}
$$

for $s \in (0,T]$. Under our standing assumptions **A1** on $\mathbf{E}$, we have

$$
\|\partial_t \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\|_{L^\infty((0,T)\times\Omega_e^\varepsilon)} \le C.
$$

Applying the trace and Korn inequalities [33] and using extension properties of u_e^ε, we obtain

$$
\|u_e^\varepsilon(s)\|_{L^2(\partial\Omega)} \le C\big[\|u_e^\varepsilon(s)\|_{L^2(\Omega_e^\varepsilon)} + \|\mathbf{e}(u_e^\varepsilon(s))\|_{L^2(\Omega_e^\varepsilon)}\big]. \tag{19}
$$

Our assumptions **A5** on the initial conditions ensure

$$
\|u_e^\varepsilon(s)\|_{L^2(\Omega_e^\varepsilon)} \le \|\partial_t u_e^\varepsilon\|_{L^2(\Omega_{e,s}^\varepsilon)} + \|u_{e0}^\varepsilon\|_{L^2(\Omega_e^\varepsilon)} \le C + \|\partial_t u_e^\varepsilon\|_{L^2(\Omega_{e,s}^\varepsilon)} \tag{20}
$$

for $s \in (0,T]$. Then applying the trace and Gronwall inequalities in (18) yields the following estimate:

$$
\begin{aligned}
&\|\partial_t u_e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|\mathbf{e}(u_e^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|p_e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} \\
&+ \|\nabla p_e^\varepsilon\|_{L^2(\Omega_{e,T}^\varepsilon)} + \|\partial_t u_f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_f^\varepsilon))} + \varepsilon\|\mathbf{e}(\partial_t u_f^\varepsilon)\|_{L^2(0,T;L^2(\Omega_f^\varepsilon))} \le C,
\end{aligned}
\tag{21}
$$

where the constant C is independent of ε. Using the Korn inequality [33] for deformation and velocity, together with a scaling argument, we obtain

$$
\begin{aligned}
&\|u_e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|\nabla u_e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} \\
&\qquad \le C_1\big(\|\mathbf{e}(u_e^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|u_e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))}\big) \le C, \\
&\|\partial_t u_f^\varepsilon\|_{L^2(\Omega_{f,T}^\varepsilon)} + \varepsilon\|\nabla\partial_t u_f^\varepsilon\|_{L^2(\Omega_{f,T}^\varepsilon)} \\
&\qquad \le C_2\big(\varepsilon\|\mathbf{e}(\partial_t u_f^\varepsilon)\|_{L^2(\Omega_{f,T}^\varepsilon)} + \|\partial_t u_f^\varepsilon\|_{L^2(\Omega_{f,T}^\varepsilon)}\big) \le C.
\end{aligned}
\tag{22}
$$

Differentiating all equations in (7) with respect to time t and taking $(\partial_t^2 u_e^\varepsilon,\, \partial_t p_e^\varepsilon,\, \partial_t^2 u_f^\varepsilon)$ as test functions in the resulting equations, we obtain

$$
\begin{aligned}
&\rho_e\|\partial_t^2 u_e^\varepsilon(s)\|^2_{L^2(\Omega_e^\varepsilon)} + \langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(\partial_t u_e^\varepsilon(s)), \mathbf{e}(\partial_t u_e^\varepsilon(s))\rangle_{\Omega_e^\varepsilon} - \rho_e\|\partial_t^2 u_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)} \\
&+ \rho_p\|\partial_t p_e^\varepsilon(s)\|^2_{L^2(\Omega_e^\varepsilon)} + 2\langle K_p^\varepsilon\nabla\partial_t p_e^\varepsilon, \nabla\partial_t p_e^\varepsilon\rangle_{\Omega_{e,s}^\varepsilon} - \rho_p\|\partial_t p_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)} \\
&+ \rho_f\|\partial_t^2 u_f^\varepsilon(s)\|^2_{L^2(\Omega_f^\varepsilon)} + 2\,\varepsilon^2\mu\|\mathbf{e}(\partial_t^2 u_f^\varepsilon)\|^2_{L^2(\Omega_{f,s}^\varepsilon)} - \rho_f\|\partial_t^2 u_f^\varepsilon(0)\|^2_{L^2(\Omega_f^\varepsilon)} \\
&= \big\langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon(0))\mathbf{e}(\partial_t u_e^\varepsilon(0)), \mathbf{e}(\partial_t u_e^\varepsilon(0))\big\rangle_{\Omega_e^\varepsilon} + 2\big\langle \partial_t\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon(s)), \mathbf{e}(\partial_t u_e^\varepsilon(s))\big\rangle_{\Omega_e^\varepsilon} \\
&- 2\big\langle \partial_t\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon(0))\mathbf{e}(u_e^\varepsilon(0)), \mathbf{e}(\partial_t u_e^\varepsilon(0))\big\rangle_{\Omega_e^\varepsilon} + 2\langle \partial_t F_u, \partial_t^2 u_e^\varepsilon\rangle_{(\partial\Omega)_s} \\
&- \big\langle 2\partial_t^2\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\,\mathbf{e}(u_e^\varepsilon) + \partial_t\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\,\mathbf{e}(\partial_t u_e^\varepsilon), \mathbf{e}(\partial_t u_e^\varepsilon)\big\rangle_{\Omega_{e,s}^\varepsilon} + 2\langle \partial_t F_p, \partial_t p_e^\varepsilon\rangle_{(\partial\Omega)_s}
\end{aligned}
\tag{23}
$$

for $s \in (0,T]$. Here we used the following equality:

$$
\begin{aligned}
\big\langle \partial_t\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\,\mathbf{e}(u_e^\varepsilon), \mathbf{e}(\partial_t^2 u_e^\varepsilon)\big\rangle_{\Omega_{e,s}^\varepsilon} &= \big\langle \partial_t\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon(s))\mathbf{e}(u_e^\varepsilon(s)), \mathbf{e}(\partial_t u_e^\varepsilon(s))\big\rangle_{\Omega_e^\varepsilon} \\
&\quad - \big\langle \partial_t\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon(0))\mathbf{e}(u_e^\varepsilon(0)), \mathbf{e}(\partial_t u_e^\varepsilon(0))\big\rangle_{\Omega_e^\varepsilon} \\
- \big\langle \partial_t^2\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\,\mathbf{e}(u_e^\varepsilon) &+ \partial_t\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\,\mathbf{e}(\partial_t u_e^\varepsilon), \mathbf{e}(\partial_t u_e^\varepsilon)\big\rangle_{\Omega_{e,s}^\varepsilon}.
\end{aligned}
$$

Assumptions **A5** on the initial conditions together with the microscopic equations in (7) ensure that

$$\|\partial_t^2 u_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)} + \|\partial_t p_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)} + \|\partial_t^2 u_f^\varepsilon(0)\|^2_{L^2(\Omega_f^\varepsilon)} \le C, \tag{24}$$

where the constant C is independent of ε. To justify (24), first we consider the Galerkin approximations of u_e^ε and $\partial_t u_f^\varepsilon$ and a function ϕ^k in the corresponding finite dimensional subspace, with $\phi^k = 0$ on $\partial\Omega$ and $\operatorname{div}\phi^k = 0$ in Ω_f^ε,

$$\begin{aligned}&\langle \rho_e\, \partial_t^2 u_e^{\varepsilon,k}, \phi^k\rangle_{\Omega_e^\varepsilon} + \big\langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^{\varepsilon,k}), \mathbf{e}(\phi^k)\big\rangle_{\Omega_e^\varepsilon} + \langle \nabla p_e^{\varepsilon,k}, \phi^k\rangle_{\Omega_e^\varepsilon}\\ &+ \langle \rho_f\, \partial_t^2 u_f^{\varepsilon,k}, \phi^k\rangle_{\Omega_f^\varepsilon} + \varepsilon^2\mu\,\langle \mathbf{e}(\partial_t u_f^{\varepsilon,k}), \mathbf{e}(\phi^k)\rangle_{\Omega_f^\varepsilon} + \langle p_e^{\varepsilon,k}, \phi^k\cdot n\rangle_{\Gamma^\varepsilon} = 0.\end{aligned}$$

Taking $t \to 0$ and using the regularity of $u_e^{\varepsilon,k}$, $\partial_t u_f^{\varepsilon,k}$, and $b_{e,3}^\varepsilon$ with respect to the time variable, we obtain

$$\begin{aligned}&\langle \rho_e\, \partial_t^2 u_e^{\varepsilon,k}(0), \phi^k\rangle_{\Omega_e^\varepsilon} + \big\langle \mathbf{E}^\varepsilon(b_{e0,3})\mathbf{e}(u_e^{\varepsilon,k}(0)), \mathbf{e}(\phi^k)\big\rangle_{\Omega_e^\varepsilon} + \langle \nabla p_e^{\varepsilon,k}(0), \phi^k\rangle_{\Omega_e^\varepsilon}\\ &+ \langle \rho_f\, \partial_t^2 u_f^{\varepsilon,k}(0), \phi^k\rangle_{\Omega_f^\varepsilon} + \varepsilon^2\mu\,\langle \mathbf{e}(\partial_t u_f^{\varepsilon,k}(0)), \mathbf{e}(\phi^k)\rangle_{\Omega_f^\varepsilon} + \langle p_e^{\varepsilon,k}(0), \phi^k\cdot n\rangle_{\Gamma^\varepsilon} = 0.\end{aligned}$$

Then the integration by parts in the last two terms and the assumptions on the initial values ensure

$$\begin{aligned}&|\langle \partial_t^2 u_e^{\varepsilon,k}(0), \phi^k\rangle_{\Omega_e^\varepsilon}| + |\langle \partial_t^2 u_f^{\varepsilon,k}(0), \phi^k\rangle_{\Omega_f^\varepsilon}| \le |\big\langle f_u, \phi^k\big\rangle_{\Omega_e^\varepsilon}| + |\langle \nabla p_{e0}^{\varepsilon,k}, \phi^k\rangle_\Omega|\\ &+ \varepsilon^2\mu\,|\langle \operatorname{div}\mathbf{e}(\partial_t u_{f0}^{1,k}), \phi^k\rangle_{\Omega_f^\varepsilon}| + \varepsilon^2\mu\,\|\nabla^2\partial_t u_{f0}^{1,k}\|_{L^2(\Omega)}\|\phi^k\|_{L^2(\Omega_f^\varepsilon)} \le C\|\phi^k\|_{L^2(\Omega)},\end{aligned}$$

and hence

$$\|\partial_t^2 u_e^{\varepsilon,k}(0)\|_{L^2(\Omega_e^\varepsilon)} + \|\partial_t^2 u_f^{\varepsilon,k}(0)\|_{L^2(\Omega_f^\varepsilon)} \le C,$$

where the constant C is independent of k and $\operatorname{div}\partial_t^2 u_f^{\varepsilon,k}(0) = 0$ in Ω_f^ε. In a similar way, we also obtain the boundedness of $\|\partial_t p_e^{\varepsilon,k}(0)\|_{L^2(\Omega_e^\varepsilon)}$ uniformly in k.

Then the estimates similar to (23) for the Galerkin approximations of u_e^ε, p_e^ε, and $\partial_t u_f^\varepsilon$ imply that $p_e^\varepsilon \in C([0,T]; L^2(\Omega_e^\varepsilon))$, $\nabla p_e^\varepsilon, \partial_t u_e^\varepsilon \in C([0,T]; L^2(\Omega_e^\varepsilon))^3$, $\mathbf{e}(u_e^\varepsilon) \in C([0,T]; L^2(\Omega_e^\varepsilon))^{3\times 3}$, $\partial_t u_f^\varepsilon \in C([0,T]; L^2(\Omega_f^\varepsilon))^3$, $\mathbf{e}(\partial_t u_f^\varepsilon) \in C([0,T]; L^2(\Omega_f^\varepsilon))^{3\times 3}$.

Then from the equations for u_e^ε and p_e^ε and the continuity of $\mathbf{e}(u_e^\varepsilon)$, $\partial_t u_e^\varepsilon$, and ∇p_e^ε with respect to the time variable, we obtain the continuity of $\partial_t^2 u_e^\varepsilon$ and $\partial_t p_e^\varepsilon$ with respect to the time variable. Then the assumptions on u_{e0}^ε, u_{e0}^1, and p_{e0}^ε ensure the boundedness of $\|\partial_t^2 u_e^\varepsilon(0)\|_{L^2(\Omega_e^\varepsilon)}$ and $\|\partial_t p_e^\varepsilon(0)\|_{L^2(\Omega_e^\varepsilon)}$ uniformly in ε.

For $\phi \in H_0^1(\Omega)$, with $\operatorname{div}\phi = 0$ in Ω_f^ε, we have

$$\begin{aligned}&\langle \rho_e\, \partial_t^2 u_e^\varepsilon, \phi\rangle_{\Omega_e^\varepsilon} + \big\langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon), \mathbf{e}(\phi)\big\rangle_{\Omega_e^\varepsilon} + \langle \nabla p_e^\varepsilon, \phi\rangle_{\Omega_e^\varepsilon}\\ &+ \langle \rho_f \partial_t^2 u_f^\varepsilon, \phi\rangle_{\Omega_f^\varepsilon} + \varepsilon^2\mu\,\langle \mathbf{e}(\partial_t u_f^\varepsilon), \mathbf{e}(\phi)\rangle_{\Omega_f^\varepsilon} + \langle p_e^\varepsilon, \phi\cdot n\rangle_{\Gamma^\varepsilon} = 0.\end{aligned}$$

Considering the continuity of $\mathbf{e}(u_e^\varepsilon)$, $\partial_t^2 u_e^\varepsilon$, ∇p_e^ε, and $\mathbf{e}(\partial_t u_f^\varepsilon)$ with respect to the time variable and taking $t \to 0$, we obtain the continuity of $\partial_t^2 u_f^\varepsilon$ and

$$\begin{aligned}&\langle \rho_e\, \partial_t^2 u_e^\varepsilon(0), \phi\rangle_{\Omega_e^\varepsilon} + \langle \mathbf{E}^\varepsilon(b_{e0,3})\mathbf{e}(u_{e0}^\varepsilon), \mathbf{e}(\phi)\rangle_{\Omega_e^\varepsilon} + \langle \nabla p_{e0}^\varepsilon, \phi\rangle_{\Omega_e^\varepsilon}\\ &+ \langle \rho_f\, \partial_t^2 u_f^\varepsilon(0), \phi\rangle_{\Omega_f^\varepsilon} + \varepsilon^2\mu\,\langle \mathbf{e}(u_{f0}^1), \mathbf{e}(\phi)\rangle_{\Omega_f^\varepsilon} + \langle p_{e0}^\varepsilon, \phi\cdot n\rangle_{\Gamma^\varepsilon} = 0.\end{aligned}$$

The integration by parts, the boundary conditions for u^ε_{e0}, and the assumptions on ϕ imply

$$\langle \rho_f \partial_t^2 u_f^\varepsilon(0), \phi\rangle_{\Omega_f^\varepsilon} = -\langle \rho_e\, \partial_t^2 u_e^\varepsilon(0), \phi\rangle_{\Omega_e^\varepsilon} + \langle \operatorname{div}(\mathbf{E}^\varepsilon(b_{e0,3})\mathbf{e}(u^\varepsilon_{e0})), \phi\rangle_{\Omega_e^\varepsilon} - \langle \nabla p^\varepsilon_{e0}, \phi\rangle_\Omega \\ + \langle \varepsilon^2 \mu \operatorname{div}(\mathbf{e}(u^1_{f0})), \phi\rangle_{\Omega_f^\varepsilon} - \varepsilon^2\langle \mu\, n\cdot \mathbf{e}(u^1_{f0})n, \phi\cdot n\rangle_{\Gamma^\varepsilon}.$$

From the assumptions on u^ε_{e0} and p^ε_{e0} we have that $\operatorname{div}(\mathbf{E}^\varepsilon(b_{e0,3})\mathbf{e}(u^\varepsilon_{e0})) = f_u$, with $f_u \in L^2(\Omega)$, and $\|\nabla p^\varepsilon_{e0}\|_{L^2(\Omega)} \le C_1$, where C_1 is independent of ε. The assumptions on u^1_{f0} ensure that $\varepsilon^2\mu\|\operatorname{div}(\mathbf{e}(u^1_{f0}))\|_{L^2(\Omega_f^\varepsilon)} \le C_2$ and there exists $\psi^\varepsilon \in H^1(\Omega_f^\varepsilon)$, such that $\|\nabla\psi^\varepsilon\|_{L^2(\Omega_f^\varepsilon)} \le C_3$ and

$$|\varepsilon^2\langle \mu\, n\cdot \mathbf{e}(u^1_{f0})n, \phi\cdot n\rangle_{\Gamma^\varepsilon}| = |\langle \nabla\psi^\varepsilon, \phi\rangle_{\Omega_f^\varepsilon}| \le C_4\|\phi\|_{L^2(\Omega_f^\varepsilon)},$$

where the constants C_2, C_3, and C_4 are independent of ε. Using the density of ϕ in $\mathcal{H} = \{v \in L^2(\Omega_f^\varepsilon) : \operatorname{div} v = 0 \ \text{ in } \Omega_f^\varepsilon\}$, we obtain the boundedness of $\partial_t^2 u_f^\varepsilon(0)$ in $\mathcal{H}$ uniformly in ε.

Then considering assumptions **A1**–**A2** and applying the Hölder and Gronwall inequalities in (23), we obtain the estimates for $\partial_t^2 u_e^\varepsilon$, $\partial_t p_e^\varepsilon$, and $\partial_t^2 u_f^\varepsilon$ stated in (15). Here we used the fact that assumptions **A1** on **E** imply the following upper bound:

$$\|\partial_t^2 \mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\|_{L^\infty((0,T)\times\Omega_e^\varepsilon)} \le C.$$

Testing the first and third equations in (7) with $\phi \in L^2(0,T;H^1(\Omega))^3$ and using the a priori estimates for u_e^ε, p_e^ε, and $\partial_t u_f^\varepsilon$, we obtain

$$\begin{aligned}\langle p_f^\varepsilon, \operatorname{div}\phi\rangle_{\Omega^\varepsilon_{f,T}} + \langle p_e^\varepsilon, \operatorname{div}\phi\rangle_{\Omega^\varepsilon_{e,T}} &= \langle \varepsilon^2\mu\,\mathbf{e}(\partial_t u_f^\varepsilon), \mathbf{e}(\phi)\rangle_{\Omega^\varepsilon_{f,T}} + \rho_f\langle \partial_t^2 u_f^\varepsilon, \phi\rangle_{\Omega^\varepsilon_{f,T}} \\ &+ \rho_e\langle \partial_t^2 u_e^\varepsilon, \phi\rangle_{\Omega^\varepsilon_{e,T}} + \langle \mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\mathbf{e}(u_e^\varepsilon), \mathbf{e}(\phi)\rangle_{\Omega^\varepsilon_{e,T}} + \langle p_e^\varepsilon n - F_u, \phi\rangle_{(\partial\Omega)_T} \\ &\le C\|\phi\|_{L^2(0,T;H^1(\Omega))^3}.\end{aligned} \tag{25}$$

Here we used the properties of an extension of p_e^ε from Ω_e^ε to Ω (see Lemma 3.1) and the trace estimate $\|p_e^\varepsilon\|_{L^2((0,T)\times\partial\Omega)} \le C_1\|p_e^\varepsilon\|_{L^2(0,T;H^1(\Omega))} \le C_2\|p_e^\varepsilon\|_{L^2(0,T;H^1(\Omega_e^\varepsilon))}$.

For any $q \in L^2(\Omega_T)$ there exists $\phi \in L^2(0,T;H^1(\Omega))^3$ satisfying $\operatorname{div}\phi = q$ in Ω, $\phi\cdot n = \frac{1}{|\partial\Omega|}\int_\Omega q(\cdot,x)dx$ on $\partial\Omega$, and $\|\phi\|_{L^2(0,T;H^1(\Omega))^3} \le C\|q\|_{L^2(\Omega_T)}$. Thus for

$$\widetilde{p^\varepsilon} = \begin{cases} p_f^\varepsilon & \text{in } \ (0,T)\times\Omega_f^\varepsilon, \\ p_e^\varepsilon & \text{in } \ (0,T)\times(\Omega\setminus\Omega_f^\varepsilon) \end{cases}$$

using (25) we obtain

$$\langle \widetilde{p^\varepsilon}, q\rangle_{\Omega_T} \le C\|q\|_{L^2((0,T)\times\Omega)},$$

where the constant C is independent of ε. This implies, by the definition of the L^2-norm and the estimates for p_e^ε, that $\|p_f^\varepsilon\|_{L^2((0,T)\times\Omega_f^\varepsilon)} \le C$.

To justify estimates (16) we take b_e^ε and c^ε as test functions in (10) and (11), respectively. Using assumptions **A3**–**A5**, we obtain

$$\begin{aligned}&\|b_e^\varepsilon(s)\|^2_{L^2(\Omega_e^\varepsilon)} + \|\nabla b_e^\varepsilon\|^2_{L^2(\Omega^\varepsilon_{e,s})} \\ &\quad\le \|b_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)} + C_1\|\mathbf{e}(u_e^\varepsilon)\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}\|b_e^\varepsilon\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))} \\ &\quad+ C_2\big[\|c_e^\varepsilon\|^2_{L^2(\Omega^\varepsilon_{e,s})} + \|b_e^\varepsilon\|^2_{L^2(\Omega^\varepsilon_{e,s})}\big] + C_3\big[1 + \varepsilon\|b_e^\varepsilon\|^2_{L^2(\Gamma^\varepsilon_s)} + \|b_e^\varepsilon\|^2_{L^2((0,s)\times\partial\Omega)}\big]\end{aligned}$$

and

$$
\begin{aligned}
&\|c_e^\varepsilon(s)\|^2_{L^2(\Omega_e^\varepsilon)} + \|c_f^\varepsilon(s)\|^2_{L^2(\Omega_f^\varepsilon)} + \|\nabla c_e^\varepsilon\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + \|\nabla c_f^\varepsilon\|^2_{L^2(\Omega_{f,s}^\varepsilon)} \\
&\quad \le \|c_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)} + \|c_f^\varepsilon(0)\|^2_{L^2(\Omega_f^\varepsilon)} \\
&\quad + C_1\|\mathbf{e}(u_e^\varepsilon)\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}\big[\|b_e^\varepsilon\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))} + \|c_e^\varepsilon\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))}\big] \\
&\quad + C_2\big[1 + \|\mathcal{G}(\partial_t u_f^\varepsilon)\|^2_{L^\infty(\Omega_{f,s}^\varepsilon)}\big]\|c_f^\varepsilon\|^2_{L^2(\Omega_{f,s}^\varepsilon)} \\
&\quad + C_3\big[\|b_e^\varepsilon\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + \|c_e^\varepsilon\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + \|c_e^\varepsilon\|^2_{L^2((0,s)\times\partial\Omega)}\big].
\end{aligned}
$$

The Gagliardo–Nirenberg and trace inequalities, together with the extension properties of b_e^ε and c^ε (see Lemma 3.1), yield

$$
\begin{aligned}
&\|b_e^\varepsilon\|^2_{L^4(\Omega_e^\varepsilon)} \le \|b_e^\varepsilon\|^2_{L^4(\Omega)} \le \delta_1\|\nabla b_e^\varepsilon\|^2_{L^2(\Omega)} + C_{\delta_1}\|b_e^\varepsilon\|^2_{L^2(\Omega)} \\
&\qquad\qquad\qquad\qquad\quad \le \delta_2\|\nabla b_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)} + C_{\delta_2}\|b_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)}, \\
&\|c_e^\varepsilon\|^2_{L^4(\Omega_e^\varepsilon)} + \|c_f^\varepsilon\|^2_{L^4(\Omega_f^\varepsilon)} \le \delta\left[\|\nabla c_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)} + \|\nabla c_f^\varepsilon\|^2_{L^2(\Omega_f^\varepsilon)}\right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad + C_\delta\left[\|c_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)} + \|c_f^\varepsilon\|^2_{L^2(\Omega_f^\varepsilon)}\right], \\
&\|b_e^\varepsilon\|^2_{L^2(\partial\Omega)} \le \delta\|\nabla b_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)} + C_\delta\|b_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)}, \\
&\|c_e^\varepsilon\|^2_{L^2(\partial\Omega)} \le \delta\|\nabla c_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)} + C_\delta\|c_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)}, \\
&\varepsilon\|b_e^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \le C\big[\varepsilon^2\|\nabla b_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)} + \|b_e^\varepsilon\|^2_{L^2(\Omega_e^\varepsilon)}\big]
\end{aligned}
\tag{26}
$$

for an arbitrary $\delta > 0$, and C_δ depending on δ and independent of ε. Notice that since the Gagliardo–Nirenberg inequality is applied to the extension of b_e^ε and c^ε defined in Ω, the constant in the Gagliardo–Nirenberg inequality is independent of ε. Then applying the Gronwall inequality and using the assumptions **A3** on $\mathcal{G}$ yields

$$
\begin{aligned}
&\|b_e^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|\nabla b_e^\varepsilon\|_{L^2((0,T)\times\Omega_e^\varepsilon)} \le C, \\
&\|c_j^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_j^\varepsilon))} + \|\nabla c_j^\varepsilon\|_{L^2((0,T)\times\Omega_j^\varepsilon)} \le C, \quad j = e, f.
\end{aligned}
\tag{27}
$$

The uniform boundedness of b_e^ε, i.e.,

$$
\|b_e^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_e^\varepsilon))} \le C,
\tag{28}
$$

with a constant C independent of ε, is proved by applying the Alikakos iteration lemma [2, Lemma 3.2]. Since the derivation of estimate (28) is rather involved, we present the detailed proof of this estimate in the appendix; see Lemma 10.1. In the same lemma in the appendix we also prove the estimate

$$
\|c_e^\varepsilon\|_{L^\infty(0,T;L^4(\Omega_e^\varepsilon))} + \|c_f^\varepsilon\|_{L^\infty(0,T;L^4(\Omega_f^\varepsilon))} \le C,
$$

where the constant C does not depend on ε.

To justify the last estimate (17), we integrate the equation for b_e^ε in (6) over

$(t, t+h)$ and consider $\theta_h b_e^\varepsilon - b_e^\varepsilon$ as a test function:

$$\begin{aligned}
&\|\theta_h b_e^\varepsilon - b_e^\varepsilon\|^2_{L^2((0,\tilde{T})\times\Omega_e^\varepsilon)} + \left\langle D_b \int_t^{t+h} \nabla b_e^\varepsilon(s)\, ds, \nabla(\theta_h b_e^\varepsilon) - \nabla b_e^\varepsilon \right\rangle_{(0,\tilde{T})\times\Omega_e^\varepsilon} \\
&\quad = \left\langle \int_t^{t+h} g_b(b_e^\varepsilon(s), c_e^\varepsilon(s), \mathbf{e}(u_e^\varepsilon(s)))\, ds, \theta_h b_e^\varepsilon - b_e^\varepsilon \right\rangle_{(0,\tilde{T})\times\Omega_e^\varepsilon} \\
&\quad + \varepsilon \left\langle \int_t^{t+h} P(b_e^\varepsilon(s))\, ds, \theta_h b_e^\varepsilon - b_e^\varepsilon \right\rangle_{(0,\tilde{T})\times\Gamma^\varepsilon} \\
&\quad + \left\langle \int_t^{t+h} F_b(b_e^\varepsilon)\, ds, \theta_h b_e^\varepsilon - b_e^\varepsilon \right\rangle_{(0,\tilde{T})\times\partial\Omega}
\end{aligned}$$

for all $\tilde{T} \in (0, T-h]$. Then using the a priori estimates for u_e^ε, b_e^ε, and c_e^ε in (15) and (16) together with the Hölder inequality implies the estimate for $b_e^\varepsilon(t+h, x) - b_e^\varepsilon(t, x)$. Similar calculations yield the estimates for $c_e^\varepsilon(t+h, x) - c_e^\varepsilon(t, x)$ and $c_f^\varepsilon(t+h, x) - c_f^\varepsilon(t, x)$. ∎

THEOREM 3.3. *Under assumptions* **A1**–**A5**, *for every* $\varepsilon > 0$ *there exists a unique weak solution of the coupled problem* (6)–(8).

Proof. We shall use a contraction argument to show the existence of a solution of the coupled system. We consider an operator $\mathcal{K}$ over $L^\infty(0, s; H^1(\Omega_e^\varepsilon)^3) \times L^\infty(0, s; L^2(\Omega_f^\varepsilon)^3)$ defined by $(u_e^{\varepsilon,j}, \partial_t u_f^{\varepsilon,j}) = \mathcal{K}(u_e^{\varepsilon,j-1}, \partial_t u_f^{\varepsilon,j-1})$, where for given $(u_e^{\varepsilon,j-1}, \partial_t u_f^{\varepsilon,j-1})$ we first define $(b_e^{\varepsilon,j}, c_e^{\varepsilon,j}, c_f^{\varepsilon,j})$ as a solution of system (6) with functions $(u_e^{\varepsilon,j-1}, \partial_t u_f^{\varepsilon,j-1})$ in place of $(u_e^\varepsilon, \partial_t u_f^\varepsilon)$ and with external boundary conditions in (8), and then $(u_e^{\varepsilon,j}, p_e^{\varepsilon,j}, \partial_t u_f^{\varepsilon,j}, p_f^{\varepsilon,j})$ are solutions of (7) with $b_e^{\varepsilon,j}$ in place of b_e^ε.

For each $j = 2, 3, \dots$, the proof of existence and uniqueness of $(b_e^{\varepsilon,j}, c_e^{\varepsilon,j}, c_f^{\varepsilon,j})$ for given $(u_e^{\varepsilon,j-1}, \partial_t u_f^{\varepsilon,j-1})$ follows the same arguments (with a number of simplifications) as the proof that $\mathcal{K}$ is a contraction for $(u_e^{\varepsilon,j}, \partial_t u_f^{\varepsilon,j})$, i.e., using the Galerkin method and fixed-point arguments. Notice that the fixed-point argument for the system for $b_e^{\varepsilon,j}$ and $c^{\varepsilon,j}$ allows us to consider the equations for $b_e^{\varepsilon,j}$ and $c^{\varepsilon,j}$ recursively. Thus using the nonnegativity of initial data b_{e0}, c_{e0}, and c_{f0} and assumptions **A4** on the reaction and boundary terms and applying iteratively the theorem on positively invariant regions [40, 47], we obtain the nonnegativity of all components of $b_e^{\varepsilon,j}$ and $c^{\varepsilon,j}$.

We choose the first iteration $(u_e^{\varepsilon,1}, p_e^{\varepsilon,1}, \partial_t u_f^{\varepsilon,1}, p_f^{\varepsilon,1})$ to satisfy the initial and boundary conditions in (7) and (8). Then applying the Galerkin method (using the basis functions for $H^1(\Omega_e^\varepsilon) \times H^1(\Omega \setminus \widetilde{\Gamma}^\varepsilon)$) and fixed-point argument, we obtain the existence of solutions $(b_e^{\varepsilon,2}, c_e^{\varepsilon,2}, c_f^{\varepsilon,2})$ of system (6) with external boundary conditions in (8) and have

$$\begin{aligned}
&\|b_e^{\varepsilon,2}\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|\nabla b_e^{\varepsilon,2}\|_{L^2(\Omega_{e,T}^\varepsilon)} + \|b_e^{\varepsilon,2}\|_{L^\infty(0,T;L^\infty(\Omega_e^\varepsilon))} \le C, \\
&\|c_l^{\varepsilon,2}\|_{L^\infty(0,T;L^2(\Omega_l^\varepsilon))} + \|\nabla c_l^{\varepsilon,2}\|_{L^2(\Omega_{l,T}^\varepsilon)} + \|c_l^{\varepsilon,2}\|_{L^\infty(0,T;L^4(\Omega_l^\varepsilon))} \le C, \quad l = e, f,
\end{aligned} \tag{29}$$

where the constant C depends only on $\|\mathbf{e}(u_e^{\varepsilon,1})\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))}$ and the constants in assumptions **A1**–**A4**. The estimates (29) can be justified in the same way as those in (16).

Next we consider system (7) with $b_e^{\varepsilon,2}$ in place of b_e^ε. To show the existence result we use the Galerkin method with the basis functions $\{\phi_j, \psi_j, \eta_j\}_{j\in\mathbb{N}}$ for the space

$$\begin{aligned} W = \{(v,p,w) \in H^1(\Omega_e^\varepsilon)^3 \times H^1(\Omega_e^\varepsilon) \times H^1(\Omega_f^\varepsilon)^3 : \ \operatorname{div} w = 0 \ \text{ in } \ \Omega_f^\varepsilon, \\ \Pi_\tau v = \Pi_\tau w \ \text{ on } \ \Gamma^\varepsilon, \quad \operatorname{div}(K_p^\varepsilon \nabla p) \in L^2(\Omega_e^\varepsilon), \\ \langle (v - K_p^\varepsilon \nabla p - w)\cdot n, \psi\rangle_{H^{-1/2}(\Gamma^\varepsilon), H^{1/2}(\Gamma^\varepsilon)} = 0\}, \end{aligned}$$

and consider the approximate solutions in the form

$$u_{e,k}^{\varepsilon,2} = \sum_{j=1}^k q_j^k(t)\phi_j, \quad p_{e,k}^{\varepsilon,2} = \sum_{j=1}^k \frac{d}{dt} q_j^k(t)\psi_j, \quad \partial_t u_{f,k}^{\varepsilon,2} = \sum_{j=1}^k \frac{d}{dt} q_j^k(t)\eta_j, \quad k \in \mathbb{N}.$$

The linearity of equations for $(u_e^{\varepsilon,2},\, p_e^{\varepsilon,2},\, \partial_t u_f^{\varepsilon,2})$ ensures the existence of unique solutions $q_j^k(t)$ of the corresponding linear system of second order ordinary differential equations with initial conditions $q_j^k(0) = \alpha_j^k$ and $\frac{d}{dt}q_j^k(0) = \beta_j^k$, where α_j^k and β_j^k are derived from the initial conditions in (7), and hence, the existence of a unique solution $(u_{e,k}^{\varepsilon,2}, p_{e,k}^{\varepsilon,2}, \partial_t u_{f,k}^{\varepsilon,2})$ for $k \in \mathbb{N}$. Then using the a priori estimates derived in the same way as in Lemma 3.2 (by considering assumptions **A1**, **A2**, and **A5**) and taking the limit as $k \to \infty$, we obtain the existence of $u_e^{\varepsilon,2} \in [H^1(0,T;H^1(\Omega_e^\varepsilon)) \cap H^2(0,T;L^2(\Omega_e^\varepsilon))]^3$, $p_e^{\varepsilon,2} \in H^1(0,T;H^1(\Omega_e^\varepsilon))$, and $\partial_t u_f^{\varepsilon,2} \in H^1(0,T;H^1(\Omega_f^\varepsilon))^3 \cap L^2(0,T;V)$, with $V = \{v \in H^1(\Omega_f^\varepsilon)^3 : \operatorname{div} v = 0 \text{ in } \Omega_f^\varepsilon\}$, satisfying (9) with $b_{e,3}^{\varepsilon,2}$ in place of $b_{e,3}^\varepsilon$. Taking $\psi \in L^2(0,T;H_0^1(\Omega_e^\varepsilon))$, $\phi \in L^2(0,T;H_0^1(\Omega_e^\varepsilon))^3$, and $\eta \in L^2(0,T;V_0)$, where $V_0 = \{v \in H_0^1(\Omega_f^\varepsilon)^3 : \operatorname{div} v = 0 \text{ in } \Omega_f^\varepsilon\}$, as test functions in the weak formulation, we obtain the equations for $u_e^{\varepsilon,2}$ and $p_e^{\varepsilon,2}$ in (7) and $\langle \rho_f \partial_t^2 u_f^{\varepsilon,2} - \varepsilon^2\mu \operatorname{div}(\mathbf{e}(\partial_t u_f^{\varepsilon,2})), \eta\rangle = 0$ for any $\eta \in L^2(0,T;V_0)$. Then De Rham's theorem applied to $-\rho_f \partial_t^2 u_f^{\varepsilon,2} + \varepsilon^2\mu \operatorname{div}(\mathbf{e}(\partial_t u_f^{\varepsilon,2}))$ implies the existence of $p_f^{\varepsilon,2} \in L^2((0,T)\times\Omega_f^\varepsilon)$ such that $-\rho_f \partial_t^2 u_f^{\varepsilon,2} + \varepsilon^2\mu \operatorname{div}(\mathbf{e}(\partial_t u_f^{\varepsilon,2})) = \nabla p_f^{\varepsilon,2}$. Using first $\psi = 0$, $\phi = 0$, and $\eta \in L^2(0,T;H^1(\Omega_f^\varepsilon))^3$, with $\Pi_\tau \eta = 0$ on $(0,T)\times\Gamma^\varepsilon$, as a test function in the weak formulation of the equations for $(u_e^{\varepsilon,2},\, p_e^{\varepsilon,2},\, \partial_t u_f^{\varepsilon,2})$ we obtain the transmission condition $-n\cdot\varepsilon^2\mu\,\mathbf{e}(\partial_t u_f^{\varepsilon,2})\,n + p_f^{\varepsilon,2} = p_e^{\varepsilon,2}$ on $(0,T)\times\Gamma^\varepsilon$, satisfied in the distribution sense. Choosing $\psi = 0$, $\phi \in L^2(0,T;H^1(\Omega_e^\varepsilon))^3$, and $\eta \in L^2(0,T;H^1(\Omega_f^\varepsilon))^3$, with $\phi = \eta$ on $(0,T)\times\Gamma^\varepsilon$, as test functions and using the equations for $u_e^{\varepsilon,2}$ and $\partial_t u_f^{\varepsilon,2}$ ensure $(\varepsilon^2\mu\,\mathbf{e}(\partial_t u_f^{\varepsilon,2}) - p_f^{\varepsilon,2} I)n = (\mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,2})\mathbf{e}(u_e^{\varepsilon,2}) - p_e^{\varepsilon,2} I)n$ on $(0,T)\times\Gamma^\varepsilon$. Then, using the equations for $u_e^{\varepsilon,2}$, $p_e^{\varepsilon,2}$, and $\partial_t u_f^{\varepsilon,2}$ and considering $\psi \in L^2(0,T;H^1(\Omega_e^\varepsilon))$, $\phi \in L^2(0,T;H^1(\Omega_e^\varepsilon))^3$, and $\eta \in L^2(0,T;H^1(\Omega_f^\varepsilon))^3$, with $\Pi_\tau\phi = \Pi_\tau\eta$ on $(0,T)\times\Gamma^\varepsilon$ and $\psi = 0$, $\phi = 0$ on $(0,T)\times\partial\Omega$, as test functions, we obtain the transmission condition $(-K_p^\varepsilon \nabla p_e^{\varepsilon,2} + \partial_t u_e^{\varepsilon,2})\cdot n = \partial_t u_f^{\varepsilon,2}\cdot n$ on $(0,T)\times\Gamma^\varepsilon$ in the distribution sense. Taking $\psi \in L^2(0,T;H^1(\Omega_e^\varepsilon))$, $\phi \in L^2(0,T;H^1(\Omega_e^\varepsilon))^3$, and $\eta \in L^2(0,T;H^1(\Omega_f^\varepsilon))^3$, with $\Pi_\tau\phi = \Pi_\tau\eta$ on $(0,T)\times\Gamma^\varepsilon$, as test functions, we obtain the boundary conditions on $(0,T)\times\partial\Omega$. Hence we obtain that $(u_e^{\varepsilon,2},\, p_e^{\varepsilon,2},\, \partial_t u_f^{\varepsilon,2}, p_f^{\varepsilon,2})$ is a weak solution of (7), with $b_{e,3}^{\varepsilon,2}$ in place of $b_{e,3}^\varepsilon$, together with the corresponding external boundary conditions in (8). Standard arguments pertaining to the consideration of two solutions of (7) imply the uniqueness of a weak solution of (7), (8). The transmission condition $-n\cdot\varepsilon^2\mu\,\mathbf{e}(\partial_t u_f^{\varepsilon,2})\,n + p_f^{\varepsilon,2} = p_e^{\varepsilon,2}$ on $(0,T)\times\Gamma^\varepsilon$ ensures that $p_f^{\varepsilon,2}$ is defined uniquely.

Also, we obtain that the estimates similar to (15) are valid for the functions $(u_e^{\varepsilon,2},\, p_e^{\varepsilon,2},\, \partial_t u_f^{\varepsilon,2},\, p_f^{\varepsilon,2})$ uniformly with respect to solutions of (6) with boundary conditions in (8):

$$
\begin{aligned}
&\|\partial_t u_e^{\varepsilon,2}\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|\nabla u_e^{\varepsilon,2}\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} \le C,\\
&\|p_e^{\varepsilon,2}\|_{L^\infty(0,T;L^2(\Omega_e^\varepsilon))} + \|\nabla p_e^{\varepsilon,2}\|_{L^2(\Omega_{e,T}^\varepsilon)} \le C,\\
&\|\partial_t u_f^{\varepsilon,2}\|_{L^\infty(0,T;L^2(\Omega_f^\varepsilon))} + \varepsilon\|\nabla \partial_t u_f^{\varepsilon,2}\|_{L^2(\Omega_{f,T}^\varepsilon)} + \|p_f^{\varepsilon,2}\|_{L^2(\Omega_{f,T}^\varepsilon)} \le C.
\end{aligned}
\tag{30}
$$

Iterating this step, we conclude the existence of a solution $(b_e^{\varepsilon,j},\, c_e^{\varepsilon,j},\, c_f^{\varepsilon,j})$ of (6) with $(u_e^{\varepsilon,j-1}, \partial_t u_f^{\varepsilon,j-1})$ instead of $(u_e^\varepsilon, \partial_t u_f^\varepsilon)$ and a solution $(u_e^{\varepsilon,j},\, p_e^{\varepsilon,j},\, \partial_t u_f^{\varepsilon,j},\, p_f^{\varepsilon,j})$ of system (7) with $b_e^{\varepsilon,j}$ instead of b_e^ε, and that the estimates similar to (29) and (30) are fulfilled for $(b_e^{\varepsilon,j},\, c_e^{\varepsilon,j},\, c_f^{\varepsilon,j})$ and $(u_e^{\varepsilon,j},\, p_e^{\varepsilon,j},\, \partial_t u_f^{\varepsilon,j},\, p_f^{\varepsilon,j})$, with $j \ge 2$.

To show the contraction property of $\mathcal{K}$, we consider two iterations

$$
(b_e^{\varepsilon,j-1}, c_e^{\varepsilon,j-1}, c_f^{\varepsilon,j-1}),\ (\partial_t u_e^{\varepsilon,j-2}, \partial_t u_f^{\varepsilon,j-2}) \ \text{ and } \ (b_e^{\varepsilon,j}, c_e^{\varepsilon,j}, c_f^{\varepsilon,j}),\ (\partial_t u_e^{\varepsilon,j-1}, \partial_t u_f^{\varepsilon,j-1}).
$$

Then the differences $\widetilde b_e^{\varepsilon,j} = b_e^{\varepsilon,j-1} - b_e^{\varepsilon,j}$, $\widetilde c_e^{\varepsilon,j} = c_e^{\varepsilon,j-1} - c_e^{\varepsilon,j}$, and $\widetilde c_f^{\varepsilon,j} = c_f^{\varepsilon,j-1} - c_f^{\varepsilon,j}$ satisfy the following equations:

$$
\begin{aligned}
&\partial_t \widetilde b_e^{\varepsilon,j} - \operatorname{div}(D_b \nabla \widetilde b_e^{\varepsilon,j})\\
&\qquad = g_b(c_e^{\varepsilon,j-1}, b_e^{\varepsilon,j-1}, \mathbf{e}(u_e^{\varepsilon,j-2})) - g_b(c_e^{\varepsilon,j}, b_e^{\varepsilon,j}, \mathbf{e}(u_e^{\varepsilon,j-1})) && \text{in } \Omega_{e,T}^\varepsilon,\\
&\partial_t \widetilde c_e^{\varepsilon,j} - \operatorname{div}(D_c \nabla \widetilde c_e^{\varepsilon,j})\\
&\qquad = g_e(c_e^{\varepsilon,j-1}, b_e^{\varepsilon,j-1}, \mathbf{e}(u_e^{\varepsilon,j-2})) - g_e(c_e^{\varepsilon,j}, b_e^{\varepsilon,j}, \mathbf{e}(u_e^{\varepsilon,j-1})) && \text{in } \Omega_{e,T}^\varepsilon,\\
&\partial_t \widetilde c_f^{\varepsilon,j} - \operatorname{div}\big(D_f \nabla \widetilde c_f^{\varepsilon,j} - \mathcal{G}(\partial_t u_f^{\varepsilon,j-2})\widetilde c_f^{\varepsilon,j}\big)\\
&\qquad + \operatorname{div}\big(c_f^{\varepsilon,j}\big[\mathcal{G}(\partial_t u_f^{\varepsilon,j-2}) - \mathcal{G}(\partial_t u_f^{\varepsilon,j-1})\big]\big)\\
&\qquad = g_f(c_f^{\varepsilon,j-1}) - g_f(c_f^{\varepsilon,j}) && \text{in } \Omega_{f,T}^\varepsilon,
\end{aligned}
\tag{31}
$$

together with the boundary conditions

$$
\begin{aligned}
&D_b \nabla \widetilde b_e^{\varepsilon,j}\cdot n = \varepsilon\big(P(b_e^{\varepsilon,j-1}) - P(b_e^{\varepsilon,j})\big) && \text{on } \Gamma_T^\varepsilon,\\
&\widetilde c_f^{\varepsilon,j} = \widetilde c_e^{\varepsilon,j} && \text{on } \Gamma_T^\varepsilon \setminus \widetilde\Gamma_T^\varepsilon,\\
&D_e \nabla \widetilde c_e^{\varepsilon,j}\cdot n = \big[D_f \nabla \widetilde c_f^{\varepsilon,j} - \mathcal{G}(\partial_t u_f^{\varepsilon,j-2})\widetilde c_f^{\varepsilon,j}\big]\cdot n\\
&\qquad\qquad - \big[\big(\mathcal{G}(\partial_t u_f^{\varepsilon,j-2}) - \mathcal{G}(\partial_t u_f^{\varepsilon,j-1})\big)c_f^{\varepsilon,j}\big]\cdot n && \text{on } \Gamma_T^\varepsilon \setminus \widetilde\Gamma_T^\varepsilon,\\
&D_e \nabla \widetilde c_e^{\varepsilon,j}\cdot n = 0 && \text{on } \widetilde\Gamma_T^\varepsilon,\\
&\big[D_f \nabla \widetilde c_f^{\varepsilon,j} - (\mathcal{G}(\partial_t u_f^{\varepsilon,j-2})c_f^{\varepsilon,j-1} - \mathcal{G}(\partial_t u_f^{\varepsilon,j-1})c_f^{\varepsilon,j})\big]\cdot n = 0 && \text{on } \widetilde\Gamma_T^\varepsilon,\\
&D_b \nabla \widetilde b_e^{\varepsilon,j}\cdot n = F_b(b_e^{\varepsilon,j-1}) - F_b(b_e^{\varepsilon,j}) && \text{on } (\partial\Omega)_T,\\
&D_e \nabla \widetilde c_e^{\varepsilon,j}\cdot n = F_c(c_e^{\varepsilon,j-1}) - F_c(c_e^{\varepsilon,j}) && \text{on } (\partial\Omega)_T.
\end{aligned}
\tag{32}
$$

Using $\widetilde b_e^{\varepsilon,j}$, $\widetilde c_e^{\varepsilon,j}$, and $\widetilde c_f^{\varepsilon,j}$ as test functions in the weak formulation of (31) and (32), we obtain, for any $\delta_1 > 0$,

$$
\begin{aligned}
\partial_t \|\widetilde b_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} + \|\nabla \widetilde b_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} &\le (\varepsilon^2 + \delta_1)\|\nabla \widetilde b_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)}\\
&+ C_1\big(\|b_e^{\varepsilon,j-1}\|_{L^\infty(\Omega_e^\varepsilon)} + C_{\delta_1}\big)\Big[\|\widetilde b_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} + \|\widetilde c_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)}\Big]\\
&+ C_2\|b_e^{\varepsilon,j-1}\|_{L^\infty(\Omega_e^\varepsilon)}\Big[\|\mathbf{e}(\widetilde u_e^{\varepsilon,j-1})\|^2_{L^2(\Omega_e^\varepsilon)} + \|\widetilde b_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)}\Big]\\
&+ C_3\big[\|c_e^{\varepsilon,j-1}\|_{L^2(\Omega_e^\varepsilon)} + \|\mathbf{e}(u_e^{\varepsilon,j-2})\|_{L^2(\Omega_e^\varepsilon)}\big]\|\widetilde b_e^{\varepsilon,j}\|^2_{L^4(\Omega_e^\varepsilon)}\\
&+ C_4\big[\varepsilon\|\widetilde b_e^{\varepsilon,j}\|^2_{L^2(\Gamma^\varepsilon)} + \|\widetilde b_e^{\varepsilon,j}\|^2_{L^2(\partial\Omega)}\big]
\end{aligned}
\tag{33}
$$

and

$$
\begin{aligned}
&\partial_t \|\widetilde{c}_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} + \|\nabla \widetilde{c}_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} + \partial_t \|\widetilde{c}_f^{\varepsilon,j}\|^2_{L^2(\Omega_f^\varepsilon)} + \|\nabla \widetilde{c}_f^{\varepsilon,j}\|^2_{L^2(\Omega_f^\varepsilon)} \\
&\le C_1 \left(\|b_e^{\varepsilon,j}\|_{L^\infty(\Omega_e^\varepsilon)} + 1\right) \Big(\|\widetilde{c}_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} + \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|^2_{L^2(\Omega_e^\varepsilon)}\Big) \\
&+ C_2 \left(\|\mathbf{e}(u_e^{\varepsilon,j-2})\|_{L^2(\Omega_e^\varepsilon)} + \|c_e^{\varepsilon,j-1}\|_{L^2(\Omega_e^\varepsilon)}\right) \Big(\|\widetilde{c}_e^{\varepsilon,j}\|^2_{L^4(\Omega_e^\varepsilon)} + \|\widetilde{b}_e^{\varepsilon,j}\|^2_{L^4(\Omega_e^\varepsilon)}\Big) \\
&+ C_3 \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|_{L^2(\Omega_e^\varepsilon)} \|c_e^{\varepsilon,j-1}\|_{L^4(\Omega_e^\varepsilon)} \|\widetilde{c}_e^{\varepsilon,j}\|_{L^4(\Omega_e^\varepsilon)} + C_4 \|\widetilde{c}_e^{\varepsilon,j}\|^2_{L^2(\partial\Omega)} \\
&+ C_5 \Big(\|c_f^{\varepsilon,j-1}\|^2_{L^4(\Omega_f^\varepsilon)} \|\partial_t \widetilde{u}_f^{\varepsilon,j-1}\|^2_{L^4(\Omega_f^\varepsilon)} + \|\mathcal{G}(\partial_t u_f^{\varepsilon,j-1})\|^2_{L^4(\Omega_f^\varepsilon)} \|\widetilde{c}_f^{\varepsilon,j}\|^2_{L^4(\Omega_f^\varepsilon)}\Big),
\end{aligned} \tag{34}
$$

where $\widetilde{u}_e^{\varepsilon,j-1} = u_e^{\varepsilon,j-1} - u_e^{\varepsilon,j-2}$ and $\widetilde{u}_f^{\varepsilon,j-1} = u_f^{\varepsilon,j-1} - u_f^{\varepsilon,j-2}$. Using the trace and the Gagliardo–Nirenberg inequalities, we estimate $\|\widetilde{b}_e^{\varepsilon,j}\|^2_{L^4(\Omega_e^\varepsilon)}$, $\|\widetilde{c}_e^{\varepsilon,j}\|^2_{L^4(\Omega_e^\varepsilon)}$, and $\|\widetilde{c}_f^{\varepsilon,j}\|^2_{L^4(\Omega_f^\varepsilon)}$, as well as the boundary terms $\varepsilon\|\widetilde{b}_e^{\varepsilon,j}\|^2_{L^2(\Gamma^\varepsilon)}$, $\|\widetilde{b}_e^{\varepsilon,j}\|^2_{L^2(\partial\Omega)}$, and $\|\widetilde{c}_e^{\varepsilon,j}\|^2_{L^2(\partial\Omega)}$, in the same way as in (26). The estimates for $c_e^{\varepsilon,j-1}$ in $L^\infty(0,T;L^4(\Omega_e^\varepsilon))$ and for $c_f^{\varepsilon,j-1}$ in $L^\infty(0,T;L^4(\Omega_f^\varepsilon))$ ensure

$$
\begin{aligned}
&\int_0^s \Big[\|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|_{L^2(\Omega_e^\varepsilon)} \|c_e^{\varepsilon,j-1}\|_{L^4(\Omega_e^\varepsilon)} \|\widetilde{c}_e^{\varepsilon,j}\|_{L^4(\Omega_e^\varepsilon)} + \|c_f^{\varepsilon,j-1}\|^2_{L^4(\Omega_f^\varepsilon)} \|\partial_t \widetilde{u}_f^{\varepsilon,j-1}\|^2_{L^4(\Omega_f^\varepsilon)}\Big] dt \\
&\le \|c_e^{\varepsilon,j-1}\|_{L^\infty(0,s;L^4(\Omega_e^\varepsilon))} \big[C_1 \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + C_\delta \|\widetilde{c}_e^{\varepsilon,j}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + \delta \|\nabla \widetilde{c}_e^{\varepsilon,j}\|^2_{L^2(\Omega_{e,s}^\varepsilon)}\big] \\
&\qquad + \|c_f^{\varepsilon,j-1}\|^2_{L^\infty(0,s;L^4(\Omega_f^\varepsilon))} \big[C_\delta \|\partial_t \widetilde{u}_f^{\varepsilon,j-1}\|^2_{L^2(\Omega_{f,s}^\varepsilon)} + \delta \|\mathbf{e}(\partial_t \widetilde{u}_f^{\varepsilon,j-1})\|^2_{L^2(\Omega_{f,s}^\varepsilon)}\big]
\end{aligned}
$$

for any $\delta > 0$. Then combining (33) and (34) and applying the Gronwall inequality, we obtain

$$
\begin{aligned}
&\|\widetilde{b}_e^{\varepsilon,j}\|^2_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \|\nabla \widetilde{b}_e^{\varepsilon,j}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + \|\widetilde{c}_e^{\varepsilon,j}\|^2_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \|\nabla \widetilde{c}_e^{\varepsilon,j}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \\
&\qquad + \|\widetilde{c}_f^{\varepsilon,j}\|^2_{L^\infty(0,s;L^2(\Omega_f^\varepsilon))} + \|\nabla \widetilde{c}_f^{\varepsilon,j}\|^2_{L^2(\Omega_{f,s}^\varepsilon)} \\
&\le C_1 \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + C_\delta \|\partial_t \widetilde{u}_f^{\varepsilon,j-1}\|^2_{L^2(\Omega_{f,s}^\varepsilon)} + \delta \|\mathbf{e}(\partial_t \widetilde{u}_f^{\varepsilon,j-1})\|^2_{L^2(\Omega_{f,s}^\varepsilon)}.
\end{aligned} \tag{35}
$$

Notice that $C_1 = C_2 e^{C_3 s} \le C_2 e^{C_3 T}$ and $C_\delta = C_4 e^{C_5 s} \le C_4 e^{C_5 T}$ for $s \in (0,T]$, and we can consider C_1 and C_δ to be independent of s.

Considering $|\widetilde{b}_e^{\varepsilon,j}|^{p-1}$, with $p = 2^k$, $k = 2, 3, \dots$, as a test function in the weak formulation of (31) and (32), applying the Gagliardo–Nirenberg inequality to $|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}$, and using the iteration in $p = 2^k$ with $k \in \mathbb{N}$ (see [2, Lemma 3.2]), we derive the estimate

$$
\begin{aligned}
\|\widetilde{b}_e^{\varepsilon,j}\|_{L^\infty(0,s;L^\infty(\Omega_e^\varepsilon))} &\le C_1 \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|_{L^{1+\frac{1}{\sigma}}(0,s;L^2(\Omega_e^\varepsilon))} \\
&\quad + C_\delta \|\partial_t \widetilde{u}_f^{\varepsilon,j-1}\|_{L^2(\Omega_{f,s}^\varepsilon)} + \delta \|\mathbf{e}(\partial_t \widetilde{u}_f^{\varepsilon,j-1})\|_{L^2(\Omega_{f,s}^\varepsilon)}
\end{aligned}
$$

for $s \in (0,T]$, an arbitrary $0 < \delta < 1$, and any $0 < \sigma < 1/9$. For more details see the proof of Lemma 10.2 in the appendix. Notice that C_1 and C_δ depend on T and are independent of s.

Now letting $\widetilde{u}_e^{\varepsilon,j} = u_e^{\varepsilon,j-1} - u_e^{\varepsilon,j}$, $\widetilde{p}_e^{\varepsilon,j} = p_e^{\varepsilon,j-1} - p_e^{\varepsilon,j}$, $\widetilde{u}_f^{\varepsilon,j} = u_f^{\varepsilon,j-1} - u_f^{\varepsilon,j}$, considering the equations for $(\widetilde{u}_e^{\varepsilon,j}, \widetilde{p}_e^{\varepsilon,j}, \partial_t \widetilde{u}_f^{\varepsilon,j})$, and using $(\partial_t \widetilde{u}_e^{\varepsilon,j}, \widetilde{p}_e^{\varepsilon,j}, \partial_t \widetilde{u}_f^{\varepsilon,j})$ as test functions in the integral formulation of these equations, we arrive at the following inequality:

$$\begin{aligned}
&\frac{1}{2}\rho_e\, \partial_t \|\partial_t \widetilde{u}_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} + \frac{1}{2}\partial_t \langle \mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j-1})\mathbf{e}(\widetilde{u}_e^{\varepsilon,j}), \mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\rangle_{\Omega_e^\varepsilon} \\
&+ \frac{1}{2}\rho_p\, \partial_t \|\widetilde{p}_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} + \|\nabla \widetilde{p}_e^{\varepsilon,j}\|^2_{L^2(\Omega_e^\varepsilon)} \\
&+ \frac{1}{2}\rho_f\, \partial_t \|\partial_t \widetilde{u}_f^{\varepsilon,j}\|^2_{L^2(\Omega_f^\varepsilon)} + \mu\, \varepsilon^2\, \|\mathbf{e}(\partial_t \widetilde{u}_f^{\varepsilon,j})\|^2_{L^2(\Omega_f^\varepsilon)} \\
&\le \big\langle \partial_t \mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j-1})\mathbf{e}(\widetilde{u}_e^{\varepsilon,j}), \mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\big\rangle_{\Omega_e^\varepsilon} \\
&+ \big\langle (\mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j-1}) - \mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j}))\, \mathbf{e}(u_e^{\varepsilon,j-1}), \partial_t \mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\big\rangle_{\Omega_e^\varepsilon} \\
&\le C_1 \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\|^2_{L^2(\Omega_e^\varepsilon)} + \partial_t \big\langle (\mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j-1}) - \mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j}))\mathbf{e}(u_e^{\varepsilon,j-1}), \mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\big\rangle_{\Omega_e^\varepsilon} \\
&- \big\langle \partial_t(\mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j-1}) - \mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j}))\mathbf{e}(u_e^{\varepsilon,j-1}), \mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\big\rangle_{\Omega_e^\varepsilon} \\
&- \big\langle (\mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j-1}) - \mathbf{E}^\varepsilon(b_{e,3}^{\varepsilon,j}))\, \partial_t \mathbf{e}(u_e^{\varepsilon,j-1}), \mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\big\rangle_{\Omega_e^\varepsilon}.
\end{aligned} \tag{36}$$

Thus using a priori estimates for $u_e^{\varepsilon,j}$, $\partial_t u_e^{\varepsilon,j}$, $\partial_t u_f^{\varepsilon,j}$, $b_e^{\varepsilon,j}$, and $\widetilde{b}_e^{\varepsilon,j}$, we have that

$$\begin{aligned}
&\|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \|\partial_t \widetilde{u}_f^{\varepsilon,j}\|_{L^\infty(0,s;L^2(\Omega_f^\varepsilon))} + \|\mathbf{e}(\partial_t \widetilde{u}_f^{\varepsilon,j})\|_{L^2(\Omega_{f,s}^\varepsilon)} \\
&+ \|\widetilde{p}_e^{\varepsilon,j}\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \|\nabla \widetilde{p}_e^{\varepsilon,j}\|_{L^2(\Omega_{e,s}^\varepsilon)} \le C_1 \|\widetilde{b}_e^{\varepsilon,j}\|_{L^\infty(0,s;L^\infty(\Omega_e^\varepsilon))} \\
&\le C_2 \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|_{L^{\sigma_1}(0,s;L^2(\Omega_e^\varepsilon))} + C_\delta \|\partial_t \widetilde{u}_f^{\varepsilon,j-1}\|_{L^2(\Omega_{f,s}^\varepsilon)} + \delta \|\mathbf{e}(\partial_t \widetilde{u}_f^{\varepsilon,j-1})\|_{L^2(\Omega_{f,s}^\varepsilon)}
\end{aligned}$$

for $s \in (0,T]$, $0 < \delta < 1$, and $\sigma_1 > 10$, with the constants C_1, C_2, and C_δ depending on T and the model parameters but independent of the solutions, initial data, and of $s \in (0,T]$. Considering $\delta < 1$ and sufficiently small time intervals s in the inequality

$$\begin{aligned}
&\|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j})\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \|\partial_t \widetilde{u}_f^{\varepsilon,j}\|_{L^\infty(0,s;L^2(\Omega_f^\varepsilon))} + \|\mathbf{e}(\partial_t \widetilde{u}_f^{\varepsilon,j})\|_{L^2(\Omega_{f,s}^\varepsilon)} \\
&\le C_2 s^{1/\sigma_1} \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} \\
&+ C_\delta s^{1/2} \|\partial_t \widetilde{u}_f^{\varepsilon,j-1}\|_{L^\infty(0,s;L^2(\Omega_f^\varepsilon))} + \delta \|\mathbf{e}(\partial_t \widetilde{u}_f^{\varepsilon,j-1})\|_{L^2(\Omega_{f,s}^\varepsilon)},
\end{aligned}$$

we obtain by the contraction arguments the existence of a fixed point of $\mathcal{K}$ and hence the existence of a unique weak solution of the microscopic problem (6)–(8) in $(0,s)$. Since the constants C_2 and C_δ depend only on T and the model parameters and do not depend on s, iterating over time intervals we obtain the existence and uniqueness result in the whole time interval $(0,T)$. ∎

4. Convergence results. The a priori estimates proved in Lemma 3.2 imply convergence results for the components of solutions of the microscopic problem (6)–(8).

LEMMA 4.1. *There exist functions* $u_e \in H^1(0,T;H^1(\Omega)) \cap H^2(0,T;L^2(\Omega))$, $p_e \in H^1(0,T;H^1(\Omega))$, $u_e^1, \partial_t u_e^1 \in L^2(\Omega_T; H^1_{\rm per}(Y_e)/\mathbb{R})$, $p_e^1 \in L^2(\Omega_T; H^1_{\rm per}(Y_e)/\mathbb{R})$, $\partial_t u_f$, $\partial_t^2 u_f \in L^2(\Omega_T; H^1_{\rm per}(Y_f))$, *and* $p_f \in L^2(\Omega_T \times Y_f)$ *such that, up to a subsequence,*

$$\begin{aligned}
&u_e^\varepsilon \to u_e && \textit{strongly in } H^1(0,T;L^2(\Omega)), \\
&p_e^\varepsilon \to p_e && \textit{strongly in } L^2(\Omega_T), \\
&\partial_t^2 u_e^\varepsilon \rightharpoonup \partial_t^2 u_e,\ \partial_t p_e^\varepsilon \rightharpoonup \partial_t p_e && \textit{weakly two-scale}, \\
&\nabla u_e^\varepsilon \rightharpoonup \nabla u_e + \nabla_y u_e^1 && \textit{weakly two-scale}, \\
&\nabla p_e^\varepsilon \rightharpoonup \nabla p_e + \nabla_y p_e^1 && \textit{weakly two-scale},
\end{aligned} \tag{37}$$

and for fluid velocity and pressure we have

$$\begin{aligned} &\partial_t u_f^\varepsilon \rightharpoonup \partial_t u_f, \quad p_f^\varepsilon \rightharpoonup p_f && \textit{weakly two-scale}, \\ &\varepsilon \nabla \partial_t u_f^\varepsilon \rightharpoonup \nabla_y \partial_t u_f && \textit{weakly two-scale}. \end{aligned} \tag{38}$$

Additionally, we have weak two-scale convergence $\partial_t u_e^\varepsilon \rightharpoonup \partial_t u_e$ *and* $\partial_t u_f^\varepsilon \rightharpoonup \partial_t u_f$ *on* Γ_T^ε.

Proof. Applying standard extension arguments (see, e.g., [1, 16] or Lemma 3.1) and using the same notation for the original and extended sequences, from estimates (15) in Lemma 3.2 we obtain a priori estimates, uniform in ε, for u_e^ε, ∇u_e^ε, $\partial_t u_e^\varepsilon$, $\partial_t^2 u_e^\varepsilon$, and $\nabla \partial_t u_e^\varepsilon$, as well as p_e^ε, ∇p_e^ε, and $\partial_t p_e^\varepsilon$ in $L^2(\Omega_T)$. Then the convergence results for u_e^ε and p_e^ε follow directly from the compactness of the embedding of $H^1(0,T;L^2(\Omega)) \cap L^2(0,T;H^1(\Omega))$ in $L^2(\Omega_T)$, the a priori estimates (15), and the compactness theorems for the two-scale convergence; see, e.g., [3, 31]. The a priori estimates (15) and the compactness theorems for the two-scale convergence ensure the convergence results for $\partial_t u_f^\varepsilon$ and p_f^ε. Using the trace inequality and a scaling argument together with a priori estimates (15), we obtain

$$\begin{aligned} \varepsilon \|\partial_t u_e^\varepsilon\|^2_{L^2(\Gamma_T^\varepsilon)} &\le C\big(\|\partial_t u_e^\varepsilon\|^2_{L^2(\Omega^\varepsilon_{e,T})} + \varepsilon^2 \|\nabla \partial_t u_e^\varepsilon\|^2_{L^2(\Omega^\varepsilon_{e,T})}\big) \le C, \\ \varepsilon \|\partial_t u_f^\varepsilon\|^2_{L^2(\Gamma_T^\varepsilon)} &\le C\big(\|\partial_t u_f^\varepsilon\|^2_{L^2(\Omega^\varepsilon_{f,T})} + \varepsilon^2 \|\nabla \partial_t u_f^\varepsilon\|^2_{L^2(\Omega^\varepsilon_{f,T})}\big) \le C, \end{aligned}$$

where the constant C is independent of ε. Then the compactness theorem for the two-scale convergence on oscillating surfaces [4, 30] ensures the weak two-scale convergence of $\partial_t u_e^\varepsilon$ and $\partial_t u_f^\varepsilon$ on Γ_T^ε. ◻

In what follows we shall use the same notation for b_e^ε, c_e^ε and their extensions to Ω, whereas the extension of c^ε from $\widetilde{\Omega}_{ef}^\varepsilon$ to Ω will be denoted by $\overline{c}^\varepsilon$. Then for b_e^ε and c^ε we have the following convergence results.

LEMMA 4.2. *There exist functions*

$$b_e,\, c \in L^2(0,T;H^1(\Omega)), \quad b_e^1 \in L^2(\Omega_T; H^1_{\mathrm{per}}(Y_e)/\mathbb{R}), \quad c^1 \in L^2(\Omega_T; H^1_{\mathrm{per}}(Y \setminus \widetilde{\Gamma})/\mathbb{R}),$$

such that, up to a subsequence,

$$\begin{aligned} &b_e^\varepsilon \to b_e, \quad c_e^\varepsilon \to c, \quad \overline{c}^\varepsilon \to c && \textit{strongly in } L^2(\Omega_T), \\ &\nabla b_e^\varepsilon \rightharpoonup \nabla b_e + \nabla_y b_e^1 && \textit{weakly two-scale}, \\ &\nabla c^\varepsilon \rightharpoonup \nabla c + \nabla_y c^1 && \textit{weakly two-scale}. \end{aligned} \tag{39}$$

Proof. Using estimates (16) and the extensions of b_e^ε, c_e^ε, and c^ε, defined in Lemma 3.1, we obtain

$$\begin{aligned} &\|b_e^\varepsilon\|_{L^2(\Omega_T)} + \|\nabla b_e^\varepsilon\|_{L^2(\Omega_T)} + \|c_e^\varepsilon\|_{L^2(\Omega_T)} + \|\nabla c_e^\varepsilon\|_{L^2(\Omega_T)} \le C, \\ &\|\overline{c}^\varepsilon\|_{L^2(\Omega_T)} + \|\nabla \overline{c}^\varepsilon\|_{L^2(\Omega_T)} \le C, \end{aligned} \tag{40}$$

where the constant C is independent of ε. The estimates (40), the compactness of the embedding of $H^1(\Omega)$ in $L^2(\Omega)$, along with the estimate (17) and the Kolmogorov compactness theorem [29] yield the strong convergence of $b_e^\varepsilon \to b_e$, $c_e^\varepsilon \to c_e$, and $\overline{c}^\varepsilon \to c$ in $L^2(\Omega_T)$. Since $\Omega^\varepsilon_{e,T} \cap \widetilde{\Omega}^\varepsilon_{ef,T} \neq \emptyset$ and $c_e^\varepsilon(t,x) = \overline{c}^\varepsilon(t,x)$ in $\Omega^\varepsilon_{e,T} \cap \widetilde{\Omega}^\varepsilon_{ef,T}$, along with the fact that c_e and c are independent of the microscopic variables y, we obtain that $c_e(t,x) = c(t,x)$ in Ω_T.

From the estimates for c^ε, applying the compactness theorem for the two-scale convergence, we obtain that there exists $c^1 \in L^2(\Omega_T; H^1_{\mathrm{per}}(Y \setminus \widetilde{\Gamma})/\mathbb{R})$ such that $\nabla c^\varepsilon \rightharpoonup \nabla c + \nabla_y c^1$ weakly two-scale [54]. ◻

5. Derivation of macroscopic equations for the flow velocity and elastic deformations. This section focuses on homogenization of the microscopic problem (7)–(8). First we define the effective tensors $\mathbf{E}^{\rm hom}$, $K_p^{\rm hom}$, and K_u.

The macroscopic elasticity tensor $\mathbf{E}^{\rm hom} = (E^{\rm hom}_{ijkl})$, permeability tensor $K_p^{\rm hom} = (K^{\rm hom}_{p,ij})$, and $K_u = (K_{u,ij})$ are defined by

$$
\begin{aligned}
E^{\rm hom}_{ijkl}(b_{e,3}) &= \frac{1}{|Y|}\int_{Y_e}\Big(E_{ijkl}(y,b_{e,3}) + E_{ij}(y,b_{e,3})\mathbf{e}_y(w^{kl})\Big)dy,\\
K^{\rm hom}_{p,ij}(x) &= \frac{1}{|Y|}\int_{Y_e}\Big(K_{p,ij}(x,y) + K_{p,i}(x,y)\nabla_y w_p^j\Big)dy,\\
K_{u,ij}(x) &= \frac{1}{|Y|}\int_{Y_e}\Big(\delta_{ij} - K_{p,i}(x,y)\nabla_y w_e^j\Big)dy,
\end{aligned}
\tag{41}
$$

where $w^{kl} = w^{kl}(b_{e,3},\cdot)$, for $k,l = 1,2,3$, are Y-periodic solutions of the unit cell problems

$$
\begin{aligned}
&\operatorname{div}_y\big(\mathbf{E}(y,b_{e,3})(\mathbf{e}_y(w^{kl}) + \mathbf{b}_{kl})\big) = 0 && \text{in } Y_e,\\
&\mathbf{E}(y,b_{e,3})(\mathbf{e}_y(w^{kl}) + \mathbf{b}_{kl})\, n = 0 && \text{on } \Gamma,\\
&\int_{Y_e} w^{kl}\, dy = 0,
\end{aligned}
\tag{42}
$$

functions $w_p^k = w_p^k(x,\cdot)$, for $k = 1,2,3$, are Y-periodic solutions of the unit cell problems

$$
\begin{aligned}
&\operatorname{div}_y\big(K_p(x,y)(\nabla_y w_p^k + e_k)\big) = 0 && \text{in } Y_e,\\
&K_p(x,y)(\nabla_y w_p^k + e_k)\cdot n = 0 && \text{on } \Gamma,\\
&\int_{Y_e} w_p^k\, dy = 0,
\end{aligned}
\tag{43}
$$

and $w_e^k = w_e^k(x,\cdot)$, for $k = 1,2,3$, are Y-periodic solutions of the unit cell problems

$$
\begin{aligned}
&\operatorname{div}_y(K_p(x,y)\nabla_y w_e^k - e_k) = 0 && \text{in } Y_e,\\
&(K_p(x,y)\nabla_y w_e^k - e_k)\cdot n = 0 && \text{on } \Gamma,\\
&\int_{Y_e} w_e^k\, dy = 0.
\end{aligned}
\tag{44}
$$

Here $\mathbf{b}_{kl} = e_k \otimes e_l$ and $\{e_j\}_{j=1}^3$ is the canonical basis of $\mathbb{R}^3$.

LEMMA 5.1. *Periodic cell problems* (42)–(44) *are well-posed and have a unique solution. The tensors* $\mathbf{E}^{\rm hom}$ *and* $K_p^{\rm hom}$ *are positive definite. Moreover,* $\mathbf{E}^{\rm hom}$ *possesses the symmetries declared in* **A1**.

Sketch of proof. Assumptions **A1** on **E** and the Korn inequality for periodic functions ensure the existence of a unique solution of the unit cell problems (42) for a given $b_{e,3} \in L^2(\Omega_T)$; see, e.g., [33]. Assumptions **A2** on K_p yield the existence of unique solutions of the unit cell problems (43) and (44). The positive definiteness of **E** and K_p, the definition of $\mathbf{E}^{\rm hom}$ and $K_p^{\rm hom}$, and the fact that w^{kl} and w_p^k, for $k,l = 1,2,3$, are solutions of (42) and (43) ensure in the standard way (see [7]) that $\mathbf{E}^{\rm hom}$ and $K_p^{\rm hom}$ are positive definite. The definition of $\mathbf{E}^{\rm hom}$ implies that $\mathbf{E}^{\rm hom}$ satisfies the same symmetry assumptions in **A1** as **E**. ∎

Applying the method of the two-scale convergence and using the convergence results in Lemmas 4.1 and 4.2, we derive the homogenized equations for displacement gradient, pressure, and flow velocity for a given $\{b_e^\varepsilon\}$ such that $b_e^\varepsilon \to b_e$ strongly in $L^2(\Omega_T)^3$ as $\varepsilon \to 0$. It should be emphasized that we have not yet derived the equation for the limit function b_e. We only use the strong convergence of $\{b_e^\varepsilon\}$.

In the formations of the macroscopic problem for $(u_e, p_e, \partial_t u_f)$ we shall use the function $Q(x, \partial_t u_f)$ defined as

$$Q(x,\partial_t u_f) = \frac{1}{|Y|}\bigg(\int_{Y_f}\partial_t u_f\,dy - \int_{Y_e} K_p(x,y)\nabla_y q(x,y,\partial_t u_f)\,dy\bigg), \tag{45}$$

where for $(t,x)\in\Omega_T$ the function q is a Y-periodic solution of the problem

$$\begin{aligned} &\operatorname{div}_y(K_p(x,y)\nabla_y q) = 0 && \text{in } Y_e,\\ &-K_p(x,y)\nabla_y q\cdot n = \partial_t u_f\cdot n && \text{on } \Gamma,\\ &\int_{Y_e} q(x,y,\partial_t u_f)\,dy = 0. \end{aligned} \tag{46}$$

THEOREM 5.2. *A sequence of solutions $\{u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon, p_f^\varepsilon\}$ of microscopic problem* (7) *and* (8) *converges, as $\varepsilon\to 0$, to a solution $(u_e, p_e, \partial_t u_f, \pi_f)$ of the macroscopic equations*

$$\begin{aligned} &\vartheta_e\rho_e\,\partial_t^2 u_e - \operatorname{div}(\mathbf{E}^{\rm hom}(b_{e,3})\mathbf{e}(u_e)) + \nabla p_e + \vartheta_f\rho_f ⨍_{Y_f}\partial_t^2 u_f\,dy = 0 && \text{in } \Omega_T,\\ &\vartheta_e\rho_p\,\partial_t p_e - \operatorname{div}\big(K_p^{\rm hom}\nabla p_e - K_u\,\partial_t u_e - Q(x,\partial_t u_f)\big) = 0 && \text{in } \Omega_T, \end{aligned} \tag{47}$$

with boundary and initial conditions

$$\begin{aligned} &\mathbf{E}^{\rm hom}(b_{e,3})\mathbf{e}(u_e)\,n = F_u && \text{on } (\partial\Omega)_T,\\ &(K_p^{\rm hom}\nabla p_e - K_u\,\partial_t u_e)\cdot n = F_p + Q(x,\partial_t u_f)\cdot n && \text{on } (\partial\Omega)_T,\\ &u_e(0) = u_{e0},\quad \partial_t u_e(0) = u^1_{e0},\quad p_e(0) = p_{e0} && \text{in } \Omega, \end{aligned} \tag{48}$$

and the two-scale problem for the fluid flow velocity and pressure

$$\begin{aligned} &\rho_f\,\partial_t^2 u_f - \operatorname{div}_y(\mu\,\mathbf{e}_y(\partial_t u_f) - \pi_f I) + \nabla p_e = 0,\quad \operatorname{div}_y\partial_t u_f = 0 && \text{in } \Omega_T\times Y_f,\\ &\Pi_\tau\partial_t u_f = \Pi_\tau\partial_t u_e && \text{on } \Omega_T\times\Gamma,\\ &n\cdot(\mu\,\mathbf{e}_y(\partial_t u_f) - \pi_f I)\,n = -p_e^1 && \text{on } \Omega_T\times\Gamma,\\ &\partial_t u_f(0) = u^1_{f0} && \text{in } \Omega\times Y_f, \end{aligned} \tag{49}$$

where $\vartheta_e = |Y_e|/|Y|$, $\vartheta_f = |Y_f|/|Y|$, and

$$p_e^1(t,x,y) = \sum_{k=1}^3 \partial_{x_k} p_e(t,x)\,w_p^k(x,y) + \sum_{k=1}^3 \partial_t u_e^k(t,x)\,w_e^k(x,y) + q(x,y,\partial_t u_f), \tag{50}$$

with w_p^k, w_e^k, and q being solutions of (43), (44), *and* (46), *respectively.*

We have $u_e\in H^2(0,T;L^2(\Omega))\cap H^1(0,T;H^1(\Omega))$, $p_e\in H^1(0,T;H^1(\Omega))$, $\partial_t u_f\in L^2(\Omega_T;H^1(Y_f))\cap H^1(0,T;L^2(\Omega\times Y_f))$, and $\pi_f\in L^2(\Omega_T\times Y_f)$ and the convergence in the following sense:

$$\begin{aligned} &u_e^\varepsilon \to u_e \quad \text{in } H^1(0,T;L^2(\Omega)), && p_e^\varepsilon\to p_e \quad \text{in } L^2(\Omega_T),\\ &\nabla u_e^\varepsilon \rightharpoonup \nabla u_e + \nabla_y u_e^1, && \nabla p_e^\varepsilon \rightharpoonup \nabla p_e + \nabla_y p_e^1 \qquad \textit{weakly two-scale},\\ &\partial_t u_f^\varepsilon \rightharpoonup \partial_t u_f,\quad p_f^\varepsilon \rightharpoonup p_e, && \varepsilon\nabla\partial_t u_f^\varepsilon \rightharpoonup \nabla_y\partial_t u_f \qquad \textit{weakly two-scale}. \end{aligned}$$

Remark. In the original microscopic problem the equations of poroelasticity and the Stokes system are coupled through the transmission conditions. The limit system shows the strong coupling in the whole domain Ω_T. Namely, the equations for macroscopic displacement and pressure defined in the whole domain Ω_T are coupled with the two-scale equations for the fluid flow defined on $\Omega_T \times Y_f$. This coupling in the limit problem can be observed through both the lower order terms in the equations and the boundary conditions.

Proof of Theorem 5.2. Considering $(\varepsilon\phi(t,x,x/\varepsilon),\ \varepsilon\psi(t,x,x/\varepsilon),\ \varepsilon\eta(t,x,x/\varepsilon))$ with $\phi \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y_e))^3$, $\psi \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y_e))$, and $\eta \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y_f))^3$ as test functions in the weak formulation of (7), with the corresponding boundary conditions in (8), we obtain

$$
\begin{aligned}
&\langle \rho_e \partial_t^2 u_e^\varepsilon, \varepsilon\phi\rangle_{\Omega^\varepsilon_{e,T}} + \langle \mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\mathbf{e}(u^\varepsilon_e), \varepsilon \mathbf{e}(\phi)\rangle_{\Omega^\varepsilon_{e,T}} + \langle \nabla p_e^\varepsilon, \varepsilon\phi\rangle_{\Omega^\varepsilon_{e,T}}\\
&+ \langle \rho_p \partial_t p_e^\varepsilon, \varepsilon\psi\rangle_{\Omega^\varepsilon_{e,T}} + \langle K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t u_e^\varepsilon, \varepsilon\nabla\psi\rangle_{\Omega^\varepsilon_{e,T}}\\
&+ \langle \rho_f \partial_t^2 u_f^\varepsilon, \varepsilon\eta\rangle_{\Omega^\varepsilon_{f,T}} + \varepsilon^2\mu\,\langle \mathbf{e}(\partial_t u_f^\varepsilon), \varepsilon\,\mathbf{e}(\eta)\rangle_{\Omega^\varepsilon_{f,T}} - \langle p_f^\varepsilon, \varepsilon\,\mathrm{div}\,\eta\rangle_{\Omega^\varepsilon_{f,T}}\\
&+ \langle \partial_t u_f^\varepsilon\cdot n, \varepsilon\psi\rangle_{\Gamma^\varepsilon_T} - \langle p_e^\varepsilon, \varepsilon\eta\cdot n\rangle_{\Gamma^\varepsilon_T} = \langle F_u, \varepsilon\phi\rangle_{(\partial\Omega)_T} + \langle F_p, \varepsilon\psi\rangle_{(\partial\Omega)_T}.
\end{aligned}
\tag{51}
$$

Letting $\varepsilon \to 0$ and using the convergence results in Lemmas 4.1 and 4.2 yields

$$
\begin{aligned}
&\langle \mathbf{E}(y, b_{e,3})(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)), \mathbf{e}_y(\phi)\rangle_{\Omega_T\times Y_e}\\
&+ \langle K_p(\nabla p_e + \nabla_y p_e^1) - \partial_t u_e, \nabla_y\psi\rangle_{\Omega_T\times Y_e} + \langle \partial_t u_f\cdot n, \psi\rangle_{\Omega_T\times\Gamma}\\
&- \langle p_f, \mathrm{div}_y\eta\rangle_{\Omega_T\times Y_f} - \langle p_e, \eta\cdot n\rangle_{\Omega_T\times\Gamma} = 0.
\end{aligned}
\tag{52}
$$

Considering first

(i) $\phi \in C_0^\infty(\Omega_T; C_0^\infty(Y_e))^3$, $\psi \in C_0^\infty(\Omega_T; C_0^\infty(Y_e))$, and $\eta \in C_0^\infty(\Omega_T; C_0^\infty(Y_f))^3$ and then

(ii) $\phi \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y_e))^3$, $\psi \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y_e))$, and $\eta \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y_f))^3$ with $\Pi_\tau\phi = \Pi_\tau\eta$ and $\eta\cdot n = 0$ on $\Omega_T\times\Gamma$, we obtain

$$
\langle p_f,\ \mathrm{div}_y\eta\rangle_{\Omega_T\times Y_f} = 0 \tag{53}
$$

and the equations for correctors

$$
\begin{aligned}
\mathrm{div}_y\big(\mathbf{E}(y, b_{e,3})(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1))\big) &= 0 && \text{in } \ \Omega_T\times Y_e,\\
\mathbf{E}(y, b_{e,3})(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1))\, n &= 0 && \text{on } \Omega_T\times\Gamma
\end{aligned}
\tag{54}
$$

and

$$
\begin{aligned}
\mathrm{div}_y(K_p(\nabla p_e + \nabla_y p_e^1) - \partial_t u_e) &= 0 && \text{in } \ \Omega_T\times Y_e,\\
(-K_p(\nabla p_e + \nabla_y p_e^1) + \partial_t u_e)\cdot n &= \partial_t u_f\cdot n && \text{on } \Omega_T\times\Gamma.
\end{aligned}
\tag{55}
$$

Considering $\eta \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y_f))^3$ with $\Pi_\tau\phi = \Pi_\tau\eta$ on $\Omega_T\times\Gamma$, from (52) and (53) it follows that

$$
p_f = p_f(t,x) \qquad \text{in } \Omega_T\times Y_f \quad \text{ and } \quad p_f = p_e \quad \text{ on } \Omega_T\times\Gamma.
$$

Thus we have $p_f = p_e$ in Ω_T. Taking $(\phi(t,x), \psi(t,x), \eta(t,x,x/\varepsilon))$, where

- $\phi \in C^\infty(\overline{\Omega}_T)^3$ and $\psi \in C^\infty(\overline{\Omega}_T)$,
- $\eta \in C^\infty(\overline{\Omega}_T; C^\infty_{\rm per}(Y_f))^3$ with $\Pi_\tau\eta = \Pi_\tau\phi$ on $\Omega_T\times\Gamma$ and $\mathrm{div}_y\eta(t,x,y) = 0$ in $\Omega_T\times Y_f$,

as test functions in the weak formulation of (7), with external boundary conditions in (8), yields

$$
\begin{aligned}
&\langle \rho_e\, \partial_t^2 u_e^\varepsilon, \phi\rangle_{\Omega_{e,T}^\varepsilon} + \langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon), \mathbf{e}(\phi)\rangle_{\Omega_{e,T}^\varepsilon} + \langle \nabla p_e^\varepsilon, \phi\rangle_{\Omega_{e,T}^\varepsilon}\\
&+ \langle \rho_p\, \partial_t p_e^\varepsilon, \psi\rangle_{\Omega_{e,T}^\varepsilon} + \langle K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t u_e^\varepsilon, \nabla\psi\rangle_{\Omega_{e,T}^\varepsilon}\\
&+ \langle \rho_f\, \partial_t^2 u_f^\varepsilon, \eta\rangle_{\Omega_{f,T}^\varepsilon} + \mu\,\varepsilon^2\langle \mathbf{e}(\partial_t u_f^\varepsilon), \mathbf{e}(\eta) + \varepsilon^{-1}\mathbf{e}_y(\eta)\rangle_{\Omega_{f,T}^\varepsilon} - \langle p_f^\varepsilon, \operatorname{div}_x \eta\rangle_{\Omega_{f,T}^\varepsilon}\\
&+ \langle \partial_t u_f^\varepsilon\cdot n, \psi\rangle_{\Gamma_T^\varepsilon} - \langle p_e^\varepsilon, \eta\cdot n\rangle_{\Gamma_T^\varepsilon} = \langle F_u, \phi\rangle_{(\partial\Omega)_T} + \langle F_p, \psi\rangle_{(\partial\Omega)_T}.
\end{aligned}
\tag{56}
$$

Letting $\varepsilon \to 0$ and using the two-scale convergence of u_e^ε, p_e^ε, and $\partial_t u_f^\varepsilon$, we obtain

$$
\begin{aligned}
&\langle \rho_e \partial_t^2 u_e, \phi\rangle_{\Omega_T\times Y_e} + \langle \mathbf{E}(y, b_{e,3})\big(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)\big), \mathbf{e}(\phi)\rangle_{\Omega_T\times Y_e}\\
&+ \langle \nabla p_e + \nabla_y p_e^1, \phi\rangle_{\Omega_T\times Y_e} - \langle \partial_t u_f, \nabla\psi\rangle_{\Omega_T\times Y_f}\\
&+ \langle \rho_p\, \partial_t p_e, \psi\rangle_{\Omega_T\times Y_e} + \langle K_p(\nabla p_e + \nabla_y p_e^1) - \partial_t u_e, \nabla\psi\rangle_{\Omega_T\times Y_e}\\
&+ \langle \rho_f\, \partial_t^2 u_f, \eta\rangle_{\Omega_T\times Y_f} + \mu\langle \mathbf{e}_y(\partial_t u_f), \mathbf{e}_y(\eta)\rangle_{\Omega_T\times Y_f} + \langle \nabla p_e, \eta\rangle_{\Omega_T\times Y_f}\\
&- \langle p_e^1, \eta\cdot n\rangle_{\Omega_T\times\Gamma} = |Y|\langle F_u, \phi\rangle_{(\partial\Omega)_T} + |Y|\langle F_p, \psi\rangle_{(\partial\Omega)_T}.
\end{aligned}
\tag{57}
$$

Here we used the relation $p_e = p_f$ a.e. in Ω_T, as well as the fact that due to the relation $\operatorname{div}\partial_t u_f^\varepsilon = 0$ and the two-scale convergence of $\partial_t u_f^\varepsilon$, we have

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\langle \partial_t u_f^\varepsilon\cdot n, \psi\rangle_{\Gamma_T^\varepsilon} &= \lim_{\varepsilon\to 0}\Big(-\langle \operatorname{div}\partial_t u_f^\varepsilon, \psi\rangle_{\Omega_{f,T}^\varepsilon} - \langle \partial_t u_f^\varepsilon, \nabla\psi\rangle_{\Omega_{f,T}^\varepsilon}\Big)\\
&= -\lim_{\varepsilon\to 0}\langle \partial_t u_f^\varepsilon, \nabla\psi\rangle_{\Omega_{f,T}^\varepsilon} = -|Y|^{-1}\langle \partial_t u_f, \nabla\psi\rangle_{\Omega_T\times Y_f}.
\end{aligned}
\tag{58}
$$

To show the convergence of $\langle p_e^\varepsilon, \eta\cdot n\rangle_{\Gamma_T^\varepsilon}$ we use $\operatorname{div}_y\eta = 0$ and the fact that p_e^1 is well defined on Γ:

$$
\begin{aligned}
\lim_{\varepsilon\to 0}\langle p_e^\varepsilon, \eta\cdot n\rangle_{\Gamma_T^\varepsilon} &= \lim_{\varepsilon\to 0}\Big(-\langle \nabla p_e^\varepsilon, \eta\rangle_{\Omega_{f,T}^\varepsilon} - \langle p_e^\varepsilon, \operatorname{div}_x\eta\rangle_{\Omega_{f,T}^\varepsilon}\Big)\\
&= -|Y|^{-1}\langle \nabla p_e + \nabla_y p_e^1, \eta\rangle_{\Omega_T\times Y_f} - |Y|^{-1}\langle p_e, \operatorname{div}_x\eta\rangle_{\Omega_T\times Y_f}\\
&= |Y|^{-1}\big(\langle p_e^1, \eta\cdot n\rangle_{\Omega_T\times\Gamma} - \langle \nabla p_e, \eta\rangle_{\Omega_T\times Y_f} - \langle p_e, \operatorname{div}_x\eta\rangle_{\Omega_T\times Y_f}\big).
\end{aligned}
\tag{59}
$$

Notice that n is the internal for Y_f normal at the boundary Γ.

Also, for an arbitrary test function $\eta_1 \in C_0^\infty(\Omega_T; C_0^\infty(Y_f))$, from the two-scale convergence of $\partial_t u_f^\varepsilon$ and the fact that $\partial_t u_f^\varepsilon$ is divergence-free, it follows that

$$
\begin{aligned}
0 = \lim_{\varepsilon\to 0}\langle \operatorname{div}\, \partial_t u_f^\varepsilon, \varepsilon\eta_1(t,x,x/\varepsilon)\rangle_{\Omega_{f,T}^\varepsilon} &= -\lim_{\varepsilon\to 0}\langle \partial_t u_f^\varepsilon, \varepsilon\nabla_x\eta_1 + \nabla_y\eta_1\rangle_{\Omega_{f,T}^\varepsilon}\\
&= -|Y|^{-1}\langle \partial_t u_f, \nabla_y\eta_1\rangle_{\Omega_T\times Y_f} = |Y|^{-1}\langle \operatorname{div}_y\partial_t u_f, \eta_1\rangle_{\Omega_T\times Y_f}.
\end{aligned}
$$

Thus $\operatorname{div}_y \partial_t u_f = 0$ in $\Omega_T \times Y_f$.

Considering $\phi \equiv 0$ and $\psi \equiv 0$, and taking first $\eta \in C_0^\infty(\Omega_T; C_0^\infty(Y_f))^3$ with $\operatorname{div}_y \eta = 0$ and then $\eta \in C_0^\infty(\Omega_T; C_{\mathrm{per}}^\infty(Y_f))^3$ with $\Pi_\tau\eta = 0$ on $\Omega_T \times \Gamma$, we obtain the two-scale problem (49) for $\partial_t u_f$. From the boundary conditions $\Pi_\tau\partial_t u_e^\varepsilon = \Pi_\tau\partial_t u_f^\varepsilon$ on Γ_T^ε and the two-scale convergence of $\partial_t u_e^\varepsilon$ and $\partial_t u_f^\varepsilon$ on Γ_T^ε (see Lemma 4.1), we obtain

$$
\begin{aligned}
&\frac{1}{|Y|}\int_{\Omega_T}\int_\Gamma \Pi_\tau\partial_t u_e(t,x)\,\psi(t,x,y)\,d\gamma_y dx dt = \lim_{\varepsilon\to 0}\varepsilon\int_{\Gamma_T^\varepsilon}\Pi_\tau\partial_t u_e^\varepsilon(t,x)\,\psi\Big(t,x,\frac{x}{\varepsilon}\Big)d\gamma dt\\
&= \lim_{\varepsilon\to 0}\varepsilon\int_{\Gamma_T^\varepsilon}\Pi_\tau\partial_t u_f^\varepsilon(t,x)\,\psi\Big(t,x,\frac{x}{\varepsilon}\Big)d\gamma dt\\
&= \frac{1}{|Y|}\int_{\Omega_T}\int_\Gamma \Pi_\tau\partial_t u_f(t,x,y)\,\psi(t,x,y)\,d\gamma_y dx dt
\end{aligned}
$$

for all $\psi \in C_0(\Omega_T; C_{\rm per}(Y))$. Thus $\Pi_\tau \partial_t u_e = \Pi_\tau \partial_t u_f$ on $\Omega_T \times \Gamma$.

Considering first $\phi \in C_0^\infty(\Omega_T)^3$, $\psi \in C_0^\infty(\Omega_T)$, and then $\phi \in C^\infty(\overline{\Omega}_T)^3$, $\psi \in C^\infty(\overline{\Omega}_T)$, together with $\eta \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y_f))^3$ and $\Pi_\tau \eta = \Pi_\tau \phi$ on Γ, and using the equations (49) for $\partial_t u_f$, we obtain the limit equations for u_e and p_e:

$$
\begin{aligned}
&\vartheta_e \rho_e\, \partial_t^2 u_e - \operatorname{div}\big(\mathbf{E}^{\rm hom}(b_{e,3})\, \mathbf{e}(u_e)\big) + \vartheta_e \nabla p_e + \frac{1}{|Y|}\int_{Y_e} \nabla_y p_e^1\, dy \\
&\qquad\qquad - \frac{1}{|Y|}\big\langle \mu\, \Pi_\tau(\mathbf{e}(\partial_t u_f)\, n), 1\big\rangle_{H^{-1/2}, H^{1/2}(\Gamma)} = 0 \quad \text{in } \Omega_T, \\
&\qquad\qquad\qquad\qquad \mathbf{E}^{\rm hom}(b_{e,3})\, \mathbf{e}(u_e)\, n = F_u \quad \text{on } (\partial\Omega)_T,
\end{aligned}
\tag{60}
$$

where $\vartheta_e = |Y_e|/|Y|$ and the effective elasticity tensor $\mathbf{E}^{\rm hom}$ is defined by (41), and

$$
\begin{aligned}
&\vartheta_e \rho_p\, \partial_t p_e \\
&- \frac{1}{|Y|}\operatorname{div}\Big[\int_{Y_e} [K_p(\nabla p_e + \nabla_y p_e^1) - \partial_t u_e] dy - \int_{Y_f} \partial_t u_f\, dy\Big] = 0 \quad \text{in } \Omega_T, \\
&\frac{1}{|Y|}\Big[\int_{Y_e} \big[K_p(\nabla p_e + \nabla_y p_e^1) - \partial_t u_e\big]\, dy - \int_{Y_f} \partial_t u_f\, dy\Big] \cdot n = F_p \quad \text{on } (\partial\Omega)_T,
\end{aligned}
\tag{61}
$$

with p_e^1 defined by the two-scale problem (55). Considering the weak formulation of (49) with the test function $\eta = 1$ yields

$$
\begin{aligned}
\rho_f \int_{Y_f} \partial_t^2 u_f\, dy + |Y_f| \nabla p_e = -\langle \mu\, (\mathbf{e}_y(\partial_t u_f) - \pi_f I) n, 1\rangle_{H^{-1/2}, H^{1/2}(\Gamma)} = \int_\Gamma p_e^1 n\, d\gamma_y \\
- \langle \mu\, \Pi_\tau(\mathbf{e}_y(\partial_t u_f) n), 1\rangle_{H^{-1/2}, H^{1/2}(\Gamma)}.
\end{aligned}
$$

Using the Y-periodicity of p_e^1, we obtain

$$
-\langle \mu\, \Pi_\tau(\mathbf{e}_y(\partial_t u_f)\, n), 1\rangle_{H^{-1/2}, H^{1/2}(\Gamma)} = \rho_f \int_{Y_f} \partial_t^2 u_f dy + |Y_f| \nabla p_e - \int_{Y_e} \nabla_y p_e^1 dy.
$$

Thus we can rewrite the equation for u_e as

$$
\vartheta_e \rho_e\, \partial_t^2 u_e - \operatorname{div}\big(\mathbf{E}^{\rm hom}(b_{e,3})\, \mathbf{e}(u_e)\big) + \nabla p_e + \vartheta_f \rho_f ⨍_{Y_f} \partial_t^2 u_f dy = 0 \quad \text{in } \Omega_T,
\tag{62}
$$

where $\vartheta_f = |Y_f|/|Y|$. Considering the structure of problem (55), we represent p_e^1 in the form

$$
p_e^1(t,x,y) = \sum_{k=1}^3 \partial_{x_k} p_e(t,x)\, w_p^k(x,y) + \sum_{k=1}^3 \partial_t u_e^k(t,x)\, w_e^k(x,y) + q(x,y,\partial_t u_f),
\tag{63}
$$

where w_p^k and w_e^k are solutions of unit cell problems (43) and (44), and q is a solution of the two-scale problem (46). Incorporating the expression (63) for p_e^1 into (61) and considering (62), we obtain that p_e and u_e satisfy the macroscopic problem (47)–(48), where $\mathbf{E}^{\rm hom}$, $K_p^{\rm hom}$, and K_u are defined by (41). The coupling with the flow velocity $\partial_t u_f$ is reflected in the interaction function Q, defined by (45). Notice that since $\operatorname{div} \partial_t u_f = 0$ in Y_f, we have that $\int_\Gamma \partial_t u_f \cdot n\, d\gamma = 0$ and the problem (46) is well-posed, i.e., the compatibility condition is satisfied. ∎

6. Strong two-scale convergence of $\mathbf{e}(u_e^\varepsilon)$, ∇p_e^ε, and $\partial_t u_f^\varepsilon$.

LEMMA 6.1. *For a subsequence of solutions of microscopic problem (7)–(8), $\{u_e^\varepsilon\}$, $\{p_e^\varepsilon\}$, and $\{\partial_t u_f^\varepsilon\}$ (denoted again by $\{u_e^\varepsilon\}$, $\{p_e^\varepsilon\}$, and $\{\partial_t u_f^\varepsilon\}$), and the limit functions u_e, u_e^1, p_e, p_e^1, and $\partial_t u_f$ as in Lemma 4.1, we have*

$$
\begin{aligned}
&\nabla u_e^\varepsilon \to \nabla u_e + \nabla_y u_e^1 && \textit{strongly two-scale,}\\
&\nabla p_e^\varepsilon \to \nabla p_e + \nabla_y p_e^1 && \textit{strongly two-scale,}\\
&\partial_t u_f^\varepsilon \to \partial_t u_f && \textit{strongly two-scale,}\\
&\varepsilon\, \mathbf{e}(\partial_t u_f^\varepsilon) \to \mathbf{e}_y(\partial_t u_f) && \textit{strongly two-scale.}
\end{aligned}
\tag{64}
$$

Proof. To show the strong two-scale convergence, we prove the convergence of the energy functional related to (7) for u_e^ε, p_e^ε, and $\partial_t u_f^\varepsilon$. Because of the dependence of $\mathbf{E}$ on the temporal variable, we have to consider a modified form of the energy functional. We consider a monotone decreasing function $\zeta : \mathbb{R}_+ \to \mathbb{R}_+$, e.g., $\zeta(t) = e^{-\gamma t}$ for $t \in \mathbb{R}_+$, and define the energy functional for the microscopic problem (7)–(8) as

$$
\begin{aligned}
\mathcal{E}^\varepsilon(u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon) = {}& \frac{1}{2}\rho_e \|\partial_t u_e^\varepsilon(s)\zeta(s)\|^2_{L^2(\Omega_e^\varepsilon)} - \langle \zeta'\zeta, \rho_e |\partial_t u_e^\varepsilon|^2\rangle_{\Omega^\varepsilon_{e,s}}\\
&+ \frac{1}{2}\langle \mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\, \mathbf{e}(u_e^\varepsilon(s))\, \zeta(s), \mathbf{e}(u_e^\varepsilon(s))\, \zeta(s)\rangle_{\Omega_e^\varepsilon}\\
&- \frac{1}{2}\left\langle \left[2\zeta'\zeta\, \mathbf{E}^\varepsilon(b^\varepsilon_{e,3}) + \zeta^2\, \partial_t \mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\right] \mathbf{e}(u_e^\varepsilon), \mathbf{e}(u_e^\varepsilon)\right\rangle_{\Omega^\varepsilon_{e,s}}\\
&+ \frac{1}{2}\rho_p \|p_e^\varepsilon(s)\zeta(s)\|^2_{L^2(\Omega_e^\varepsilon)} - \langle \zeta'\zeta, \rho_p |p_e^\varepsilon|^2\rangle_{\Omega^\varepsilon_{e,s}} + \langle K_p^\varepsilon \nabla p_e^\varepsilon\, \zeta, \nabla p_e^\varepsilon\, \zeta\rangle_{\Omega^\varepsilon_{e,s}}\\
&+ \frac{1}{2}\rho_f \|\partial_t u_f^\varepsilon(s)\zeta(s)\|^2_{L^2(\Omega_f^\varepsilon)} - \langle \zeta'\zeta, \rho_f |\partial_t u_f^\varepsilon|^2\rangle_{\Omega^\varepsilon_{f,s}} + \mu \|\varepsilon\zeta\, \mathbf{e}(\partial_t u_f^\varepsilon)\|^2_{L^2(\Omega^\varepsilon_{f,s})}
\end{aligned}
\tag{65}
$$

for $s \in (0, T]$. Considering $(\partial_t u_e^\varepsilon\, \zeta^2, p_e^\varepsilon\, \zeta^2, \partial_t u_f^\varepsilon\, \zeta^2)$ as a test function in (9) we obtain the equality

$$
\begin{aligned}
\mathcal{E}^\varepsilon(u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon) = {}& \frac{1}{2}\rho_e \|\partial_t u_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)}\\
&+ \frac{1}{2}\left\langle \mathbf{E}^\varepsilon(b^\varepsilon_{e,3})\, \mathbf{e}(u_e^\varepsilon(0)), \mathbf{e}(u_e^\varepsilon(0))\right\rangle_{\Omega_e^\varepsilon} + \frac{1}{2}\rho_p \|p_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)}\\
&+ \frac{1}{2}\rho_f \|\partial_t u_f^\varepsilon(0)\|^2_{L^2(\Omega_f^\varepsilon)} + \langle F_u, \partial_t u_e^\varepsilon\rangle_{(\partial\Omega)_s} + \langle F_p, p_e^\varepsilon\rangle_{(\partial\Omega)_s}.
\end{aligned}
\tag{66}
$$

Due to the assumptions on $\mathbf{E}$ and $\partial_t \mathbf{E}$, there exists a positive constant γ such that

$$
\begin{gathered}
(2\gamma \mathbf{E}(y,\xi) - \partial_t \mathbf{E}(y,\xi))\, A \cdot A \geq 0 \;\text{ for all symmetric matrices } A,\\
\text{all continuous bounded functions } \xi, \text{ and } y \in Y.
\end{gathered}
$$

Since $\{b_e^\varepsilon\}$ converges strongly in $L^2(\Omega_T)$, $\mathbf{e}(u_e^\varepsilon)$ converges weakly two-scale, and $\mathbf{E}^\varepsilon(b^\varepsilon_{e,3})$ is uniformly bounded, we have the weak two-scale convergence of the sequence $(\mathbf{E}^\varepsilon(b^\varepsilon_{e,3}))^{\frac{1}{2}}\mathbf{e}(u_e^\varepsilon)$ to $(\mathbf{E}(y, b_{e,3}))^{\frac{1}{2}}(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1))$ and of $(2\gamma \mathbf{E}^\varepsilon(b^\varepsilon_{e,3}) - \partial_t \mathbf{E}^\varepsilon(b^\varepsilon_{e,3}))^{\frac{1}{2}}\mathbf{e}(u_e^\varepsilon)$ to $(2\gamma \mathbf{E}(y, b_{e,3}) - \partial_t \mathbf{E}(y, b_{e,3}))^{\frac{1}{2}}(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1))$ as $\varepsilon \to 0$. Using in (65) and (66) the lower semicontinuity of the corresponding norms, the initial conditions for u_e^ε, p_e^ε, and

$\partial_t u_f^\varepsilon$, and the convergence of $\partial_t u_e^\varepsilon$, p_e^ε, and $\partial_t u_f^\varepsilon$ implies

$$
\begin{aligned}
&\rho_e\|\partial_t u_e(s)\zeta(s)\|^2_{L^2(\Omega\times Y_e)} + 2\gamma\rho_e\|\partial_t u_e\,\zeta\|^2_{L^2(\Omega_s\times Y_e)}\\
&+ \big\langle \mathbf{E}(y,b_{e,3})\zeta^2(s)(\mathbf{e}(u_e(s)) + \mathbf{e}_y(u_e^1(s)), \mathbf{e}(u_e(s)) + \mathbf{e}_y(u_e^1(s))\big\rangle_{\Omega\times Y_e}\\
&+ \langle \zeta^2\big(2\gamma\,\mathbf{E}(y,b_{e,3}) - \partial_t\mathbf{E}(y,b_{e,3})\big)(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)), \mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)\rangle_{\Omega_s\times Y_e}\\
&+ \rho_p\|p_e(s)\zeta(s)\|^2_{L^2(\Omega\times Y_e)} + 2\gamma\rho_p\|p_e\,\zeta\|^2_{L^2(\Omega_s\times Y_e)}\\
&+ 2\langle \zeta^2 K_p(\nabla p_e + \nabla_y p_e^1), \nabla p_e + \nabla_y p_e^1\rangle_{\Omega_s\times Y_e} + \rho_f\|\partial_t u_f(s)\zeta(s)\|^2_{L^2(\Omega\times Y_f)}\\
&+ 2\gamma\rho_f\|\partial_t u_f\,\zeta\|^2_{L^2(\Omega_s\times Y_f)} + 2\mu\|\mathbf{e}_y(\partial_t u_f)\,\zeta\|^2_{L^2(\Omega_s\times Y_f)}\\
&\le 2|Y|\liminf_{\varepsilon\to 0}\mathcal{E}^\varepsilon(u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon) \le 2|Y|\limsup_{\varepsilon\to 0}\mathcal{E}^\varepsilon(u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon)\\
&= \langle \mathbf{E}(y,b_{e,3})(\mathbf{e}(u_{e0}) + \mathbf{e}_y(\hat u_{e0})), \mathbf{e}(u_{e0}) + \mathbf{e}_y(\hat u_{e0})\rangle_{\Omega\times Y_e}\\
&+ \rho_e\|u_{e0}^1\|^2_{L^2(\Omega\times Y_e)} + \rho_p\|p_{e0}\|^2_{L^2(\Omega\times Y_e)} + \rho_f\|u_{f0}^1\|^2_{L^2(\Omega\times Y_f)}\\
&+ 2|Y|\langle F_u, \partial_t u_e\zeta^2\rangle_{(\partial\Omega)_s} + 2|Y|\langle F_p, p_e\zeta^2\rangle_{(\partial\Omega)_s}
\end{aligned}
\tag{67}
$$

for $s \in (0,T]$. Here we used the weak and the weak two-scale convergences of $\partial_t u_e^\varepsilon$, $\mathbf{e}(u_e^\varepsilon)$, $\mathbf{e}(\partial_t u_e^\varepsilon)$, p_e^ε, and ∇p_e^ε, and the weak two-scale convergence of $\partial_t u_f^\varepsilon$ and $\varepsilon\,\mathbf{e}(\partial_t u_f^\varepsilon)$. Considering the limit equations (47)–(49) for u_e, p_e, and $\partial_t u_f$ and taking $(\partial_t u_e\,\zeta^2, p_e\,\zeta^2, \partial_t u_f\,\zeta^2)$ as a test function yields

$$
\begin{aligned}
&\frac12\rho_e\|\partial_t u_e(s)\zeta(s)\|^2_{L^2(\Omega\times Y_e)} - \frac12\rho_e\|\partial_t u_e(0)\|^2_{L^2(\Omega\times Y_e)} + \gamma\rho_e\|\partial_t u_e\zeta\|^2_{L^2(\Omega_s\times Y_e)}\\
&+ \big\langle \mathbf{E}(y,b_{e,3})\big(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)\big), \mathbf{e}(\partial_t u_e)\,\zeta^2\big\rangle_{\Omega_s\times Y_e} + \langle \nabla p_e + \nabla_y p_e^1, \partial_t u_e\,\zeta^2\rangle_{\Omega_s\times Y_e}\\
&+ \frac12\rho_p\|p_e(s)\zeta^2(s)\|^2_{L^2(\Omega\times Y_e)} - \frac12\rho_p\|p_e(0)\|^2_{L^2(\Omega\times Y_e)} + \gamma\rho_p\|p_e\zeta\|^2_{L^2(\Omega_s\times Y_e)}\\
&+ \langle K_p(x,y)(\nabla p_e + \nabla_y p_e^1) - \partial_t u_e, \nabla p_e\zeta^2\rangle_{\Omega_s\times Y_e} + \frac12\rho_f\|\partial_t u_f(s)\zeta(s)\|^2_{L^2(\Omega\times Y_f)}\\
&- \frac12\rho_f\|\partial_t u_f(0)\|^2_{L^2(\Omega\times Y_f)} + \gamma\rho_f\|\partial_t u_f\zeta\|^2_{L^2(\Omega_s\times Y_f)} - \langle p_e^1, \partial_t u_f\cdot n\,\zeta^2\rangle_{\Omega_s\times\Gamma}\\
&+ \mu\langle \mathbf{e}_y(\partial_t u_f), \mathbf{e}_y(\partial_t u_f)\zeta^2\rangle_{\Omega_s\times Y_f} = |Y|\langle F_u, \partial_t u_e\,\zeta^2\rangle_{(\partial\Omega)_s} + |Y|\langle F_p, p_e\,\zeta^2\rangle_{(\partial\Omega)_s}
\end{aligned}
\tag{68}
$$

for $s \in (0,T]$. From (55) for the corrector p_e^1 we obtain

$$
-\langle p_e^1, \partial_t u_f\cdot n\,\zeta^2\rangle_{\Omega_s\times\Gamma} = \langle K_p(x,y)(\nabla p_e + \nabla_y p_e^1) - \partial_t u_e, \nabla_y p_e^1\,\zeta^2\rangle_{\Omega_s\times Y_e}.
\tag{69}
$$

Considering (54) for the corrector u_e^1 and taking $\partial_t u_e^1\,\zeta^2$ as a test function yields

$$
\begin{aligned}
&\big\langle \mathbf{E}(y,b_{e,3})\big(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)\big), \mathbf{e}(\partial_t u_e)\zeta^2\big\rangle_{\Omega_s\times Y_e}\\
&= \big\langle \mathbf{E}(y,b_{e,3})\big(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)\big), (\mathbf{e}(\partial_t u_e) + \mathbf{e}_y(\partial_t u_e^1))\,\zeta^2\big\rangle_{\Omega_s\times Y_e}\\
&= \frac12\,\big\langle \mathbf{E}(y,b_{e,3})\big(\mathbf{e}(u_e(s)) + \mathbf{e}_y(u_e^1(s))\big)\zeta^2(s), \mathbf{e}(u_e(s)) + \mathbf{e}_y(u_e^1(s))\big\rangle_{\Omega\times Y_e}\\
&- \frac12\,\big\langle \mathbf{E}(y,b_{e,3})\big(\mathbf{e}(u_e(0)) + \mathbf{e}_y(u_e^1(0))\big), \mathbf{e}(u_e(0)) + \mathbf{e}_y(u_e^1(0))\big\rangle_{\Omega\times Y_e}\\
&+ \frac12\,\big\langle \big(2\gamma\mathbf{E}(y,b_{e,3}) - \partial_t\mathbf{E}(y,b_{e,3})\big)\zeta^2\big(\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)\big), \mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)\big\rangle_{\Omega_s\times Y_e}.
\end{aligned}
\tag{70}
$$

Combining (68)–(70) with (67) and using that $\mathbf{e}_y(u_e^1(0)) = \mathbf{e}_y(\hat u_{e0})$ in $\Omega\times Y_e$, we obtain

$$
\mathcal{E}(u_e, p_e, \partial_t u_f) \le \liminf_{\varepsilon\to 0}\mathcal{E}^\varepsilon(u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon) \le \limsup_{\varepsilon\to 0}\mathcal{E}^\varepsilon(u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon) = \mathcal{E}(u_e, p_e, \partial_t u_f)
$$

and thus conclude that $\lim_{\varepsilon\to 0} \mathcal{E}^\varepsilon(u_e^\varepsilon, p_e^\varepsilon, \partial_t u_f^\varepsilon) = \mathcal{E}(u_e, p_e, \partial_t u_f)$. Then the strong two-scale convergence relations stated in the lemma follow by lower semicontinuity arguments. ∎

7. Derivation of macroscopic equations for reaction-diffusion-convection problem. The homogenized coefficients in the reaction-diffusion-convection equations, which will be obtained in the derivation of the macroscopic problem, are defined by

$$
\begin{aligned}
D^{\hom}_{b,ij} &= \frac{1}{|Y|}\int_{Y_e} \big[D_b^{ij} + \big(D_b \nabla_y \omega_b^j(y)\big)_i\big] dy,\\
D^{\hom}_{ij} &= \fint_Y \big[D^{ij}(y) + \big(D(y)\nabla_y \omega^j(y)\big)_i\big] dy,\\
\mathrm{v}_f(t,x) &= \frac{1}{|Y|}\int_{Y_f} \big[\mathcal{G}(\partial_t u_f(t,x,y)) - D_f \nabla_y z(t,x,y)\big] dy,
\end{aligned}
\tag{71}
$$

with ω_b and ω being Y-periodic solutions of the unit cell problems

$$
\begin{aligned}
\operatorname{div}\big(D_b(\nabla_y \omega_b^j(y) + e_j)\big) &= 0 \quad \text{in } Y_e,\\
D_b(\nabla_y \omega_b^j(y) + e_j)\cdot n &= 0 \quad \text{on } \Gamma
\end{aligned}
\tag{72}
$$

and

$$
\begin{aligned}
&\operatorname{div}_y(D(y)(\nabla_y \omega^j + e_j)) = 0 && \text{in } Y\setminus\tilde\Gamma,\\
&D_e(\nabla_y \omega_e^j + e_j)\cdot n = 0, \quad D_f(\nabla_y \omega_f^j + e_j)\cdot n = 0 && \text{on } \widetilde\Gamma,
\end{aligned}
\tag{73}
$$

where $\omega_e^j(y) = \omega^j(y)$ for $y\in Y_e$ and $\omega_f^j(y) = \omega^j(y)$ for $y \in Y_f$, and z is a Y-periodic solution of

$$
\begin{aligned}
&\operatorname{div}_y(D_f\nabla_y z - \mathcal{G}(\partial_t u_f)) = 0 && \text{in } Y_f,\\
&(D_f\nabla_y z - \mathcal{G}(\partial_t u_f))\cdot n = 0 && \text{on } \Gamma.
\end{aligned}
\tag{74}
$$

Here

$$
D(y) = \begin{cases} D_e & \text{in } Y_e,\\ D_f & \text{in } Y_f.\end{cases}
$$

Notice that the definition of $D_b^{\hom}$ and $D^{\hom}$ and the fact that $D_b^{jj} > 0$, with $j = 1,2,3$, $D_e > 0$, $D_f > 0$, and ω_b^j, ω^j are solutions of the unit cell problems (72) and (73) ensure that $D_b^{\hom}$ and $D^{\hom}$ are positive definite.

Next we derive macroscopic equations for the limit functions b_e and c defined in (39). The main difficulty in the proof is to show the convergence of the nonlinear functions depending on the displacement gradient.

THEOREM 7.1. *Solutions of the microscopic problem* (6), (8) *converge to solutions*

$b_e, c \in L^2(0,T;H^1(\Omega))$ *of the macroscopic equations*

$$
\begin{aligned}
&\vartheta_e\,\partial_t b_e - \operatorname{div}(D_b^{\mathrm{hom}}\nabla b_e)\\
&\qquad\quad = \vartheta_e \fint_{Y_e} g_b(c, b_e, W(b_{e,3},y)\,\mathbf{e}(u_e))dy + \vartheta_\Gamma\, P(b_e) && \text{in } \Omega_T,\\
&\partial_t c - \operatorname{div}(D^{\mathrm{hom}}\nabla c - \mathrm{v}_f\, c)\\
&\qquad\quad = \vartheta_f\, g_f(c) + \vartheta_e \fint_{Y_e} g_e(c, b_e, W(b_{e,3},y)\,\mathbf{e}(u_e))dy && \text{in } \Omega_T,\\
&D_b^{\mathrm{hom}}\nabla b_e\cdot n = F_b(b_e) && \text{on } (\partial\Omega)_T,\\
&(D^{\mathrm{hom}}\nabla c - \mathrm{v}_f\, c)\cdot n = F_c(c) && \text{on } (\partial\Omega)_T,\\
&b(0,x) = b_0(x), \quad c(0,x) = c_0(x) && \text{in } \Omega,
\end{aligned} \tag{75}
$$

where

$$
W(b_{e,3},y) = \big\{W_{klij}(b_{e,3},y)\big\}_{k,l,i,j=1}^3 = \big\{b_{kl}^{ij} + (\mathbf{e}_y(w^{ij}(b_{e,3},y)))_{kl}\big\}_{k,l,i,j=1}^3, \tag{76}
$$

with w^{ij} *being solutions of the unit cell problems* (42)*, and* $\mathbf{b}_{kl} = e_k\otimes e_l$*,* $\{e_k\}_{k=1}^3$*, is the canonical basis of* $\mathbb{R}^3$*.*

Here $\vartheta_e = |Y_e|/|Y|$*,* $\vartheta_f = |Y_f|/|Y|$*, and* $\vartheta_\Gamma = |\Gamma|/|Y|$*. We have the convergence in the following sense:*

$$
\begin{aligned}
&b_e^\varepsilon \to b_e, && \overline{c}^\varepsilon \to c && \textit{strongly in } L^2(\Omega_T),\\
&\nabla b_e^\varepsilon \rightharpoonup \nabla b_e + \nabla_y b_e^1, && \nabla c^\varepsilon \rightharpoonup \nabla c + \nabla_y c^1 && \textit{weakly two-scale.}
\end{aligned}
$$

Proof. We can rewrite the microscopic equation for b_e^ε as

$$
\begin{aligned}
&-\langle b_e^\varepsilon\,\chi_{\Omega_e^\varepsilon}, \partial_t\varphi_1\rangle_{\Omega_T} + \langle D_b^\varepsilon\nabla b_e^\varepsilon, \nabla\varphi_1\,\chi_{\Omega_e^\varepsilon}\rangle_{\Omega_T} - \langle b_{e0}, \varphi_1\chi_{\Omega_e^\varepsilon}\rangle_{\Omega_T}\\
&\qquad = \langle g_b(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)), \varphi_1\,\chi_{\Omega_e^\varepsilon}\rangle_{\Omega_T} + \varepsilon\langle P(b_e^\varepsilon), \varphi_1\rangle_{\Gamma_T^\varepsilon} + \langle F_b(b_e^\varepsilon), \varphi_1\rangle_{(\partial\Omega)_T}
\end{aligned} \tag{77}
$$

with $\varphi_1 = \phi_1(t,x) + \varepsilon\phi_2(t,x,x/\varepsilon)$, where $\phi_1\in C^\infty(\overline{\Omega}_T)$ is such that $\phi_1(T,x)=0$ for $x\in\overline{\Omega}$, and $\phi_2\in C_0^\infty(\Omega_T; C_{\mathrm{per}}^\infty(Y))$, and $\chi_{\Omega_e^\varepsilon}$ the characteristic function of Ω_e^ε. Taking into account the strong convergence of b_e^ε and c_e^ε and the two-scale convergence of ∇b_e^ε and ∇c_e^ε (see Lemma 4.2) together with the strong two-scale convergence of $\mathbf{e}(u_e^\varepsilon)$, we obtain

$$
\begin{aligned}
&-\langle |Y_e| b_e, \partial_t\phi_1\rangle_{\Omega_T} + \langle D_b(\nabla b_e + \nabla_y b_e^1), \nabla\phi_1 + \nabla_y\phi_2\rangle_{\Omega_T\times Y_e}\\
&\quad - \langle |Y_e| b_{e0}, \phi_1\rangle_{\Omega_T} = \langle g_b(c, b_e, \mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)), \phi_1\rangle_{\Omega_T\times Y_e}\\
&\qquad\qquad + \langle P(b_e), \phi_1\rangle_{\Omega_T\times\Gamma} + |Y|\langle F_b(b_e), \phi_1\rangle_{(\partial\Omega)_T}.
\end{aligned} \tag{78}
$$

Here we used the fact that due to the strong two-scale convergence of $\mathbf{e}(u_e^\varepsilon)$, we have

$$
\lim_{\varepsilon\to 0}\|\mathcal{T}_\varepsilon^*(\mathbf{e}(u_e^\varepsilon)) - \mathbf{e}(u_e) - \mathbf{e}_y(u_e^1)\|_{L^2(\Omega_T\times Y_e)} = 0,
$$

where $\mathcal{T}_\varepsilon^*$ is the periodic unfolding operator for the perforated domain Ω_e^ε; see, e.g., [15]. Assumptions on g_b in **A4** and the a priori estimates for c^ε, b_e^ε, and u_e^ε ensure

$$
\begin{aligned}
&\|g_b(\mathcal{T}_\varepsilon^*(c_e^\varepsilon), \mathcal{T}_\varepsilon^*(b_e^\varepsilon), \mathcal{T}_\varepsilon^*(\mathbf{e}(u_e^\varepsilon))) - g_b(c, b_e, \mathbf{e}(u_e) + \mathbf{e}_y(u_e^1))\|_{L^1(\Omega_T\times Y_e)}\\
&\qquad \le C_1\Big(\|\mathcal{T}_\varepsilon^*(c_e^\varepsilon) - c\|_{L^2(\Omega_T\times Y_e)} + \|\mathcal{T}_\varepsilon^*(b_e^\varepsilon) - b_e\|_{L^2(\Omega_T\times Y_e)}\\
&\qquad\qquad + \|\mathcal{T}_\varepsilon^*(\mathbf{e}(u_e^\varepsilon)) - \mathbf{e}(u_e) - \mathbf{e}_y(u_e^1)\|_{L^2(\Omega_T\times Y_e)}\Big),\\
&\|g_b(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon))\|_{L^2(\Omega_{e,T}^\varepsilon)} \le C_2,
\end{aligned} \tag{79}
$$

where $C_1 = C_1(\|\mathcal{T}^*_\varepsilon(\mathbf{e}(u^\varepsilon_e))\|_{L^2(\Omega_T\times Y_e)}, \|\mathbf{e}(u_e)+\mathbf{e}_y(u^1_e)\|_{L^2(\Omega_T\times Y_e)}, \|\mathcal{T}^*_\varepsilon(c^\varepsilon_e)\|_{L^2(\Omega_T\times Y_e)}, \|\mathcal{T}^*_\varepsilon(b^\varepsilon_e)\|_{L^2(\Omega_T\times Y_e)}, \|c\|_{L^2(\Omega_T)}, \|b_e\|_{L^2(\Omega_T)})$ and the constants C_1 and C_2 are independent of ε. Combining the estimates in (79), the definition of Ω^ε_e, and the strong convergence of c^ε_e and b^ε_e in $L^2(\Omega_T)$ and of $\mathcal{T}^*_\varepsilon(\mathbf{e}(u^\varepsilon_e))$ in $L^2(\Omega_T\times Y_e)$, along with the properties of the unfolding operator, we obtain

$$\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon_{e,T}} g_b(c^\varepsilon_e, b^\varepsilon_e, \mathbf{e}(u^\varepsilon_e))\psi\Big(t,x,\frac{x}{\varepsilon}\Big)dxdt = \frac{1}{|Y|}\int_{\Omega_T}\int_{Y_e} g_b(c,b_e,\mathbf{e}(u_e)+\mathbf{e}_y(u^1_e))\psi dydxdt$$
$$+\frac{1}{|Y|}\lim_{\varepsilon\to 0}\int_{\Omega_T}\int_{Y_e}\big[g_b(\mathcal{T}^*_\varepsilon(c^\varepsilon_e),\mathcal{T}^*_\varepsilon(b^\varepsilon_e),\mathcal{T}^*_\varepsilon(\mathbf{e}(u^\varepsilon_e))) - g_b(c,b_e,\mathbf{e}(u_e)+\mathbf{e}_y(u^1_e))\big]\mathcal{T}^*_\varepsilon(\psi)dydxdt$$
$$= \frac{1}{|Y|}\int_{\Omega_T}\int_{Y_e} g_b(c,b_e,\mathbf{e}(u_e)+\mathbf{e}_y(u^1_e))\,\psi\,dydxdt \tag{80}$$

for all $\psi\in C^\infty_0(\Omega_T; C_{\rm per}(Y))$. Thus using the estimate for $\|g_b(c^\varepsilon_e, b^\varepsilon_e, \mathbf{e}(u^\varepsilon_e))\|_{L^2(\Omega^\varepsilon_{e,T})}$ in (79), we conclude

$$g_b(c^\varepsilon_e, b^\varepsilon_e, \mathbf{e}(u^\varepsilon_e)) \to g_b(c_e, b_e, \mathbf{e}(u_e)+\mathbf{e}_y(u^1_e)) \qquad \text{two-scale.}$$

To show the convergence of the boundary integral over Γ^ε, we used the Lipschitz continuity of P and the trace estimate

$$\begin{aligned}\varepsilon\|b^\varepsilon_e - b_e\|^2_{L^2(\Gamma^\varepsilon_T)} &\le C_1\Big(\|b^\varepsilon_e-b_e\|^2_{L^2(\Omega^\varepsilon_{e,T})} + \varepsilon^2\|\nabla(b^\varepsilon_e-b_e)\|^2_{L^2(\Omega^\varepsilon_{e,T})}\Big)\\ &\le C_2\Big(\|b^\varepsilon_e-b_e\|^2_{L^2(\Omega^\varepsilon_{e,T})} + \varepsilon^2\big[\|\nabla b^\varepsilon_e\|_{L^2(\Omega^\varepsilon_{e,T})} + \|\nabla b_e\|^2_{L^2(\Omega^\varepsilon_{e,T})}\big]\Big).\end{aligned}\tag{81}$$

Then due to the strong convergence of b^ε_e in $L^2(\Omega_T)$, the regularity of b_e, i.e., $b_e \in L^2(0,T;H^1(\Omega))$, and the boundedness of ∇b^ε_e in $L^2(\Omega^\varepsilon_{e,T})$, uniformly in ε, we obtain

$$\lim_{\varepsilon\to 0}\varepsilon\|P(b^\varepsilon_e)-P(b_e)\|^2_{L^2(\Gamma^\varepsilon_T)} \le C\lim_{\varepsilon\to 0}\varepsilon\,\|b^\varepsilon_e-b_e\|^2_{L^2(\Gamma^\varepsilon_T)} = 0.$$

Taking in (78) first $\phi_1\equiv 0$ and then $\phi_2\equiv 0$ and considering ϕ_1 such that $\phi_1(0)=0$, we obtain macroscopic equations for b_e in (75). The standard arguments for parabolic equations imply that $\partial_t b_e \in L^2(0,T;H^1(\Omega)')$. Combining this with the fact that $b_e\in L^2(0,T;H^1(\Omega))$(see Lemma 4.2), we conclude that $b_e\in C([0,T];L^2(\Omega))$. Then from (78) we obtain that b_e satisfies the initial condition.

The properties of Ω^ε_f and of the unfolding operator $\mathcal{T}^*_{\varepsilon,f}$ for the domain Ω^ε_f yield

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega^\varepsilon_{f,T}}\mathcal{G}(\partial_t u^\varepsilon_f)\,\psi(t,x,x/\varepsilon)\,dxdt &= \lim_{\varepsilon\to 0}\frac{1}{|Y|}\int_{\Omega_T}\int_{Y_f}\mathcal{G}(\mathcal{T}^*_{\varepsilon,f}(\partial_t u^\varepsilon_f))\mathcal{T}^*_{\varepsilon,f}(\psi)dydxdt\\ &= \frac{1}{|Y|}\int_{\Omega_T}\int_{Y_f}\mathcal{G}(\partial_t u_f)\psi\,dydxdt\\ &\quad+\frac{1}{|Y|}\lim_{\varepsilon\to 0}\int_{\Omega_T}\int_{Y_f}\big[\mathcal{G}(\mathcal{T}^*_{\varepsilon,f}(\partial_t u^\varepsilon_f)) - \mathcal{G}(\partial_t u_f)\big]\,\mathcal{T}^*_{\varepsilon,f}(\psi)\,dydxdt\end{aligned}$$

for all $\psi\in C^\infty_0(\Omega_T;C_{\rm per}(Y))$. Using the Lipschitz continuity of $\mathcal{G}$ and the strong convergence of $\mathcal{T}^*_{\varepsilon,f}(\partial_t u^\varepsilon_f)$, ensured by the strong two-scale convergence of $\partial_t u^\varepsilon_f$, we obtain

$$\begin{aligned}&\lim_{\varepsilon\to 0}\int_{\Omega_T\times Y_f}\big[\mathcal{G}(\mathcal{T}^*_{\varepsilon,f}(\partial_t u^\varepsilon_f)) - \mathcal{G}(\partial_t u_f)\big]\,\mathcal{T}^*_{\varepsilon,f}(\psi)dydxdt\\ &\le C\lim_{\varepsilon\to 0}\|\mathcal{T}^*_{\varepsilon,f}(\partial_t u^\varepsilon_f)-\partial_t u_f\|_{L^2(\Omega_T\times Y_f)}\|\psi\|_{L^2(\Omega_T\times Y_f)} = 0.\end{aligned}$$

Thus taking into account the boundedness of $\mathcal{G}(\partial_t u_f^\varepsilon)$, we conclude

$$\mathcal{G}(\partial_t u_f^\varepsilon) \to \mathcal{G}(\partial_t u_f) \qquad \text{two-scale.}$$

In the same way as for g_b, the assumptions in **A4** ensure that

$$\begin{aligned}
&\|g_e(\mathcal{T}_\varepsilon^*(c_e^\varepsilon), \mathcal{T}_\varepsilon^*(b_e^\varepsilon), \mathcal{T}_\varepsilon^*(\mathbf{e}(u_e^\varepsilon))) - g_e(c, b_e, \mathbf{e}(u_e) + \mathbf{e}_y(u_e^1))\|_{L^1(\Omega_T\times Y_e)} \\
&\le C\Big(\|\mathcal{T}_\varepsilon^*(c_e^\varepsilon) - c\|_{L^2(\Omega_T\times Y_e)} \\
&+ \|\mathcal{T}_\varepsilon^*(b_e^\varepsilon) - b_e\|_{L^2(\Omega_T\times Y_e)} + \|\mathcal{T}_\varepsilon^*(\mathbf{e}(u_e^\varepsilon)) - \mathbf{e}(u_e) - \mathbf{e}_y(u_e^1)\|_{L^2(\Omega_T\times Y_e)}\Big),
\end{aligned} \tag{82}$$

where $C = C(\|\mathcal{T}_\varepsilon^*(\mathbf{e}(u_e^\varepsilon))\|_{L^2(\Omega_T\times Y_e)}, \|\mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)\|_{L^2(\Omega_T\times Y_e)}, \|\mathcal{T}_\varepsilon^*(c_e^\varepsilon)\|_{L^2(\Omega_T\times Y_e)}$, $\|\mathcal{T}_\varepsilon^*(b_e^\varepsilon)\|_{L^2(\Omega_T\times Y_e)}, \|c\|_{L^2(\Omega_T)}, \|b_e\|_{L^2(\Omega_T)})$. The a priori estimates for c^ε, b_e^ε, and u_e^ε and assumptions on g in **A4** imply

$$\|g_e(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon))\|_{L^2(\Omega_{e,T}^\varepsilon)} \le C,$$

with a constant C independent of ε. Then estimate (82) and the strong convergence of c_e^ε and b_e^ε in $L^2(\Omega_T)$ and of $\mathcal{T}_\varepsilon^*(\mathbf{e}(u_e^\varepsilon))$ in $L^2(\Omega_T\times Y_e)$, together with calculations similar to (80), yield

$$g_e(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)) \to g_e(c, b_e, \mathbf{e}(u_e) + \mathbf{e}_y(u_e^1)) \qquad \text{two-scale.}$$

Considering $\varphi_2(t,x) = \psi_1(t,x) + \varepsilon\psi_2\big(t,x,\frac{x}{\varepsilon}\big)$, with $\psi_1 \in C_0^\infty(0,T; C^\infty(\overline{\Omega}))$ and $\psi_2 \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(Y\setminus\widetilde{\Gamma}))$, as a test function in (11), we obtain

$$\begin{aligned}
&-\langle c_e^\varepsilon \chi_{\Omega_e^\varepsilon}, \partial_t\varphi_2\rangle_{\Omega_T} + \langle D_c\nabla c_e^\varepsilon, \nabla\varphi_2\chi_{\Omega_e^\varepsilon}\rangle_{\Omega_T} - \langle g_c(c_e^\varepsilon, b_e^\varepsilon, \mathbf{e}(u_e^\varepsilon)), \varphi_2\chi_{\Omega_e^\varepsilon}\rangle_{\Omega_T} \\
&-\langle c_f^\varepsilon \chi_{\Omega_f^\varepsilon}, \partial_t\varphi_2\rangle_{\Omega_T} + \langle D_f\nabla c_f^\varepsilon - \mathcal{G}(\partial_t u_f^\varepsilon)c_f^\varepsilon, \nabla\varphi_2\chi_{\Omega_f^\varepsilon}\rangle_{\Omega_T} - \langle g_f(c_f^\varepsilon), \varphi_2\chi_{\Omega_f^\varepsilon}\rangle_{\Omega_T} \\
&\qquad = \langle F_c(c_e^\varepsilon), \varphi_2\rangle_{(\partial\Omega)_T}.
\end{aligned}$$

The two-scale and the strong convergences of c_e^ε and c_f^ε together with strong two-scale convergence of $\mathbf{e}(u_e^\varepsilon)$ and $\partial_t u_f^\varepsilon$ ensure that

$$\begin{aligned}
&-\langle |Y_e| c, \partial_t\psi_1\rangle_{\Omega_T} + \langle D_c(\nabla c + \nabla_y c^1), \nabla\psi_1 + \nabla_y\psi_2\rangle_{\Omega_T\times Y_e} \\
&-\langle |Y_f| c, \partial_t\psi_1\rangle_{\Omega_T} + \langle D_f(\nabla c + \nabla_y c^1) - \mathcal{G}(\partial_t u_f)c, \nabla\psi_1 + \nabla_y\psi_2\rangle_{\Omega_T\times Y_f} \\
&-\langle g_c(c, b_e, \mathbf{e}(u_e) + \mathbf{e}(u_e^1)), \psi_1\rangle_{\Omega_T\times Y_e} - \langle g_f(c), \psi_1\rangle_{\Omega_T\times Y_f} = |Y|\langle F_c(c), \psi_1\rangle_{(\partial\Omega)_T}.
\end{aligned}$$

Letting $\psi_1 = 0$ yields

$$\begin{aligned}
\langle D_c(\nabla c + \nabla_y c_e^1), \nabla_y\psi_2\rangle_{\Omega_T\times Y_e} &+ \langle D_f(\nabla c + \nabla_y c_f^1), \nabla_y\psi_2\rangle_{\Omega_T\times Y_f} \\
&- \langle \mathcal{G}(\partial_t u_f)c, \nabla_y\psi_2\rangle_{\Omega_T\times Y_f} = 0,
\end{aligned} \tag{83}$$

where $c_l^1(t,x,y) = c^1(t,x,y)$ for $y \in Y_l$ and $(t,x)\in\Omega_T$, with $l = e, f$. Taking into account the structure of (83), we represent c^1 in the form

$$\begin{aligned}
c_e^1(t,x,y) &= \sum_{j=1}^3 \partial_{x_j} c(t,x)\,\omega^j(y) && \text{for} \quad (t,x)\in\Omega_T,\ y\in Y_e, \\
c_f^1(t,x,y) &= \sum_{j=1}^3 \partial_{x_j} c(t,x)\,\omega^j(y) + c(t,x)\,z(t,x,y) && \text{for} \quad (t,x)\in\Omega_T,\ y\in Y_f,
\end{aligned}$$

where ω^j, with $j = 1, 2, 3$, and z are solutions of the unit cell problems (73) and (74), respectively. Then choosing $\psi_2 = 0$, we obtain the macroscopic equations for c in (75). ∎

8. Well-posedness of the limit problem. Uniqueness of a weak solution. To ensure that the whole sequence of solutions of microscopic problem converges, we shall prove the uniqueness of a solution of the limit problem (47)–(49), (75). In fact we are going to prove, using the contraction arguments, that the limit problem is well-posed and in particular has a unique solution.

We consider an operator $\mathcal{K}$ on $L^\infty(0,T;H^1(\Omega)) \times L^\infty(0,T;L^2(\Omega))$ given by $(u_e^j, \partial_t u_f^j) = \mathcal{K}(u_e^{j-1}, \partial_t u_f^{j-1})$, where for given $(u_e^{j-1}, \partial_t u_f^{j-1})$ we first define b_e^j, c^j as a solution of (75) with $(u_e^{j-1}, \partial_t u_f^{j-1})$ in place of $(u_e, \partial_t u_f)$ and then $(u_e^j, p_e^j, u_f^j, \pi_f^j)$ are solutions of (47)–(49) with b_e^j in place of b_e. We denote $\widetilde{c}^j = c^j - c^{j-1}$, $\widetilde{b}_e^j = b_e^j - b_e^{j-1}$, $\widetilde{u}_e^{j-1} = u_e^{j-1} - u_e^{j-2}$, $\widetilde{p}_e^{j-1} = p_e^{j-1} - p_e^{j-2}$, and $\widetilde{u}_f^{j-1} = u_f^{j-1} - u_f^{j-2}$. To prove the existence of a unique solution of problem (47)–(49), (75), we derive a contraction inequality and show that the operator $\mathcal{K}$ has a fixed point.

First we obtain estimates for solutions of the reaction-diffusion-convection system (75).

LEMMA 8.1. *Any two consecutive iterations*

$$(u_e^{j-1}, \partial_t u_f^{j-1}),\ (b_e^j, c^j) \quad \text{and} \quad (u_e^{j-2}, \partial_t u_f^{j-2}),\ (b_e^{j-1}, c^{j-1})$$

for the limit problem (47)–(49), (75) *satisfy the following estimates:*

$$\begin{aligned}
&\|b_e^j\|_{L^\infty(0,T;L^\infty(\Omega))} + \|c^j\|_{L^\infty(0,T;L^\infty(\Omega))} + \|b_e^{j-1}\|_{L^\infty(0,T;L^\infty(\Omega))} \\
&\qquad\qquad + \|c^{j-1}\|_{L^\infty(0,T;L^\infty(\Omega))} \le C, \\
&\|\widetilde{b}_e^j\|_{L^\infty(0,s;L^\infty(\Omega))} + \|\widetilde{c}^j\|_{L^\infty(0,s;L^2(\Omega))} \\
&\qquad\qquad \le C\Big[\|\mathbf{e}(\widetilde{u}_e^{j-1})\|_{L^{1+\frac{1}{\sigma}}(0,s;L^2(\Omega))} + \|\partial_t \widetilde{u}_f^{j-1}\|_{L^2(\Omega_s\times Y_f)}\Big]
\end{aligned} \tag{84}$$

with an arbitrary $s \in (0,T]$ *and any* $0 < \sigma < 1/9$, *the constant* C *being independent of* s.

Proof. The boundedness of b_e^j and b_e^{j-1} can be obtained in the same way as the corresponding estimate for b_e^ε in (16). To show the boundedness of c^j, we consider $(c^j - M)^+$, where $M \ge \max\{\|c_0\|_{L^\infty(\Omega)}, 1\}$, as a test function in the equation for c^j in (75). Using assumptions **A4** on g_e, g_f, and F_c, we obtain

$$\begin{aligned}
&\|(c^j(s) - M)^+\|^2_{L^2(\Omega)} + \|\nabla(c^j - M)^+\|^2_{L^2(\Omega_s)} \\
&\quad \le M\big[\|b_e^j\|_{L^\infty(\Omega_s)} + 1\big]\|(c^j - M)^+\|_{L^1(\Omega_s)} \\
&\quad + M\big[\|\mathrm{v}_f^{j-1}\|_{L^\infty(\Omega_s)} + 1\big]\|\nabla(c^j - M)^+\|_{L^1(\Omega_s)} \\
&\quad + C\big[\|b_e^j\|_{L^\infty(\Omega_s)} + C_\delta\|\mathrm{v}_f^{j-1}\|_{L^\infty(\Omega_s)} + 1\big]\|(c^j - M)^+\|^2_{L^2(\Omega_s)} \\
&\quad + \|\mathbf{e}(u_e^{j-1})\|_{L^\infty(0,s;L^2(\Omega))}\|W^j\|_{L^\infty(\Omega_s;L^2(Y_e))}\|(c^j - M)^+\|^2_{L^2(0,s;L^4(\Omega))} \\
&\quad + \|\mathbf{e}(u_e^{j-1})\|^2_{L^2(\Omega_s)}\|W^j\|^2_{L^\infty(\Omega_s;L^2(Y_e))}(1 + M^2)\int_0^s |\Omega_M(t)|^{\frac{1}{2}}\,dt
\end{aligned}$$

for $s \in (0,T)$, where $\Omega_M(t) = \{x \in \Omega : c^j(t,x) > M\}$ for $t \in (0,T)$. Here v_f^{j-1} is defined in the following way: first we replace $\partial_t u_f$ in the unit cell problem (74) with $\partial_t u_f^{j-1}$ to obtain z^{j-1}, and then we use the third line of (71) with z^{j-1} instead of z to obtain v_f^{j-1}. The definition of v_f^{j-1} and of $W^j = W(b_{e,3}^j, y)$ in (76) together with assumptions **A1** and **A3** on **E** and $\mathcal{G}$ ensure that $\|\mathrm{v}_f^{j-1}\|_{L^\infty(\Omega_s)} \leq C$ and $\|W^j\|_{L^\infty(\Omega_s; L^2(Y_e))} \leq C_1 \|b_{e,3}^j\|_{L^\infty(\Omega_s)} \leq C_2$. Using the embedding $H^1(\Omega) \subset L^4(\Omega)$, we obtain

$$\|(c^j - M)^+\|^2_{L^\infty(0,s;L^2(\Omega))} + \|\nabla (c^j - M)^+\|^2_{L^2(\Omega_s)} \leq C M^2 \int_0^s \Big[|\Omega_M(t)| + |\Omega_M(t)|^{\frac{1}{2}}\Big] dt$$

for some $s \in (0,T]$. Then applying Theorem II.6.1 in [22] with $q = 4(1+\gamma)$, $r = 5(1+\gamma)/2$ and iterating over time intervals yields the boundedness of c^j in $L^\infty(0,T;L^\infty(\Omega))$. The same calculations ensure also the boundedness of c^{j-1}.

Considering the equations for $\widetilde{b}_e^j$ and $\widetilde{c}^j$ and using $\widetilde{b}_e^j$ and $\widetilde{c}^j$ as test functions in these equations, we obtain

$$\begin{aligned}
&\|\widetilde{b}_e^j(s)\|^2_{L^2(\Omega)} + \|\nabla \widetilde{b}_e^j\|^2_{L^2(\Omega_s)} \leq C_1 \|c^{j-1}\|_{L^\infty(0,s;L^2(\Omega))} \|\widetilde{b}_e^j\|^2_{L^2(0,s;L^4(\Omega))} \\
&+ \|\widetilde{b}_e^j\|^2_{L^2(\Omega_s)} + C_2 \|b_e^j\|_{L^\infty(\Omega_s)} \big[\|\widetilde{b}_e^j\|^2_{L^2(\Omega_s)} + \|\widetilde{c}^j\|^2_{L^2(\Omega_s)}\big] \\
&+ C_3 \|b_e^{j-1}\|_{L^\infty(\Omega_s)} \Big[\|\mathbf{e}(\widetilde{u}_e^{j-1})\|^2_{L^2(\Omega_s)} + \|\widetilde{W}^j\|^2_{L^2(0,s;L^4(\Omega;L^2(Y_e)))} + \|\widetilde{b}_e^j\|^2_{L^2(\Omega_s)} \\
&+ \|\widetilde{b}_e^j\|^2_{L^2(0,s;L^4(\Omega))}\Big] + C_4 \|\mathbf{e}(u_e^{j-1})\|_{L^\infty(0,s;L^2(\Omega))} \|\widetilde{b}_e^j\|^2_{L^2(0,s;L^4(\Omega))},
\end{aligned} \tag{85}$$

where $\widetilde{W}^j = W(b_{e,3}^j, y) - W(b_{e,3}^{j-1}, y)$, and

$$\begin{aligned}
&\|\widetilde{c}^j(s)\|^2_{L^2(\Omega)} + \|\nabla \widetilde{c}^j\|^2_{L^2(\Omega_s)} \\
&\leq C_1 \big[1 + \|b_e^{j-1}\|_{L^\infty(\Omega_s)} + \|c^j\|_{L^\infty(\Omega_s)}\big] \big[\|\widetilde{c}^j\|^2_{L^2(\Omega_s)} + \|\widetilde{b}_e^j\|^2_{L^2(\Omega_s)}\big] \\
&+ C_2 \|\mathbf{e}(u_e^{j-1})\|_{L^\infty(0,s;L^2(\Omega))} \big[\|\widetilde{b}_e^j\|^2_{L^2(0,s;L^4(\Omega))} + \|\widetilde{c}^j\|^2_{L^2(0,s;L^4(\Omega))}\big] \\
&+ C_3 \big(\|b_e^{j-1}\|_{L^\infty(\Omega_s)} + \|c^{j-1}\|_{L^\infty(\Omega_s)}\big) \Big[\|\mathbf{e}(\widetilde{u}_e^{j-1})\|^2_{L^2(\Omega_s)} \\
&\qquad + \|\widetilde{W}^j\|^2_{L^2(0,s;L^4(\Omega;L^2(Y_e)))} + \|\widetilde{c}^j\|^2_{L^2(\Omega_s)} + \|\widetilde{c}^j\|^2_{L^2(0,s;L^4(\Omega))}\Big] \\
&+ C_\mu \Big[\|\mathrm{v}_f^{j-1}\|^2_{L^\infty(\Omega_s)} \|\widetilde{c}^j\|^2_{L^2(\Omega_s)} + \|c^{j-1}\|^2_{L^\infty(\Omega_s)} \|\widetilde{\mathrm{v}}_f^{j-1}\|^2_{L^2(\Omega_s)}\Big]
\end{aligned} \tag{86}$$

for $s \in (0,T]$. Here we used assumptions **A4** on the nonlinear functions g_b, g_e, g_f, P, F_b, and F_c. From the definition of v_f^{j-1} and W^{j-1}, the Lipschitz continuity of $\mathcal{G}$ and assumptions on **E**, it follows that

$$\|\widetilde{\mathrm{v}}_f^{j-1}\|_{L^2(\Omega_s)} \leq C \|\partial_t \widetilde{u}_f^{j-1}\|_{L^2(\Omega_s \times Y_f)}, \quad \|\widetilde{W}^j\|_{L^2(0,s;L^4(\Omega;L^2(Y_e)))} \leq C \|\widetilde{b}_e^j\|_{L^2(0,s;L^4(\Omega))}.$$

Adding the inequalities (85) and (86), considering the compactness of embedding $H^1(\Omega) \subset L^4(\Omega)$, and using the Hölder and Gronwall inequalities yields

$$\begin{aligned}
&\|\widetilde{b}_e^j\|_{L^\infty(0,s;L^2(\Omega))} + \|\nabla \widetilde{b}_e^j\|_{L^2(\Omega_s)} + \|\widetilde{c}^j\|_{L^\infty(0,s;L^2(\Omega))} + \|\nabla \widetilde{c}^j\|_{L^2(\Omega_s)} \\
&\qquad \leq C \Big[\|\mathbf{e}(\widetilde{u}_e^{j-1})\|_{L^2(\Omega_s)} + \|\partial_t \widetilde{u}_f^{j-1}\|_{L^2(\Omega_s \times Y_f)}\Big].
\end{aligned} \tag{87}$$

To derive the estimate for the L^∞-norm of $\widetilde{b}_e^j$ we use $(\widetilde{b}_e^j)^{p-1}$ as a test function in (75):

$$
\begin{aligned}
&\frac{1}{p}\|\widetilde{b}_e^j(s)\|^p_{L^p(\Omega)} + \frac{4(p-1)}{p^2}\|\nabla|\widetilde{b}_e^j|^{\frac{p}{2}}\|^2_{L^2(\Omega_s)} \leq C_1\Big[\|c^j\|_{L^\infty(0,s;L^2(\Omega))} \\
&+ \|\mathbf{e}(u_e^{j-1})\|_{L^\infty(0,s;L^2(\Omega))}\|W^{j-1}\|_{L^\infty(\Omega_s;L^2(Y_e))}\Big]\||\widetilde{b}_e^j|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega))} \\
&+ C_2\|\widetilde{b}_e^j\|^p_{L^p(\Omega_s)} + C_3\|b_e^{j-1}\|^2_{L^\infty(\Omega_s)}\big\langle|\mathbf{e}(\widetilde{u}_e^{j-1})|, |\widetilde{b}_e^j|^{p-1}\big\rangle_{\Omega_s} \\
&+ C_4\|b_e^{j-1}\|_{L^\infty(\Omega_s)}\big\langle|\mathbf{e}(u_e^{j-1})|\|\widetilde{W}^j\|_{L^2(Y_e)}, |\widetilde{b}_e^j|^{p-1}\big\rangle_{\Omega_s} \\
&+ C_5\|b_e^{j-1}\|_{L^\infty(\Omega_s)}\left[\frac{1}{p}\|\widetilde{c}^j\|^p_{L^\infty(0,s;L^2(\Omega))} + \frac{p-1}{p}\||\widetilde{b}_e^j|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega))}\right]
\end{aligned}
\tag{88}
$$

for $s \in (0,T]$. Using the Gagliardo–Nirenberg inequality

$$
\|w\|_{L^4(\Omega)} \leq C\|\nabla w\|^\alpha_{L^2(\Omega)}\|w\|^{1-\alpha}_{L^1(\Omega)},
$$

with $\alpha = 9/10$, and making calculations similar to those in (118) in the appendix, we obtain the following estimate:

$$
\begin{aligned}
\langle|\mathbf{e}(\widetilde{u}_e^{j-1})|, |\widetilde{b}_e^j|^{p-1}\rangle_{\Omega_s} &\leq \delta\frac{p-1}{p^2}\|\nabla|\widetilde{b}_e^j|^{\frac{p}{2}}\|^2_{L^2(\Omega_s)} \\
&+ C_\delta\frac{(p-1)p^\beta}{p}\||\widetilde{b}_e^j|^{\frac{p}{2}}\|^2_{L^\infty(0,s;L^1(\Omega))} + C\frac{1}{p}\|\mathbf{e}(\widetilde{u}_e^{j-1})\|^p_{L^{1+\frac{1}{\sigma}}(0,s;L^2(\Omega))},
\end{aligned}
\tag{89}
$$

where $\beta = \frac{\alpha}{1-\alpha}$, $0 < \sigma < 1/9$, and $\delta > 0$ can be chosen arbitrarily. The definition of $\widetilde{W}^j$ implies

$$
\langle|\mathbf{e}(u_e^{j-1})|\,\|\widetilde{W}^j\|_{L^2(Y_e)}, |\widetilde{b}_e^j|^{p-1}\rangle_{\Omega_s} \leq C\|\mathbf{e}(u_e^{j-1})\|_{L^\infty(0,s;L^2(\Omega))}\||\widetilde{b}_e^j|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega))}. \tag{90}
$$

Incorporating the estimate (87) for $\|\widetilde{c}^j\|_{L^\infty(0,\tau;L^2(\Omega))}$ together with (89) and (90) into (88), using the Gagliardo–Nirenberg inequality to estimate $\|\widetilde{b}_e^j\|^p_{L^p(\Omega_s)}$ by $\||\widetilde{b}_e^j|^{\frac{p}{2}}\|^2_{L^1(\Omega_s)}$, and iterating over $p = 2^k$, with $k = 2, 3, \dots$, as in the Alikakos lemma [2, Lemma 3.2], we finally get

$$
\|\widetilde{b}_e^j\|_{L^\infty(0,s;L^\infty(\Omega))} \leq C\Big[\|\mathbf{e}(\widetilde{u}_e^{j-1})\|_{L^{1+\frac{1}{\sigma}}(0,s;L^2(\Omega))} + \|\partial_t\widetilde{u}_f^{j-1}\|_{L^2(\Omega_s\times Y_f)}\Big]
$$

for $s \in (0,T]$ and any $0 < \sigma < 1/9$. □

The macroscopic equations for elastic deformation and pressure are coupled with the two-scale problem for fluid flow velocity. Thus the derivation of the estimates for u_e and $\partial_t u_f$ is nonstandard and is shown in the following lemma.

Lemma 8.2. *For two iterations*

$$
(u_e^{j-1}, p_e^{j-1}, \partial_t u_f^{j-1}, \pi_f^{j-1}),\ (b_e^{j-1}, c^{j-1}) \quad \textit{and} \quad (u_e^j, p_e^j, \partial_t u_f^j, \pi_f^j),\ (b_e^j, c^j)
$$

for limit problem (47)*–*(49)*,* (75)*, we have the following estimates:*

$$
\begin{aligned}
&\|\partial_t\widetilde{u}_e^j\|_{L^\infty(0,s;L^2(\Omega))} + \|\mathbf{e}(\widetilde{u}_e^j)\|_{L^\infty(0,s;L^2(\Omega))} \\
&\qquad + \|\widetilde{p}_e^j\|_{L^\infty(0,s;L^2(\Omega))} + \|\nabla\widetilde{p}_e^j\|_{L^2(\Omega_s)} \leq C\|\widetilde{b}_e^j\|_{L^\infty(0,s;L^\infty(\Omega))}, \\
&\|\partial_t\widetilde{u}_f^j\|_{L^\infty(0,s;L^2(\Omega\times Y_f))} + \|\mathbf{e}_y(\partial_t\widetilde{u}_f^j)\|_{L^2(\Omega_s\times Y_f)} \leq C\|\widetilde{b}_e^j\|_{L^\infty(0,s;L^\infty(\Omega))}
\end{aligned}
\tag{91}
$$

for $s \in (0,T]$, where $\widetilde{u}_e^j = u_e^j - u_e^{j-1}$, $\widetilde{p}_e^j = p_e^j - p_e^{j-1}$, $\partial_t\widetilde{u}_f^j = \partial_t u_f^j - \partial_t u_f^{j-1}$, $\widetilde{b}_e^j = b_e^j - b_e^{j-1}$, and the constant C is independent of s and solutions of the macroscopic problem.

Proof. We begin with the two-scale model for fluid flow velocity. Taking $\partial_t \widetilde{u}_f^j - \partial_t \widetilde{u}_e^j$ as a test function in the equation for the difference $\partial_t \widetilde{u}_f^j$, we obtain

$$
\begin{aligned}
&\big\langle \mathbf{E}^{\rm hom}(b_{e,3}^{j-1})\mathbf{e}(\widetilde{u}_e^j(s)), \mathbf{e}(\widetilde{u}_e^j(s))\big\rangle_\Omega - \big\langle \partial_t \mathbf{E}^{\rm hom}(b_{e,3}^{j-1})\mathbf{e}(\widetilde{u}_e^j), \mathbf{e}(\widetilde{u}_e^j)\big\rangle_{\Omega_s} \\
&+ 2\big\langle (\mathbf{E}^{\rm hom}(b_{e,3}^{j}) - \mathbf{E}^{\rm hom}(b_{e,3}^{j-1}))\, \mathbf{e}(u_e^j(s)), \mathbf{e}(\widetilde{u}_e^j(s))\big\rangle_\Omega \\
&- 2\big\langle \partial_t(\mathbf{E}^{\rm hom}(b_{e,3}^{j}) - \mathbf{E}^{\rm hom}(b_{e,3}^{j-1}))\mathbf{e}(u_e^j), \mathbf{e}(\widetilde{u}_e^j)\big\rangle_{\Omega_s} \\
&- 2\big\langle (\mathbf{E}^{\rm hom}(b_{e,3}^{j}) - \mathbf{E}^{\rm hom}(b_{e,3}^{j-1}))\partial_t \mathbf{e}(u_e^j), \mathbf{e}(\widetilde{u}_e^j)\big\rangle_{\Omega_s} \\
&+ \rho_e \|\partial_t \widetilde{u}_e^j(s)\|^2_{L^2(\Omega)} + \rho_f \|\partial_t \widetilde{u}_f^j(s)\|^2_{L^2(\Omega\times Y_f)} + 2\mu \|\mathbf{e}_y(\partial_t \widetilde{u}_f^j)\|^2_{L^2(\Omega_s\times Y_f)} \\
&+ 2\left\langle \nabla \widetilde{p}_e^j, \partial_t \widetilde{u}_e^j + \int_{Y_f} \partial_t \widetilde{u}_f^j\, dy \right\rangle_{\Omega_s} = 2\langle \widetilde{p}_e^{1,j}, \partial_t \widetilde{u}_f^j \cdot n\rangle_{\Omega_s\times\Gamma} + \rho_e \|\partial_t \widetilde{u}_e^j(0)\|^2_{L^2(\Omega)} \\
&+ \rho_f \|\partial_t \widetilde{u}_f^j(0)\|^2_{L^2(\Omega\times Y_f)} + \big\langle \mathbf{E}^{\rm hom}(b_{e,3}^{j-1})\, \mathbf{e}(\widetilde{u}_e^j(0)), \mathbf{e}(\widetilde{u}_e^j(0))\big\rangle_\Omega \\
&+ 2\big\langle (\mathbf{E}^{\rm hom}(b_{e,3}^{j}) - \mathbf{E}^{\rm hom}(b_{e,3}^{j-1}))\, \mathbf{e}(u_e^j(0)), \mathbf{e}(\widetilde{u}_e^j(0))\big\rangle_\Omega,
\end{aligned} \tag{92}
$$

where $\widetilde{p}_e^{1,j} = p_e^{1,j} - p_e^{1,j-1}$. Equation (47) for p_e^j and p_e^{j-1} yields

$$
\begin{aligned}
&\rho_p \|\widetilde{p}_e^j(s)\|^2_{L^2(\Omega)} + 2\langle K_p^{\rm hom} \nabla \widetilde{p}_e^j, \nabla \widetilde{p}_e^j\rangle_{\Omega_s} \\
&= 2\langle K_u\, \partial_t \widetilde{u}_e^j + Q(x, \partial_t u_f^j) - Q(x, \partial_t u_f^{j-1}), \nabla \widetilde{p}_e^j\rangle_{\Omega_s} + \rho_p \|\widetilde{p}_e^j(0)\|^2_{L^2(\Omega)}.
\end{aligned} \tag{93}
$$

Due to the assumptions in **A1** on **E**, we have

$$
\begin{aligned}
&\|\mathbf{E}^{\rm hom}(b_{e,3}^{j}) - \mathbf{E}^{\rm hom}(b_{e,3}^{j-1})\|_{L^\infty(\Omega_s)} + \|\partial_t(\mathbf{E}^{\rm hom}(b_{e,3}^{j}) - \mathbf{E}^{\rm hom}(b_{e,3}^{j-1}))\|_{L^\infty(\Omega_s)} \\
&\qquad \le C\|\widetilde{b}_e^j\|_{L^\infty(0,s;L^\infty(\Omega))}
\end{aligned}
$$

for $s \in (0,T]$. The expression (50) for $p_e^{1,j}$ and $p_e^{1,j-1}$ and the estimates for the H^1-norm of the solutions of the unit cell problems (43), (44), and (46) yield

$$
\|\widetilde{p}_e^{1,j}\|_{L^2(\Omega_s\times\Gamma)} \le C\left(\|\nabla \widetilde{p}_e^j\|_{L^2(\Omega_s)} + \|\partial_t \widetilde{u}_e^j\|_{L^2(\Omega_s)} + \|\partial_t \widetilde{u}_f^j\|_{L^2(\Omega_s\times\Gamma)}\right).
$$

From the compactness of the embedding $H^1(Y_f) \subset L^2(\Gamma)$ we obtain

$$
\|\partial_t \widetilde{u}_f^j\|_{L^2(\Omega_s\times\Gamma)} \le C_\delta \|\partial_t \widetilde{u}_f^j\|_{L^2(\Omega_s\times Y_f)} + \delta \|\nabla_y \partial_t \widetilde{u}_f^j\|_{L^2(\Omega_s\times Y_f)}
$$

for any $\delta > 0$. Adding (92) and (93) and applying the Hölder and Gronwall inequalities yields

$$
\begin{aligned}
&\|\partial_t \widetilde{u}_f^j\|_{L^\infty(0,s;L^2(\Omega\times Y_f))} + \|\mathbf{e}_y(\partial_t \widetilde{u}_f^j)\|_{L^2(\Omega_s\times Y_f)} + \|\partial_t \widetilde{u}_e^j\|_{L^\infty(0,s;L^2(\Omega))} \\
&+ \|\mathbf{e}(\widetilde{u}_e^j)\|_{L^\infty(0,s;L^2(\Omega))} + \|\widetilde{p}_e^j\|_{L^\infty(0,s;L^2(\Omega))} + \|\nabla \widetilde{p}_e^j\|_{L^2(\Omega_s)} \le C\|\widetilde{b}_e^j\|_{L^\infty(0,s;L^\infty(\Omega))}
\end{aligned}
$$

for all $s \in (0,T]$. ∎

The estimates in Lemmas 8.1 and 8.2 together with a fixed-point argument imply the existence of a unique solution of the strongly coupled limit problem (47)–(49), (75).

LEMMA 8.3. *There exists a unique weak solution of the limit problem* (47)–(49) *and* (75).

Proof. Considering the equations for the difference of two iterations for (47)–(49), (75) and using estimates in Lemmas 8.1 and 8.2 yields

$$
\begin{aligned}
&\|\partial_t(u_e^j - u_e^{j-1})\|_{L^\infty(0,s;L^2(\Omega))} + \|\mathbf{e}(u_e^j - u_e^{j-1})\|_{L^\infty(0,s;L^2(\Omega))} \\
&+ \|\partial_t(u_f^j - u_f^{j-1})\|_{L^\infty(0,s;L^2(\Omega\times Y_f))} + \|\mathbf{e}_y(u_f^j - u_f^{j-1})\|_{L^2(\Omega_s\times Y_f)} \\
&\le C_1\|b_e^j - b_e^{j-1}\|_{L^\infty(0,s;L^\infty(\Omega))} \\
&\le C\Big[\|\mathbf{e}(u_e^{j-1} - u_e^{j-2})\|_{L^{1+\frac{1}{\sigma}}(0,s;L^2(\Omega))} + \|\partial_t(u_f^{j-1} - u_f^{j-2})\|_{L^2(\Omega_s\times Y_f)}\Big]
\end{aligned}
\tag{94}
$$

for $s \in (0,T)$ and any $0 < \sigma < 1/9$, where C is independent of s and iterative solutions of the limit problem. Considering a time interval $(0,\tilde T)$, such that $C\tilde T^{\frac{\sigma}{1+\sigma}} < 1$ and $C\tilde T^{1/2} < 1$, and applying a fixed-point argument, we obtain the existence of a unique solution of the coupled system (47)–(49), (75) on the time interval $[0,\tilde T]$. Iterating this step over time intervals of length $\tilde T$ yields the existence and uniqueness of a solution of the macroscopic problem (47)–(49), (75) on an arbitrary time interval $[0,T]$. ◻

9. Incompressible case. Quasi-stationary poroelastic equations in Ω_e^ε. Problem (6)–(8) was derived under a number of assumptions on plant tissue. In some cases these assumptions should be changed, and system (6)–(8) should be modified accordingly.

In this section we consider two possible modifications of problem (6)–(8):

(i) the incompressible case, when the intercellular space is completely saturated with water and we have the elliptic equation for p_e^ε;

(ii) the quasi-stationary case for the displacement u_e^ε. In this case we can consider both compressible and incompressible fluid phases in the elastic part Ω_e^ε.

In the first case the equation for p_e^ε in (7) is replaced with the following elliptic equation:

$$
-\operatorname{div}(K_p^\varepsilon \nabla p_e^\varepsilon - \partial_t u_e^\varepsilon) = 0 \qquad \text{in } \Omega_{e,T}^\varepsilon. \tag{95}
$$

In the second situation we consider in (7) the quasi-stationary equations for u_e^ε,

$$
-\operatorname{div}(\mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon)) + \nabla p_e^\varepsilon = 0 \qquad \text{in } \Omega_{e,T}^\varepsilon. \tag{96}
$$

In the incompressible case, i.e., p_e^ε satisfies (95), Definition 2.4 of a weak solution of microscopic problem (6)–(8) should be modified. Namely, we assume that

$$
p_e^\varepsilon \in L^2(0,T;H^1(\Omega_e^\varepsilon)) \quad \text{with} \quad \int_{\Omega_e^\varepsilon} p_e^\varepsilon(t,x)\,dx = 0 \quad \text{for} \quad t\in(0,T) \tag{97}
$$

and no initial conditions for p_e^ε are required. Additionally we assume that

$$
\int_{\partial\Omega} F_p(t,x)\,dx = 0 \quad \text{for} \quad t \in (0,T).
$$

The analysis of the quasi-stationary problems considered in this section is very similar to the analysis of (6)–(8) presented in the previous sections. The only part that should be slightly modified is the derivation of a priori estimates.

For the incompressible case, in the same way as in the proof of Lemma 3.2, but now with (95) for p_e^ε, we obtain

$$
\begin{aligned}
&\|\partial_t u_e^\varepsilon(s)\|^2_{L^2(\Omega_e^\varepsilon)} + \|\mathbf{e}(u_e^\varepsilon(s))\|^2_{L^2(\Omega_e^\varepsilon)} + \|\nabla p_e^\varepsilon\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + \varepsilon^2\|\mathbf{e}(\partial_t u_f^\varepsilon)\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \\
&+ \|\partial_t u_f^\varepsilon(s)\|^2_{L^2(\Omega_f^\varepsilon)} \le \delta\big[\|u_e^\varepsilon(s)\|^2_{L^2(\partial\Omega)} + \|p_e^\varepsilon\|^2_{L^2((0,s)\times\partial\Omega)}\big] + C_1\|\mathbf{e}(u_e^\varepsilon)\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \\
&+ C_\delta\big[\|F_u\|^2_{L^\infty(0,s;L^2(\partial\Omega))} + \|\partial_t F_u\|^2_{L^2((0,s)\times\partial\Omega)} + \|F_p\|^2_{L^2((0,s)\times\partial\Omega)}\big] + C_2
\end{aligned}
\tag{98}
$$

for $s \in (0,T]$ and arbitrary $\delta > 0$. Then, as in the proof of Lemma 3.2, applying the trace and Korn inequalities [33] and using extension properties of u_e^ε and assumptions **A5** on initial data u_{e0}^ε, u_{e0}^1, and u_{f0}^1, we obtain estimates (19), (20), and (22). The trace and Poincare inequalities together with the constraints in (97) and properties of an extension of p_e^ε from Ω_e^ε to Ω (see Lemma 3.1) ensure that

$$\|p_e^\varepsilon\|^2_{L^2((0,s)\times\partial\Omega)} \le C\|\nabla p_e^\varepsilon\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \tag{99}$$

for $s \in (0,T]$. Then applying the Gronwall inequality, we obtain from (98) the estimates for u_e^ε, $\partial_t u_e^\varepsilon$, p_e^ε, and $\partial_t u_f^\varepsilon$ in (21).

Differentiating the equations in (7) and (95) with respect to time t and taking $(\partial_t^2 u_e^\varepsilon, \partial_t p_e^\varepsilon, \partial_t^2 u_f^\varepsilon)$ as test functions in the weak formulation of the resulting equations, we obtain

$$\begin{aligned}
&\rho_e\|\partial_t^2 u_e^\varepsilon(s)\|^2_{L^2(\Omega_e^\varepsilon)} + \langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(\partial_t u_e^\varepsilon(s)), \mathbf{e}(\partial_t u_e^\varepsilon(s))\rangle_{\Omega_e^\varepsilon} \\
&+ 2\langle K_p^\varepsilon \nabla \partial_t p_e^\varepsilon, \nabla \partial_t p_e^\varepsilon\rangle_{\Omega_{e,s}^\varepsilon} + \rho_f\|\partial_t^2 u_f^\varepsilon(s)\|^2_{L^2(\Omega_f^\varepsilon)} + 2\mu\,\varepsilon^2\|\mathbf{e}(\partial_t^2 u_f^\varepsilon)\|^2_{L^2(\Omega_{f,s}^\varepsilon)} \\
&= 2\langle \partial_t F_u, \partial_t^2 u_e^\varepsilon\rangle_{(\partial\Omega)_s} + 2\langle \partial_t F_p, \partial_t p_e^\varepsilon\rangle_{(\partial\Omega)_s} + \rho_e\|\partial_t^2 u_e^\varepsilon(0)\|^2_{L^2(\Omega_e^\varepsilon)} \\
&+ \rho_f\|\partial_t^2 u_f^\varepsilon(0)\|^2_{L^2(\Omega_f^\varepsilon)} + 2\big\langle \partial_t \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon(s)), \mathbf{e}(\partial_t u_e^\varepsilon(s))\big\rangle_{\Omega_e^\varepsilon} \\
&+ \big\langle \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(\partial_t u_e^\varepsilon(0)) - 2\partial_t \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon(0))\mathbf{e}(u_e^\varepsilon(0)), \mathbf{e}(\partial_t u_e^\varepsilon(0))\big\rangle_{\Omega_e^\varepsilon} \\
&- \big\langle 2\partial_t^2 \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(u_e^\varepsilon) + \partial_t \mathbf{E}^\varepsilon(b_{e,3}^\varepsilon)\mathbf{e}(\partial_t u_e^\varepsilon), \mathbf{e}(\partial_t u_e^\varepsilon)\big\rangle_{\Omega_{e,s}^\varepsilon}
\end{aligned} \tag{100}$$

for $s \in (0,T]$. As before, applying the Korn inequality and the Poincare inequality together with the constraint in (97), we obtain the estimates for $\partial_t^2 u_e^\varepsilon$, $\partial_t p_e^\varepsilon$, and $\partial_t^2 u_f^\varepsilon$ stated in (15). The equations for $\partial_t u_f^\varepsilon$ and u_e^ε and calculations similar to those in the proof of Lemma 3.2 ensure the estimate for p_f^ε.

To derive the a priori estimates in the second case, when u_e^ε satisfies the quasi-stationary equations (96), we have to check that the Korn inequality holds for u_e^ε.

LEMMA 9.1. *For $u_e^\varepsilon(s) \in H^1(\Omega_e^\varepsilon)$, with $s \in (0,T]$, we have the following estimate:*

$$\begin{aligned}
\|u_e^\varepsilon(s)\|_{H^1(\Omega_e^\varepsilon)} &\le C\big[\|\mathbf{e}(u_e^\varepsilon(s))\|_{L^2(\Omega_e^\varepsilon)} + \varepsilon^{\frac12}\|\Pi_\tau \partial_t u_f^\varepsilon\|_{L^2(\Gamma_T^\varepsilon)} + \|u_e^\varepsilon(0)\|_{H^1(\Omega)}\big], \\
\|\partial_t u_e^\varepsilon(s)\|_{H^1(\Omega_e^\varepsilon)} &\le C\big[\|\partial_t \mathbf{e}(u_e^\varepsilon(s))\|_{L^2(\Omega_e^\varepsilon)} + \varepsilon^{\frac12}\|\Pi_\tau \partial_t u_f^\varepsilon(s)\|_{L^2(\Gamma^\varepsilon)}\big].
\end{aligned} \tag{101}$$

Proof. Consider first Y_e and $\mathcal{V} = \{v \in H^1(Y_e)^3 : \Pi_\tau v = 0 \text{ on } \Gamma\}$. Then since $\mathcal{V} \cap \mathcal{R}(Y_e) = \{0\}$, where $\mathcal{R}(Y_e)$ is the space of all rigid displacements, we have

$$\|v\|^2_{H^1(Y_e)} \le C\big[\|\mathbf{e}(v)\|^2_{L^2(Y_e)} + \|\Pi_\tau v\|^2_{L^2(\Gamma)}\big]. \tag{102}$$

Considering scaling $x = \varepsilon y$ and summing over $\xi \in \Xi^\varepsilon$, we obtain

$$\|v\|^2_{L^2(\hat\Omega_e^\varepsilon)} + \varepsilon^2\|\nabla v\|^2_{L^2(\hat\Omega_e^\varepsilon)} \le C\big[\varepsilon^2\|\mathbf{e}(v)\|^2_{L^2(\hat\Omega_e^\varepsilon)} + \varepsilon\|\Pi_\tau v\|^2_{L^2(\Gamma^\varepsilon)}\big], \tag{103}$$

where $\hat\Omega_e^\varepsilon = \mathrm{Int}(\bigcup_{\xi\in\Xi^\varepsilon} \varepsilon(\overline{Y}_e + \xi))$. Using the fact that $\Pi_\tau \partial_t u_e^\varepsilon = \Pi_\tau \partial_t u_f^\varepsilon$ on Γ^ε and estimating u_e^ε by $\partial_t u_e^\varepsilon$ and the initial value $u_e^\varepsilon(0)$, we obtain

$$\|\Pi_\tau u_e^\varepsilon(s)\|_{L^2(\Gamma^\varepsilon)} \le C\big[\|\Pi_\tau \partial_t u_f^\varepsilon\|_{L^2(\Gamma_T^\varepsilon)} + \|u_e^\varepsilon(0)\|_{L^2(\Gamma^\varepsilon)}\big].$$

Hence applying (103) to u_e^ε and using the fact that $\varepsilon\|u_e^\varepsilon(0)\|^2_{L^2(\Gamma^\varepsilon)} \le C\|u_e^\varepsilon(0)\|^2_{H^1(\Omega)}$, we have

$$\|u_e^\varepsilon(s)\|^2_{L^2(\hat\Omega_e^\varepsilon)} \le C\big[\varepsilon^2\|\mathbf{e}(u_e^\varepsilon)(s)\|^2_{L^2(\hat\Omega_e^\varepsilon)} + \varepsilon\|\Pi_\tau \partial_t u_f^\varepsilon\|^2_{L^2(\Gamma_T^\varepsilon)} + \|u_e^\varepsilon(0)\|^2_{H^1(\Omega)}\big].$$

Then considering the extension of u_e^ε from Ω_e^ε to Ω (see, e.g., [33]) and applying the Korn inequality in Ω yields the estimate stated in the lemma. ∎

Then, in the same way as in the proof of Lemma 3.2, applying the Korn inequalities proved in Lemma 9.1 and using extension properties of u_e^ε and the regularity of the initial data $u_{f0}^1 \in H^2(\Omega)^3$, we obtain the following a priori estimates for solutions of the quasi-stationary problem:

$$
\begin{aligned}
&\|u_e^\varepsilon\|_{L^\infty(0,T;H^1(\Omega_e^\varepsilon))} + \|\partial_t u_e^\varepsilon\|_{L^\infty(0,T;H^1(\Omega_e^\varepsilon))} \le C,\\
&\|p_e^\varepsilon\|_{L^2(0,T;H^1(\Omega_e^\varepsilon))} + \|\partial_t p_e^\varepsilon\|_{L^2(0,T;H^1(\Omega_e^\varepsilon))} \le C,\\
&\|\partial_t u_f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_f^\varepsilon))} + \|\partial_t^2 u_f^\varepsilon\|_{L^\infty(0,T;L^2(\Omega_f^\varepsilon))}\\
&\qquad\qquad + \varepsilon\|\nabla\partial_t u_f^\varepsilon\|_{H^1(0,T;L^2(\Omega_f^\varepsilon))} + \|p_f^\varepsilon\|_{L^2(\Omega_{f,T}^\varepsilon)} \le C,
\end{aligned}
\tag{104}
$$

where the constant C is independent of ε. Notice that in the incompressible and quasi-stationary case, i.e., in the case of (95) and (96) for p_e^ε and u_e^ε, respectively, problem (7), (8), (95), and (96) is well-posed without the initial conditions for u_e^ε and p_e^ε. In this case $u_e^\varepsilon(0,\cdot)$ and $\partial_t u_e^\varepsilon(0,\cdot)$ are determined from the corresponding elliptic equations and the initial values for the fluid flow u_{f0}^1.

In contrast with the limit equations given by (47), in the quasi-stationary and incompressible case the macroscopic equations for effective displacement and pressure do not contain time derivatives and take the form

$$
\begin{aligned}
&-\operatorname{div}(\mathbf{E}^{\rm hom}(b_{e,3})\mathbf{e}(u_e)) + \nabla p_e + \vartheta_f\rho_f \fint_{Y_f} \partial_t^2 u_f\, dy = 0 && \text{in } \Omega_T,\\
&-\operatorname{div}\big(K_p^{\rm hom}\nabla p_e - K_u\,\partial_t u_e - Q(x,\partial_t u_f)\big) = 0 && \text{in } \Omega_T,\\
&\mathbf{E}^{\rm hom}(b_{e,3})\mathbf{e}(u_e)\, n = F_u && \text{on } (\partial\Omega)_T,\\
&(K_p^{\rm hom}\nabla p_e - K_u\,\partial_t u_e)\cdot n = F_p + Q(x,\partial_t u_f)\cdot n && \text{on } (\partial\Omega)_T,
\end{aligned}
\tag{105}
$$

together with the two-scale equations (49) for u_f and π_f.

10. Appendix. Here we provide proofs of the estimates for $\|b_e^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_e^\varepsilon))}$, $\|c^\varepsilon\|_{L^\infty(0,T;L^4(\Omega^\varepsilon))}$ and for the difference $\|\widetilde{b}_e^{\varepsilon,j}\|_{L^\infty(0,T;L^\infty(\Omega_e^\varepsilon))}$ of two iterations for system (6)–(8).

LEMMA 10.1. *Under assumptions* **A1**–**A5** *solutions of the microscopic problem* (6)–(8) *satisfy the following estimates:*

$$
\begin{aligned}
&\|b_e^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_e^\varepsilon))} \le C,\\
&\|c_e^\varepsilon\|_{L^\infty(0,T;L^4(\Omega_e^\varepsilon))} + \|c_f^\varepsilon\|_{L^\infty(0,T;L^4(\Omega_f^\varepsilon))} \le C,
\end{aligned}
\tag{106}
$$

where the constant C is independent of ε.

Proof. To show that $|b_e^\varepsilon|^p$ for $p \ge 2$ is an admissible test function for (10), we set $b_{e,N}^\varepsilon(t,x) = \min\{b_e^\varepsilon(t,x), N\}$ for $(t,x) \in \Omega_{e,T}^\varepsilon$, where $N > \|b_{e0}\|_{L^\infty(\Omega)}$, and derive estimates for $|b_{e,N}^\varepsilon|^p$ independent of N. Then letting $N \to \infty$, we obtain the desired estimates for b_e^ε. Taking $(b_{e,N}^\varepsilon)^{p-1}$ as a test function in (10) and applying simple calculations, we obtain

$$
\begin{aligned}
\|b^\varepsilon_{e,N}(s)\|^p_{L^p(\Omega^\varepsilon_e)} + \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})} \leq C_1 \Big[&\|\mathbf{e}(u^\varepsilon_e)\|_{L^\infty(0,s;L^2(\Omega^\varepsilon_e))} \\
&+ \|c^\varepsilon_e\|_{L^\infty(0,s;L^2(\Omega^\varepsilon_e))} + 1\Big] \int_0^s \|b^\varepsilon_e\|_{L^{2p}(\Omega^\varepsilon_e)} \|b^\varepsilon_{e,N}\|^{p-1}_{L^{2p}(\Omega^\varepsilon_e)} dt \\
+ C_2 \|b_{e0}\|^p_{L^p(\Omega^\varepsilon_e)} + C_3 \|b^\varepsilon_e\|_{L^\infty(0,s;L^2(\Omega^\varepsilon_e))} &\| |b^\varepsilon_{e,N}|^{\frac{p}{2}} \|^{2\frac{p-1}{p}}_{L^2(0,s;L^4(\Omega^\varepsilon_e))} \\
+ C_4 \Big[\| |b^\varepsilon_{e,N}|^{\frac{p}{2}} \|^2_{L^2(0,s;L^4(\Omega^\varepsilon_e))} &+ \|c^\varepsilon_e\|^p_{L^p(0,s;L^2(\Omega^\varepsilon_e))} + \|\mathbf{e}(u^\varepsilon_e)\|^p_{L^p(0,s;L^2(\Omega^\varepsilon_e))}\Big] \\
&+ \varepsilon \langle |P(b^\varepsilon_e)|, |b^\varepsilon_{e,N}|^{p-1}\rangle_{\Gamma^\varepsilon_s} + \langle |F_b(b^\varepsilon_e)|, |b^\varepsilon_{e,N}|^{p-1}\rangle_{(\partial\Omega)_s}
\end{aligned}
\tag{107}
$$

for $s \in (0, T]$. Here we used the fact that the definition of $b^\varepsilon_{e,N}$ implies

$$
\langle \nabla b^\varepsilon_e, \nabla (b^\varepsilon_{e,N})^{p-1}\rangle_{\Omega^\varepsilon_{e,s}} = \langle \nabla b^\varepsilon_{e,N}, \nabla (b^\varepsilon_{e,N})^{p-1}\rangle_{\Omega^\varepsilon_{e,s}}
$$

and that due to the inequality $b^\varepsilon_e \geq 0$ in $\Omega^\varepsilon_{e,T}$ we have

$$
\begin{aligned}
\langle \partial_t b^\varepsilon_e, |b^\varepsilon_{e,N}|^{p-1}\rangle_{\Omega^\varepsilon_{e,s}} &\geq \frac{1}{p}\|b^\varepsilon_{e,N}(s)\|^p_{L^p(\Omega^\varepsilon_e)} - \frac{1}{p}\|b_{e0}\|^p_{L^p(\Omega^\varepsilon_e)} - \|b_{e0}\|^p_{L^p(\Omega^\varepsilon_e)} \\
+ \langle b^\varepsilon_e(s), |b^\varepsilon_{e,N}(s)|^{p-1}\rangle_{\Omega^\varepsilon_e \setminus \Omega^{\varepsilon,N}_e(s)} &\geq \frac{1}{p}\|b^\varepsilon_{e,N}(s)\|^p_{L^p(\Omega^\varepsilon_e)} - (1+1/p)\|b_{e0}\|^p_{L^p(\Omega^\varepsilon_e)}.
\end{aligned}
$$

Here $\Omega^{\varepsilon,N}_e(t) = \{x \in \Omega^\varepsilon_{e,s} \, : \, b^\varepsilon_e(t,x) \leq N\}$ for $t \in (0,T)$. Applying the Gagliardo–Nirenberg inequality, we can estimate

$$
\| |b^\varepsilon_{e,N}|^p \|_{L^2(\Omega^\varepsilon_e)} = \| |b^\varepsilon_{e,N}|^{\frac{p}{2}} \|^2_{L^4(\Omega^\varepsilon_e)} \leq C \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|^{2\alpha}_{L^2(\Omega^\varepsilon_e)} \| |b^\varepsilon_{e,N}|^{\frac{p}{2}} \|^{1-\alpha}_{L^1(\Omega^\varepsilon_e)}
\tag{108}
$$

with $\alpha = 9/10$. Using the embedding $L^2(0,s;H^1(\Omega^\varepsilon_e)) \subset L^2(0,s;L^6(\Omega^\varepsilon_e))$, in space-dimensions two and three, and applying the Gagliardo–Nirenberg inequality to $\| |b^\varepsilon_{e,N}|^{\frac{p}{2}} \|_{L^4(\Omega^\varepsilon_e)}$ yields

$$
\begin{aligned}
&\int_0^s \|b^\varepsilon_e\|_{L^{2p}(\Omega^\varepsilon_e)} \|b^\varepsilon_{e,N}\|^{p-1}_{L^{2p}(\Omega^\varepsilon_e)} dt \\
&\leq \int_0^s \Big(\|b^\varepsilon_e\|_{L^{\frac{2p}{3}}(\Omega^\varepsilon_e)} + \|\nabla |b^\varepsilon_e|^{\frac{p}{3}}\|^{\frac{3}{p}}_{L^2(\Omega^\varepsilon_e)} \Big) \|b^\varepsilon_{e,N}\|^{\frac{p-1}{4}}_{L^p(\Omega^\varepsilon_e)} \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|^{\frac{3(p-1)}{2p}}_{L^2(\Omega^\varepsilon_e)} dt.
\end{aligned}
$$

Then using the Hölder inequality on the right-hand side of the last estimate, we obtain

$$
\begin{aligned}
&\int_0^s \|b^\varepsilon_e\|_{L^{2p}(\Omega^\varepsilon_e)} \|b^\varepsilon_{e,N}\|^{p-1}_{L^{2p}(\Omega^\varepsilon_e)} dt \\
&\leq C \Big[\int_0^s \Big(\|b^\varepsilon_e\|^{\frac{2p}{3}}_{L^{\frac{2p}{3}}(\Omega^\varepsilon_e)} + \|\nabla |b^\varepsilon_e|^{\frac{p}{3}}\|^2_{L^2(\Omega^\varepsilon_e)} \Big) dt \Big]^{\frac{3}{2p}} \\
&\qquad \times \sup_{(0,s)} \|b^\varepsilon_{e,N}\|^{\frac{p-1}{4}}_{L^p(\Omega^\varepsilon_e)} \Big[\int_0^s \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|^{\frac{3}{2}\frac{(2p-2)}{2p-3}}_{L^2(\Omega^\varepsilon_e)} dt \Big]^{\frac{2p-3}{2p}}.
\end{aligned}
\tag{109}
$$

For $p \geq 3$ we can estimate

$$
\begin{aligned}
\|\nabla |b^\varepsilon_e|^{\frac{p}{3}}\|^2_{L^2(\Omega^\varepsilon_{e,s})} &\leq \|\nabla b^\varepsilon_e\|^2_{L^2(\Omega^{\varepsilon,1}_{e,s})} + \|\nabla |b^\varepsilon_e|^{\frac{p-1}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s} \setminus \Omega^{\varepsilon,1}_{e,s})} \\
&\leq \|\nabla b^\varepsilon_e\|^2_{L^2(\Omega^\varepsilon_{e,s})} + \|\nabla |b^\varepsilon_e|^{\frac{p-1}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})},
\end{aligned}
\tag{110}
$$

where $\Omega^{\varepsilon,1}_{e,s} = \{(t,x) \in \Omega^\varepsilon_{e,s} : b^\varepsilon_e(t,x) \le 1\}$. Also notice that for $p \ge 3$ we have $\frac{3}{4}\frac{(2p-2)}{2p-3} \le 1$ and $\frac{2p}{3} \le p-1$. Thus applying the Young inequality in (109) yields

$$
\begin{aligned}
\int_0^s \|b^\varepsilon_e\|_{L^{2p}(\Omega^\varepsilon_e)} \|b^\varepsilon_{e,N}\|^{p-1}_{L^{2p}(\Omega^\varepsilon_e)} dt \le \delta_1 \sup_{(0,s)} \|b^\varepsilon_{e,N}\|^p_{L^p(\Omega^\varepsilon_e)} + \delta_2 \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})} \\
+ C_\delta \Big(1 + \|b^\varepsilon_e\|^{p-1}_{L^{p-1}(\Omega^\varepsilon_{e,s})} + \|\nabla b^\varepsilon_e\|^2_{L^2(\Omega^\varepsilon_{e,s})} + \|\nabla |b^\varepsilon_e|^{\frac{p-1}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})}\Big)^{\frac{3}{2}}
\end{aligned} \tag{111}
$$

for any $\delta_1 > 0$ and $\delta_2 > 0$. Using the trace inequality, we estimate the integral over Γ^ε as

$$
\begin{aligned}
&\varepsilon \langle |P(b^\varepsilon_e)|, |b^\varepsilon_{e,N}|^{p-1}\rangle_{\Gamma^\varepsilon_s} \le C_1 \varepsilon \langle 1 + |b^\varepsilon_e|, |b^\varepsilon_{e,N}|^{p-1}\rangle_{\Gamma^\varepsilon_s} \\
&\le C_2(\varepsilon) \int_0^s \Big[1 + \| |b^\varepsilon_e|^{\frac{p}{3}} \|^{\frac{1}{4}}_{L^2(\Omega^\varepsilon_e)} \|\nabla |b^\varepsilon_e|^{\frac{p}{3}}\|^{\frac{3}{4}}_{L^2(\Omega^\varepsilon_e)} + \| |b^\varepsilon_e|^{\frac{p}{3}} \|^{\frac{1}{6}}_{L^2(\Omega^\varepsilon_e)} \|\nabla |b^\varepsilon_e|^{\frac{p}{3}}\|^{\frac{5}{6}}_{L^2(\Omega^\varepsilon_e)} \Big]^{\frac{3}{p}} \\
&\qquad\qquad \times \Big[\| |b^\varepsilon_{e,N}|^{\frac{p}{2}} \|_{L^2(\Omega^\varepsilon_e)} \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|_{L^2(\Omega^\varepsilon_e)} \Big]^{\frac{p-1}{p}} dt \\
&\le C_3(\varepsilon) \Big[1 + \| |b^\varepsilon_e|^{\frac{p}{3}} \|^{\frac{1}{2p}}_{L^\infty(0,s;L^2(\Omega^\varepsilon_e))} \|\nabla |b^\varepsilon_e|^{\frac{p}{3}}\|^{\frac{5}{2(p+1)}}_{L^2(\Omega^\varepsilon_{e,s})} \Big] \\
&\qquad\qquad \times \sup_{(0,s)} \| |b^\varepsilon_{e,N}|^{\frac{p}{2}} \|^{\frac{p-1}{p}}_{L^2(\Omega^\varepsilon_e)} \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|^{\frac{p-1}{p}}_{L^2(\Omega^\varepsilon_{e,s})}.
\end{aligned} \tag{112}
$$

Applying the Young inequality on the right-hand side of (112) and using (110), together with the uniform estimate of $\|\nabla b^\varepsilon_e\|_{L^2(\Omega^\varepsilon_{e,s})}$ obtained in Lemma 3.2, yields

$$
\begin{aligned}
\varepsilon \langle |P(b^\varepsilon_e)|, |b^\varepsilon_{e,N}|^{p-1}\rangle_{\Gamma^\varepsilon_s} \le C(\varepsilon) \Big[1 + \| |b^\varepsilon_e|^{\frac{p}{3}} \|^{\frac{1}{2}}_{L^\infty(0,s;L^2(\Omega^\varepsilon_e))} \Big(1 + \|\nabla |b^\varepsilon_e|^{\frac{p-1}{2}}\|^{\frac{5p}{2(p+1)}}_{L^2(\Omega^\varepsilon_{e,s})} \Big) \Big] \\
+ \delta_1 \sup_{(0,s)} \| |b^\varepsilon_{e,N}|^{\frac{p}{2}} \|^2_{L^2(\Omega^\varepsilon_e)} + \delta_2 \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})}.
\end{aligned}
$$

The same calculations together with (110) ensure that

$$
\begin{aligned}
\langle |F_b(b^\varepsilon_e)|, |b^\varepsilon_{e,N}|^{p-1}\rangle_{(\partial\Omega)_s} \le C(\varepsilon) \Big[1 + \| |b^\varepsilon_e|^{\frac{p-1}{2}} \|^3_{L^\infty(0,s;L^2(\Omega^\varepsilon_e))} + \|\nabla |b^\varepsilon_e|^{\frac{p-1}{2}}\|^3_{L^2(\Omega^\varepsilon_{e,s})} \Big] \\
+ \delta_1 \sup_{(0,s)} \|b^\varepsilon_{e,N}\|^p_{L^p(\Omega^\varepsilon_e)} + \delta_2 \|\nabla |b^\varepsilon_{e,N}|^{\frac{p}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})}.
\end{aligned}
$$

Considering $p = 3$ and using the standard a priori estimates (27) for b^ε_e yields

$$
\begin{aligned}
\int_0^s \|b^\varepsilon_e\|_{L^6(\Omega^\varepsilon_e)} \|b^\varepsilon_{e,N}\|^2_{L^6(\Omega^\varepsilon_e)} dt \le C \Big[\int_0^s \Big(\|b^\varepsilon_e\|^2_{L^2(\Omega^\varepsilon_e)} + \|\nabla b^\varepsilon_e\|^2_{L^2(\Omega^\varepsilon_e)} \Big) dt \Big]^{\frac{1}{2}} \\
\times \sup_{(0,s)} \|b^\varepsilon_{e,N}\|^{\frac{1}{2}}_{L^3(\Omega^\varepsilon_e)} \Big[\int_0^s \|\nabla |b^\varepsilon_{e,N}|^{\frac{3}{2}}\|^2_{L^2(\Omega^\varepsilon_e)} dt \Big]^{\frac{1}{2}} \\
\le C_\delta + \delta_1 \sup_{(0,s)} \|b^\varepsilon_{e,N}(s)\|^3_{L^3(\Omega^\varepsilon_e)} + \delta_2 \|\nabla |b^\varepsilon_{e,N}|^{\frac{3}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})}.
\end{aligned} \tag{113}
$$

For the boundary integrals, for $p = 3$, we have

$$\begin{aligned}
&\varepsilon\langle |P(b_e^\varepsilon)|, |b_{e,N}^\varepsilon|^2\rangle_{\Gamma_s^\varepsilon} + \langle |F_b(b_e^\varepsilon)|, |b_{e,N}^\varepsilon|^2\rangle_{(\partial\Omega)_s} \\
&\le C_1(\varepsilon)\Big[1 + \|b_e^\varepsilon\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}^{\frac{1}{6}} \|\nabla b_e^\varepsilon\|_{L^2(\Omega_{e,s}^\varepsilon)}^{\frac{5}{8}}\Big] \\
&\quad \cdot \sup_{(0,s)} \||b_{e,N}^\varepsilon|^{\frac{3}{2}}\|_{L^2(\Omega_e^\varepsilon)}^{\frac{2}{3}} \|\nabla |b_{e,N}^\varepsilon|^{\frac{3}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^{\frac{2}{3}} \\
&\le C_2(\varepsilon)\Big[1 + \|b_e^\varepsilon\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}^{\frac{1}{2}} \|\nabla b_e^\varepsilon\|_{L^2(\Omega_{e,s}^\varepsilon)}^{\frac{15}{8}}\Big] \\
&\quad + \delta_1 \sup_{(0,s)} \|b_{e,N}^\varepsilon(s)\|_{L^3(\Omega_e^\varepsilon)}^3 + \delta_2 \|\nabla |b_{e,N}^\varepsilon|^{\frac{3}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^2 .
\end{aligned} \tag{114}$$

Considering (107) for $p = 3$ and using the estimates (108), (113), and (114) together with the standard a priori estimates for b_e^ε, c_e^ε, and u_e^ε shown in Lemma 3.2, we obtain

$$\begin{aligned}
&\|b_{e,N}^\varepsilon(s)\|_{L^3(\Omega_e^\varepsilon)}^3 + \|\nabla |b_{e,N}^\varepsilon|^{\frac{3}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^2 \\
&\le C(\varepsilon) + \delta_1 \sup_{(0,s)} \|b_{e,N}^\varepsilon(s)\|_{L^3(\Omega_e^\varepsilon)}^3 + \delta_2 \|\nabla |b_{e,N}^\varepsilon|^{\frac{3}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^2
\end{aligned}$$

with $s \in (0, T]$, a constant $C(\varepsilon)$ independent of N, and arbitrary $0 < \delta_1 \le \frac{1}{2}$ and $0 < \delta_2 \le \frac{1}{2}$. Considering the supremum over $(0, s)$ and taking the limit $N \to \infty$ yields that $b_e^\varepsilon \in L^\infty(0, T; L^3(\Omega_e^\varepsilon))$ and $\nabla |b_e^\varepsilon|^{\frac{3}{2}} \in L^2(\Omega_{e,T}^\varepsilon)$. Taking iteratively $p = 4, 5, \dots$ and choosing $\delta_1 > 0$ and $\delta_2 > 0$ sufficiently small for each fixed p and for fixed ε, we obtain estimates for $\|b_{e,N}^\varepsilon\|_{L^\infty(0,T;L^p(\Omega_e^\varepsilon))}$ and $\|\nabla |b_{e,N}^\varepsilon|^{\frac{p}{2}}\|_{L^2(\Omega_{e,T}^\varepsilon)}^2$ independent of N. Letting $N \to \infty$ yields that $|b_e^\varepsilon|^{\frac{p}{2}} \in L^2(0, T; H^1(\Omega_e^\varepsilon))$ and $b_e^\varepsilon \in L^\infty(0, T; L^p(\Omega_e^\varepsilon))$ for every fixed $p \ge 2$.

Now we consider $(b_e^\varepsilon)^{p-1}$ as a test function in (10) and obtain

$$\begin{aligned}
&\frac{1}{p}\|b_e^\varepsilon(s)\|_{L^p(\Omega_e^\varepsilon)}^p + \frac{4(p-1)}{p^2}\|\nabla |b_e^\varepsilon|^{\frac{p}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^2 \le \frac{1}{p}\|b_{e0}\|_{L^p(\Omega_e^\varepsilon)}^p + \|b_e^\varepsilon\|_{L^p(\Omega_{e,s}^\varepsilon)}^p \\
&\quad + C_1\big(\|c_e^\varepsilon\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \|\mathbf{e}(u_e^\varepsilon)\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + 1\big)\||b_e^\varepsilon|^{\frac{p}{2}}\|_{L^2(0,s;L^4(\Omega_e^\varepsilon))}^2 \\
&\qquad + \frac{C_2}{p}\big(\|c_e^\varepsilon\|_{L^p(0,s;L^2(\Omega_e^\varepsilon))}^p + \|\mathbf{e}(u_e^\varepsilon)\|_{L^p(0,s;L^2(\Omega_e^\varepsilon))}^p\big) \\
&\qquad + \varepsilon\langle |P(b_e^\varepsilon)|, |b_e^\varepsilon|^{p-1}\rangle_{\Gamma_s^\varepsilon} + \langle F_b(b_e^\varepsilon), |b_e^\varepsilon|^{p-1}\rangle_{(\partial\Omega)_s}
\end{aligned} \tag{115}$$

for $s \in (0, T]$. The integral over Γ^ε is estimated as

$$\begin{aligned}
\varepsilon\langle |P(b_e^\varepsilon)|, |b_e^\varepsilon|^{p-1}\rangle_{\Gamma_s^\varepsilon} &\le C_1\varepsilon\langle 1 + |b_e^\varepsilon|, |b_e^\varepsilon|^{p-1}\rangle_{\Gamma_s^\varepsilon} \\
&\le C_2\Big(1 + \||b_e^\varepsilon|^{\frac{p}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^2 + \varepsilon^2\|\nabla |b_e^\varepsilon|^{\frac{p}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^2\Big).
\end{aligned}$$

Using the properties of extension of b_e^ε from Ω_e^ε to Ω and applying the Gagliardo–Nirenberg inequality

$$\|w\|_{L^2(\Omega)} \le C\|\nabla w\|_{L^2(\Omega)}^{\alpha_1}\|w\|_{L^1(\Omega)}^{1-\alpha_1}, \qquad \|w\|_{L^4(\Omega)} \le C\|\nabla w\|_{L^2(\Omega)}^{\alpha_2}\|w\|_{L^1(\Omega)}^{1-\alpha_2},$$

with $\alpha_1 = \frac{3}{5}$ and $\alpha_2 = \frac{9}{10}$, we obtain

$$\begin{aligned}
\varepsilon\langle |P(b_e^\varepsilon)|, |b_e^\varepsilon|^{p-1}\rangle_{\Gamma_s^\varepsilon} &\le C_\delta\big(1 + \||b_e^\varepsilon|^{\frac{p}{2}}\|_{L^1(\Omega_{e,s}^\varepsilon)}^2\big) + (\varepsilon^2 + \delta)\|\nabla |b_e^\varepsilon|^{\frac{p}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^2, \\
\langle |F_b(b_e^\varepsilon)|, |b_e^\varepsilon|^{p-1}\rangle_{(\partial\Omega)_s} &\le C\big(1 + \||b_e^\varepsilon|^{\frac{p}{2}}\|_{L^2((0,s)\times\partial\Omega)}^2\big) \\
&\le C_\delta\big[1 + \||b_e^\varepsilon|^{\frac{p}{2}}\|_{L^1(\Omega_{e,s}^\varepsilon)}^2\big] + \delta\|\nabla |b_e^\varepsilon|^{\frac{p}{2}}\|_{L^2(\Omega_{e,s}^\varepsilon)}^2 .
\end{aligned}$$

Then applying the Gagliardo–Nirenberg inequality and the extension lemma, Lemma 3.1, to $\||b_e^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(\Omega_s)}$ and $\||b_e^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))}$ in (115) and using the estimates (27) yields

$$\|b_e^\varepsilon(s)\|^p_{L^p(\Omega_e^\varepsilon)} + \|\nabla|b_e^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \le C_1^p + C_2(1+p^{10})\int_0^s \||b_e^\varepsilon|^{\frac{p}{2}}\|^2_{L^1(\Omega_e^\varepsilon)}dt,$$

where the constants C_1 and C_2 are independent of ε. Then the Alikakos iteration lemma implies the boundedness of b_e^ε, uniformly in ε.

We turn to c^ε. Considering first $(c_{e,N}^\varepsilon)^{p-1}$ and $(c_{f,N}^\varepsilon)^{p-1}$, where $c_{j,N}^\varepsilon(t,x) = \min\{c_j^\varepsilon(t,x), N\}$ for $(t,x)\in\Omega_{j,T}^\varepsilon$ with $j=e,f$ and $N>0$, as test functions in (11) and performing calculations similar to those in the derivation of (107), we obtain

$$\begin{aligned}
&\|c_{e,N}^\varepsilon(s)\|^p_{L^p(\Omega_e^\varepsilon)} + \|c_{f,N}^\varepsilon(s)\|^p_{L^p(\Omega_f^\varepsilon)} + \|\nabla|c_{e,N}^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + \|\nabla|c_{f,N}^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(\Omega_{f,s}^\varepsilon)}\\
&\le \|c_e^\varepsilon(0)\|^p_{L^p(\Omega_e^\varepsilon)} + \|c_f^\varepsilon(0)\|^p_{L^p(\Omega_f^\varepsilon)} + C_1\big[1+\|\mathcal{G}(\partial_t u_f^\varepsilon)\|^2_{L^\infty(\Omega_{f,T}^\varepsilon)}\big]\|c_{f,N}^\varepsilon\|^p_{L^p(\Omega_{f,s}^\varepsilon)}\\
&+C_2\Big[\|b_e^\varepsilon\|^p_{L^p(\Omega_{e,s}^\varepsilon)} + \|c_{e,N}^\varepsilon\|^p_{L^p(\Omega_{e,s}^\varepsilon)}\Big] + C_3\int_0^s\big(1+\|c_e^\varepsilon\|_{L^p(\partial\Omega)}\big)\|c_{e,N}^\varepsilon\|^{p-1}_{L^p(\partial\Omega)}dt\\
&+C_4\|\mathbf{e}(u_e^\varepsilon)\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}\int_0^s\Big[\||b_e^\varepsilon|^p\|_{L^2(\Omega_e^\varepsilon)} + \|c_e^\varepsilon\|_{L^{2p}(\Omega_e^\varepsilon)}\|c_{e,N}^\varepsilon\|^{p-1}_{L^{2p}(\Omega_e^\varepsilon)}\\
&+\||c_{e,N}^\varepsilon|^p\|_{L^2(\Omega_e^\varepsilon)}\Big]dt + C_5\|c_e^\varepsilon\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}\||c_{e,N}^\varepsilon|^{\frac{p}{2}}\|^{2\frac{p-1}{p}}_{L^2(0,s;L^4(\Omega_e^\varepsilon))}\\
&+C_6\|c_f^\varepsilon\|_{L^\infty(0,s;L^2(\Omega_f^\varepsilon))}\||c_{f,N}^\varepsilon|^{\frac{p}{2}}\|^{2\frac{p-1}{p}}_{L^2(0,s;L^4(\Omega_f^\varepsilon))}.
\end{aligned}$$

Similar to (111) we estimate

$$\begin{aligned}
\int_0^s\|c_e^\varepsilon\|_{L^{2p}(\Omega_e^\varepsilon)}\|c_{e,N}^\varepsilon\|^{p-1}_{L^{2p}(\Omega_e^\varepsilon)}dt &\le \delta_1\sup_{(0,s)}\|c_{e,N}^\varepsilon(s)\|^p_{L^p(\Omega_e^\varepsilon)} + \delta_2\|\nabla|c_{e,N}^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)}\\
&+C_\delta\Big(1+\|c_e^\varepsilon\|^{p-1}_{L^{p-1}(\Omega_{e,s}^\varepsilon)} + \|\nabla c_e^\varepsilon\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + \|\nabla|c_e^\varepsilon|^{\frac{p-1}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)}\Big)^{\frac{3}{2}}.
\end{aligned}$$

The boundary integral can be estimated in the same way as in (112):

$$\begin{aligned}
\int_0^s(1+\|c_e^\varepsilon\|_{L^p(\partial\Omega)})\|c_{e,N}^\varepsilon\|^{p-1}_{L^p(\partial\Omega)}dt &\le \delta_1\sup_{(0,s)}\|c_{e,N}^\varepsilon(s)\|^p_{L^p(\Omega_e^\varepsilon)} + \delta_2\|\nabla|c_{e,N}^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)}\\
&+C(\varepsilon)\Big[1+\||c_e^\varepsilon|^{\frac{p-1}{2}}\|^3_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \|\nabla|c_e^\varepsilon|^{\frac{p-1}{2}}\|^3_{L^2(\Omega_{e,s}^\varepsilon)}\Big].
\end{aligned}$$

Considering $p=3,\dots,6$ iteratively, using estimates (27), and making the calculations similar to those for $b_{e,N}^\varepsilon$ yields

$$\begin{aligned}
&\|c_{e,N}^\varepsilon\|_{L^\infty(0,T;L^6(\Omega_e^\varepsilon))} + \|c_{f,N}^\varepsilon(s)\|_{L^\infty(0,T;L^6(\Omega_f^\varepsilon))}\\
&+\|\nabla|c_{e,N}^\varepsilon|^3\|_{L^2(\Omega_{e,s}^\varepsilon)} + \|\nabla|c_{f,N}^\varepsilon|^3\|_{L^2(\Omega_{f,s}^\varepsilon)} \le C,
\end{aligned}$$

where the constant C depends on p and ε and is independent of N. Letting $N\to\infty$, we obtain that $(c_j^\varepsilon)^{p-1}\in L^2(0,T;H^1(\Omega_j^\varepsilon))$ with $j=e,f$ and $p=3,\dots,6$. Thus we can consider $(c_e^\varepsilon)^{p-1}$ and $(c_f^\varepsilon)^{p-1}$, with $p = 3, 4$, as test functions in (11):

$$\begin{aligned}
&\|c_e^\varepsilon(s)\|^p_{L^p(\Omega_e^\varepsilon)} + \|c_f^\varepsilon(s)\|^p_{L^p(\Omega_f^\varepsilon)} + \|\nabla |c_e^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})} + \|\nabla |c_f^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(\Omega^\varepsilon_{f,s})} \\
&\leq \|c_e^\varepsilon(0)\|^p_{L^p(\Omega_e^\varepsilon)} + \|c_f^\varepsilon(0)\|^p_{L^p(\Omega_f^\varepsilon)} + C_1\big[1 + \|\mathcal{G}(\partial_t u_f^\varepsilon)\|^2_{L^\infty(\Omega^\varepsilon_{f,T})}\big]\|c_f^\varepsilon\|^p_{L^p(\Omega^\varepsilon_{f,s})} \\
&+ C_2\|\mathbf{e}(u_e^\varepsilon)\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}\big[\||b_e^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))} + \||c_e^\varepsilon|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))}\big] \\
&+ C_3\big[1 + \|b_e^\varepsilon\|^p_{L^p(\Omega^\varepsilon_{e,s})} + \|c_e^\varepsilon\|^p_{L^p(\Omega^\varepsilon_{e,s})} + \|c_e^\varepsilon\|^p_{L^p((0,s)\times\partial\Omega)}\big].
\end{aligned}$$

In the same way as in (115), applying the Gagliardo–Nirenberg inequality to $|c_j^\varepsilon|^{\frac{p}{2}}$ in $L^2(\Omega_j^\varepsilon)$ and $L^4(\Omega_j^\varepsilon)$ and using properties of the extension of c_e^ε from Ω_e^ε to Ω and of c^ε from $\widetilde{\Omega}_{ef}$ to Ω, we obtain

$$\|c_e^\varepsilon\|_{L^\infty(0,T;L^4(\Omega_e^\varepsilon))} + \|c_f^\varepsilon\|_{L^\infty(0,T;L^4(\Omega_f^\varepsilon))} + \|\nabla|c_e^\varepsilon|^2\|_{L^2(\Omega^\varepsilon_{e,T})} + \|\nabla|c_f^\varepsilon|^2\|_{L^2(\Omega^\varepsilon_{f,T})} \leq C,$$

where the constant C is independent of ε. ∎

Next we present the proof of the estimate for $\|\widetilde{b}_e^{\varepsilon,j}\|_{L^\infty(0,s;L^\infty(\Omega_e^\varepsilon))}$.

LEMMA 10.2. *For the difference of two iterations $\widetilde{b}_e^{\varepsilon,j} = b_e^{\varepsilon,j-1} - b_e^{\varepsilon,j}$, $\widetilde{u}_e^{\varepsilon,j-1} = u_e^{\varepsilon,j-2} - u_e^{\varepsilon,j-1}$, and $\partial_t\widetilde{u}_f^{\varepsilon,j-1} = \partial_t u_f^{\varepsilon,j-2} - \partial_t u_f^{\varepsilon,j-1}$ for the microscopic system* (6)–(8), *defined in Theorem* 3.3, *we have*

$$\begin{aligned}
\|\widetilde{b}_e^{\varepsilon,j}\|_{L^\infty(0,s;L^\infty(\Omega_e^\varepsilon))} &\leq C\|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|_{L^{1+\frac{1}{\sigma}}(0,s;L^2(\Omega_e^\varepsilon))} \\
&+ C_\delta\|\partial_t\widetilde{u}_f^{\varepsilon,j-1}\|_{L^2(\Omega^\varepsilon_{f,s})} + \delta\|\mathbf{e}(\partial_t\widetilde{u}_f^{\varepsilon,j-1})\|_{L^2(\Omega^\varepsilon_{f,s})}
\end{aligned}$$

for $s \in (0, T)$, any $\delta > 0$, and $0 < \sigma < 1/9$, where the constants C and C_δ are independent of s and j.

Proof. Considering $(\widetilde{b}_e^{\varepsilon,j})^{p-1}$ as a test function in the weak formulation of (31) yields

$$\begin{aligned}
&\frac{1}{p}\|\widetilde{b}_e^{\varepsilon,j}(s)\|^p_{L^p(\Omega_e^\varepsilon)} + \frac{2(p-1)}{p^2}\|\nabla|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(\Omega^\varepsilon_{e,s})} \leq C_1\|\widetilde{b}_e^{\varepsilon,j}\|^p_{L^p(\Omega^\varepsilon_{e,s})} \\
&+ C_2\left[\|c_e^{\varepsilon,j}\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \|\mathbf{e}(u_e^{\varepsilon,j-1})\|_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}\right]\||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))} \\
&+ C_3\|b_e^{\varepsilon,j-1}\|_{L^\infty(\Omega^\varepsilon_{e,s})}\left[\frac{1}{p}\|\widetilde{c}_e^{\varepsilon,j}\|^p_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))} + \frac{p-1}{p}\||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))}\right] \\
&+ C_4\|b_e^{\varepsilon,j-1}\|_{L^\infty(\Omega^\varepsilon_{e,s})}\langle|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})|, |\widetilde{b}_e^{\varepsilon,j}|^{p-1}\rangle_{\Omega^\varepsilon_{e,s}}
\end{aligned}\tag{116}$$

for $s \in (0, T)$. Applying the Hölder inequality, we estimate

$$\begin{aligned}
\langle|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})|, |\widetilde{b}_e^{\varepsilon,j}|^{p-1}\rangle_{\Omega^\varepsilon_{e,s}} &\leq \int_0^s \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|_{L^{\frac{2p}{p+1}}(\Omega_e^\varepsilon)}\|\widetilde{b}_e^{\varepsilon,j}\|^{p-1}_{L^{2p}(\Omega_e^\varepsilon)}dt \\
&\leq C_1\int_0^s \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|_{L^2(\Omega_e^\varepsilon)}\|\widetilde{b}_e^{\varepsilon,j}\|^{p-1}_{L^{2p}(\Omega_e^\varepsilon)}dt \\
\leq C_2\bigg(\int_0^s \|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|^{\frac{p(1+\sigma)}{(p\sigma+1)}}_{L^2(\Omega_e^\varepsilon)}dt\bigg)^{\frac{(p\sigma+1)}{p(1+\sigma)}} &\bigg(\int_0^s \||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^{2(1+\sigma)}_{L^4(\Omega_e^\varepsilon)}dt\bigg)^{\frac{(p-1)}{p(1+\sigma)}}
\end{aligned}\tag{117}$$

for some $\sigma > 0$. Applying the Gagliardo–Nirenberg inequality

$$\|w\|_{L^4(\Omega)} \leq C\|\nabla w\|^\alpha_{L^2(\Omega)}\|w\|^{1-\alpha}_{L^1(\Omega)}$$

with $\alpha = \frac{9}{10}$, we obtain for $0 < \sigma < 1/9$

$$\left(\int_0^s \||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^{2(1+\sigma)}_{L^4(\Omega_e^\varepsilon)} dt\right)^{\frac{1}{1+\sigma}} \leq C\left(\int_0^s \|\nabla|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^{2(1+\sigma)\alpha}_{L^2(\Omega_e^\varepsilon)} \||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^{2(1+\sigma)(1-\alpha)}_{L^1(\Omega_e^\varepsilon)} dt\right)^{\frac{1}{(1+\sigma)}}$$
$$\leq C_2\|\nabla|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^{2\alpha}_{L^2(\Omega_{e,s}^\varepsilon)}\left(\int_0^s \||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^{\frac{2(1+\sigma)(1-\alpha)}{1-\alpha(1+\sigma)}}_{L^1(\Omega_e^\varepsilon)} dt\right)^{\frac{1-\alpha(1+\sigma)}{1+\sigma}}$$
$$\leq \frac{\delta}{p}\|\nabla|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + C_\delta p^{\frac{\alpha}{1-\alpha}}\left(\int_0^s \||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^{\frac{2(1+\sigma)(1-\alpha)}{1-\alpha(1+\sigma)}}_{L^1(\Omega_e^\varepsilon)} dt\right)^{\frac{1-\alpha(1+\sigma)}{(1+\sigma)(1-\alpha)}} \tag{118}$$
$$\leq \frac{\delta}{p}\|\nabla|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} + C_\delta p^{\frac{\alpha}{1-\alpha}}\||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^\infty(0,s;L^1(\Omega_e^\varepsilon))}$$

for any $\delta > 0$. Hence we have the following estimate:

$$\begin{aligned}\langle|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})|, |\widetilde{b}_e^{\varepsilon,j}|^{p-1}\rangle_{\Omega_{e,s}^\varepsilon} &\leq \delta\frac{p-1}{p^2}\|\nabla|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \\ &+ C_\delta\frac{(p-1)p^\beta}{p}\||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^\infty(0,s;L^1(\Omega_e^\varepsilon))} + C\frac{1}{p}\|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|^p_{L^{1+\frac{1}{\sigma}}(0,s;L^2(\Omega_e^\varepsilon))},\end{aligned} \tag{119}$$

with $\beta = \frac{\alpha}{1-\alpha}$. We incorporate inequality (119) in (116), estimate $\|\widetilde{b}_e^{\varepsilon,j}\|^p_{L^p(\Omega_{e,s}^\varepsilon)}$ and $\||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(0,s;L^4(\Omega_e^\varepsilon))}$ in terms of $\||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(0,s;L^1(\Omega_e^\varepsilon))}$ and $\|\nabla|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)}$ by applying the Gagliardo–Nirenberg inequality, and then use the estimate (35) for $\|\widetilde{c}_e^{\varepsilon,j}\|^p_{L^\infty(0,s;L^2(\Omega_e^\varepsilon))}$ and the boundedness of $b_e^{\varepsilon,j-1}$, which can be shown in the same way as the L^∞-estimates in (106), to obtain

$$\begin{aligned}&\|\widetilde{b}_e^{\varepsilon,j}(s)\|^p_{L^p(\Omega_e^\varepsilon)} + \|\nabla|\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^2(\Omega_{e,s}^\varepsilon)} \\ &\quad\leq C_1p^{10}\sup_{(0,s)}\||\widetilde{b}_e^{\varepsilon,j}|^{\frac{p}{2}}\|^2_{L^1(\Omega_e^\varepsilon)} + C_2^p\|\mathbf{e}(\widetilde{u}_e^{\varepsilon,j-1})\|^p_{L^{1+\frac{1}{\sigma}}(0,s;L^2(\Omega_e^\varepsilon))} \\ &\quad+ \delta^p\|\mathbf{e}(\partial_t\widetilde{u}_f^{\varepsilon,j-1})\|^p_{L^2(\Omega_{f,s}^\varepsilon)} + C_\delta^p\|\partial_t\widetilde{u}_f^{\varepsilon,j-1}\|^p_{L^2(\Omega_{f,s}^\varepsilon)}.\end{aligned}$$

Using (35) and iterating in $p = 2^k$ for $k = 2, 3, \dots$, similarly to [2, Lemma 3.2], we obtain the estimate stated in the lemma. ∎

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