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HOMOGENIZATION OF A VISCOELASTIC MODEL FOR PLANT CELL WALL BIOMECHANICS *

Mariya Ptashnyk[1] and Brian Seguin[2]

Abstract. The microscopic structure of a plant cell wall is given by cellulose microfibrils embedded in a cell wall matrix. In this paper we consider a microscopic model for interactions between viscoelastic deformations of a plant cell wall and chemical processes in the cell wall matrix. We consider elastic deformations of the cell wall microfibrils and viscoelastic Kelvin–Voigt type deformations of the cell wall matrix. Using homogenization techniques (two-scale convergence and periodic unfolding methods) we derive macroscopic equations from the microscopic model for cell wall biomechanics consisting of strongly coupled equations of linear viscoelasticity and a system of reaction-diffusion and ordinary differential equations. As is typical for microscopic viscoelastic problems, the macroscopic equations governing the viscoelastic deformations of plant cell walls contain memory terms. The derivation of the macroscopic problem for the degenerate viscoelastic equations is conducted using a perturbation argument.

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1. Introduction

To obtain a better understanding of the mechanical properties and development of plant tissues it is important to model and analyse the interactions between the chemical processes and mechanical deformations of plant cells. The main feature of plant cells are their walls, which must be strong to resist high internal hydrostatic pressure (turgor pressure) and flexible to permit growth. The biomechanics of plant cell walls is determined by the cell wall microstructure, given by microfibrils, and the physical properties of the cell wall matrix. The orientation of microfibrils, their length, high tensile strength, and interactions with wall matrix macromolecules strongly influence the wall's stiffness. It is also supposed that calcium-pectin cross-linking chemistry is one of the main regulators of cell wall elasticity and extension [30]. Pectin can be modified by the enzyme pectin methylesterase (PME), which removes methyl groups by breaking ester bonds. The de-esterified pectin is able to form calcium-pectin cross-links, and so stiffen the cell wall and reduce its expansion, see *e.g.* [29]. It has been shown that the modification of pectin by PME and the control of the amount of calcium-pectin cross-links

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greatly influence the mechanical deformations of plant cell walls [23,24], and the interference with PME activity causes dramatic changes in growth behavior of plant cells and tissues [31].

To address the interactions between microstructure, chemistry and mechanics, in the microscopic model for plant cell wall biomechanics we consider the influence of the microscopic structure, associated with the cellulose microfibrils, and the calcium-pectin cross-links on the mechanical properties of plant cell walls. We model the cell wall as a three-dimensional continuum consisting of a polysaccharide matrix and cellulose microfibrils. It was observed experimentally that plant cell wall microfibrils are anisotropic, see *e.g.* [10], and the cell wall matrix, in addition to elastic deformations, exhibits viscous behaviour, see *e.g.* [14]. Hence we model the cell wall matrix as a linearly viscoelastic Kelvin–Voigt material, whereas microfibrils are modelled as an anisotropic linearly elastic material. Within the matrix, we consider the dynamics of the enzyme PME, methylesterfied pectin, demethylesterfied pectin, calcium ions, and calcium-pectin cross-links. A model for plant cell wall biomechanics in which the cell wall matrix was assumed to be linearly elastic was derived and analysed in [26]. The interplay between the mechanics and the cross-link dynamics comes in by assuming that the elastic and viscous properties of the cell wall matrix depend on the density of the cross-links and that stress within the cell wall can break calcium-pectin cross-links. The stress-dependent opening of calcium channels in the cell plasma membrane is addressed in the flux boundary conditions for calcium ions. The resulting microscopic model is a system of strongly coupled four diffusion-reaction equations, one ordinary differential equation, and the equations of linear viscoelasticity. Since only the cell wall matrix is viscoelastic we obtain degenerate elastic-viscoelastic equations. In our model we focus on the interactions between the chemical reactions within the cell wall and its deformation and, hence, do not consider the growth of the cell wall.

To analyse the macroscopic mechanical properties of the plant cell wall we rigorously derive macroscopic equations from the microscopic description of plant cell wall biomechanics. The two-scale convergence, *e.g.* [4,21], and the periodic unfolding method, *e.g.* [7,8], are applied to obtain the macroscopic equations. For the viscoelastic equations the macroscopic momentum balance equation contains a term that depends on the history of the strain represented by an integral term (fading memory effect). Due to the coupling between the viscoelastic properties and the biochemistry of a plant cell wall, the elastic and viscous tensors depend on space and time variables. This fact introduces additional complexity in the derivation and in the structure of the macroscopic equations, compered to classical viscoelastic equations.

The main novelty of this paper is the multiscale analysis and derivation of the macroscopic problem from a microscopic description of the mechanical and chemical processes. This approach allows us to take into account the complex microscopic structure of a plant cell wall and to analyse the impact of the heterogeneous distribution of cell wall structural elements on the mechanical properties of plants. The main mathematical difficulty arises from the strong coupling between the equations of linear viscoelasticity for cell wall mechanics and the system of reaction-diffusion and ordinary differential equations for the chemical processes in the wall matrix. Also the degeneracy of the viscoelastic equations, due to the fact that only the cell wall matrix is assumed to be viscoelastic and microfibrils are assumed to be elastic, induces additional technical difficulties in the multiscale analysis of the microscopic model. To derive the macroscopic equations for the viscoelastic model for cell wall biomechanics we consider perturbed equations by introducing an inertial term. Once the macroscopic problem of the perturbed equations is derived, the perturbation parameter is sent to zero. By showing that the limit problem (as the perturbation parameter tends to zero) of the two-scale macroscopic problem for the perturbed microscopic equations is the same as the two-scale macroscopic problem for the original microscopic equations, we obtain the effective homogenized equations for the original viscoelastic problem coupled with reaction-diffusion and ordinary differential equations. A perturbation approach, by considering a viscosity term multiplied by a small perturbation parameter in the elastic inclusions, was also used in [11] to derive a macroscopic model for an elastic-viscoelastic problem.

A multiscale analysis of the viscoelastic equations with time-independent coefficients was considered previously in [12,13,18,27]. Macroscopic equations for scalar elastic-viscoelastic equations with time-independent coefficients were derived in [11] by applying the H-convergence method [19]. A microscopic viscoelastic Kelvin–Voigt model with time-dependent coefficients in the context of thermo-viscoelasticity was analysed in [1] and macroscopic equations were derived by applying the method of asymptotic expansion.

The paper is organised as follows. In Section 2 we formulate a mathematical model for plant cell wall biomechanics in which the cell wall matrix is assumed to be viscoelastic. In Section 3 we summarise the main results of the paper. The well-possedness of the microscopic model is shown in Section 4. The multiscale analysis of the microscopic model is conducted in Section 5.

2. Microscopic model for viscoelastic deformations of plant cell walls

The main feature of plant cells are their walls, which must be strong to resist high internal hydrostatic pressure and flexible to permit the growth. To better understand the interplay between these in some sense conflicting functions, we consider a mathematical model describing the interactions between the mechanical properties and the chemical processes in cell walls, surrounding plant cells. Plant cell walls are separated from the inside of the cell by the plasma membrane, modelled as an internal boundary of the cell wall (see Fig. 1a). Individual cells in plant tissues are joined together by a pectin network of middle lamella. The primary wall of a plant cell consists mainly of oriented cellulose microfibrils imbedded in the cell wall matrix, which is composed of pectin, hemicellulose, structural proteins, and water. It was observed experimentally that in addition to elastic deformations the plant cell wall matrix exhibits viscoelastic behaviour [14]. Hence, in contrast to the model considered in [26], here we assume that the deformations of the plant cell wall matrix are determined by the equations of linear viscoelasticity.

To model mechanical deformations of plant cell walls, we consider a domain $\Omega = (0, a_1) \times (0, a_2) \times (0, a_3)$ representing a flat section of a cell wall, where a_i, with $i = 1, 2, 3$, are positive numbers. We assume that the microfibrils are oriented in the x_3-direction (see Fig. 1b). We shall distinguish between six disjoint parts of the boundary $\partial\Omega$ of the domain Ω. The interior boundary $\Gamma_{\mathcal{I}} = \{0\} \times (0, a_2) \times (0, a_3)$ represents the cell plasma membrane, the exterior boundary $\Gamma_{\mathcal{E}} = \{a_1\} \times (0, a_2) \times (0, a_3)$ denotes the side of the cell wall which is in contact with the middle lamella, on the top and bottom boundaries $\Gamma_{\mathcal{U}} = (0, a_1) \times \{0\} \times (0, a_3) \cup (0, a_1) \times \{a_2\} \times (0, a_3)$ we will prescribe traction boundary conditions, reflecting the turgor pressure. On the boundaries $\Gamma_{\mathcal{P}} = (0, a_1) \times (0, a_2) \times \{0\} \cup (0, a_1) \times (0, a_2) \times \{a_3\}$ we consider periodic boundary conditions.

To determine the microscopic structure of the cell wall given by cell wall microfibrils, we consider $Y = (0, 1)^2 \times (0, a_3)$ and define $\hat{Y} = (0, 1)^2$, together with the subdomain $\hat{Y}_F$, with $\overline{\hat{Y}_F} \subset \hat{Y}$, and $\hat{Y}_M = \hat{Y} \setminus \overline{\hat{Y}_F}$. Then $Y_F = \hat{Y}_F \times (0, a_3)$ and $Y_M = \hat{Y}_M \times (0, a_3)$ represent the cell wall microfibrils and cell wall matrix, rescaled to the 'unit cell' Y (see Fig. 1c). We also define $\hat{\Gamma} = \partial\hat{Y}_F \cap \partial\hat{Y}_M$ and $\Gamma = \partial Y_F \cap \partial Y_M$.

We assume that the microfibrils in the cell wall are distributed periodically and have a diameter on the order of ε, where the small parameter ε characterise the size of the microstructure, *i.e.* the ratio between the diameter of the microfibrils and the thickness of the cell wall. The domains

$$\Omega_F^\varepsilon = \bigcup_{\xi \in \mathbb{Z}^2} \left\{ \varepsilon(\hat{Y}_F + \xi) \times (0, a_3) \;\middle|\; \varepsilon(\hat{Y} + \xi) \subset (0, a_1) \times (0, a_2) \right\} \quad \text{and} \quad \Omega_M^\varepsilon = \Omega \setminus \overline{\Omega_F^\varepsilon}$$

denote the parts of Ω occupied by the microfibrils and by the cell wall matrix, respectively. The boundary between the cell wall matrix and the microfibrils is denoted by

$$\Gamma^\varepsilon = \partial\Omega_M^\varepsilon \cap \partial\Omega_F^\varepsilon.$$

We adopt the following notation: $\Omega_T = (0, T) \times \Omega$, $\Omega_{M,T}^\varepsilon = (0, T) \times \Omega_M^\varepsilon$, $\Gamma_{\mathcal{I},T} = (0, T) \times \Gamma_{\mathcal{I}}$, $\Gamma_T^\varepsilon = (0, T) \times \Gamma^\varepsilon$, $\Gamma_{\mathcal{U},T} = (0, T) \times \Gamma_{\mathcal{U}}$, $\Gamma_{\mathcal{E},T} = (0, T) \times \Gamma_{\mathcal{E}}$, and $\Gamma_{\mathcal{EU},T} = (0, T) \times (\Gamma_{\mathcal{E}} \cup \Gamma_{\mathcal{U}})$, and define

$$\mathcal{W}(\Omega) = \{\mathbf{u} \in H^1(\Omega; \mathbb{R}^3) \mid \int_\Omega \mathbf{u}\, dx = \mathbf{0}, \int_\Omega [(\nabla\mathbf{u})_{12} - (\nabla\mathbf{u})_{21}]\, dx = 0 \text{ and } \mathbf{u} \text{ is } a_3\text{-periodic in } x_3\},$$

$$\mathcal{V}(\Omega_M^\varepsilon) = \{n \in H^1(\Omega_M^\varepsilon) \mid n \text{ is } a_3\text{-periodic in } x_3\}.$$

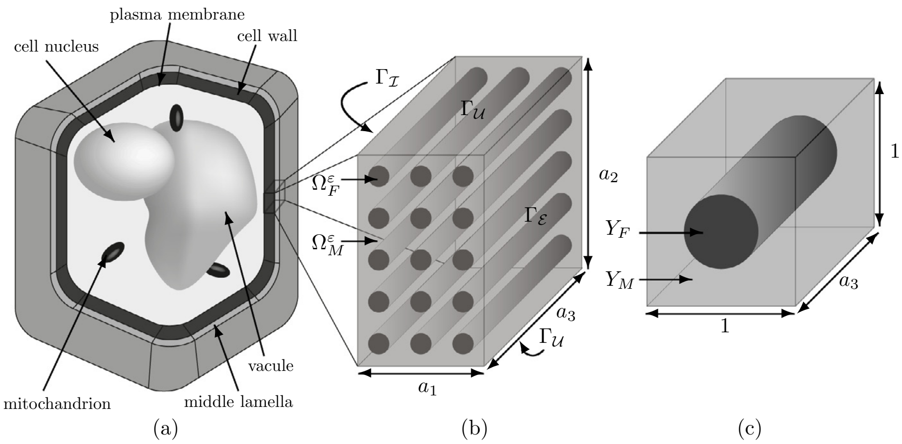

FIGURE 1. (a) A schematic of a plant cell with an indication of the domain Ω as a part of the cell wall. (b) A depiction of the domain Ω with the subsets representing the cell wall matrix Ω_M^ε and the microfibrils Ω_F^ε. The (hidden) surface $\Gamma_{\mathcal{I}}$ corresponds to the plasma membrane and is in contact with the interior of the cell, the surface $\Gamma_{\mathcal{E}}$ is facing the outside of the cell and is in contact with the middle lamella, and $\Gamma_{\mathcal{U}}$ is the union of the surfaces on the top and bottom of Ω. (c) A depiction of the 'unit cell' Y.

By Korn's second inequality, the L^2-norm of the strain defines a norm on $\mathcal{W}(\Omega)$

$$\|\mathbf{u}\|_{\mathcal{W}(\Omega)} = \|\mathbf{e}(\mathbf{u})\|_{L^2(\Omega)} \qquad \text{for all } \mathbf{u} \in \mathcal{W}(\Omega),$$

see *e.g.* [6, 17, 22]. For more details see also [26].

The microscopic model for elastic-viscoelastic deformations $\mathbf{u}^\varepsilon$ of plant cell walls and for the densities of esterified pectin p_1^ε, PME enzyme p_2^ε, de-esterified pectin n_1^ε, calcium ions n_2^ε, and calcium-pectin cross-links b^ε reads

$$\begin{cases} \operatorname{div}\left(\mathbb{E}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\mathbf{u}^\varepsilon) + \mathbb{V}^\varepsilon(b^\varepsilon, x)\partial_t\mathbf{e}(\mathbf{u}^\varepsilon)\right) = \mathbf{0} & \text{in } \Omega_T, \\ \left(\mathbb{E}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\mathbf{u}^\varepsilon) + \mathbb{V}^\varepsilon(b^\varepsilon, x)\partial_t\mathbf{e}(\mathbf{u}^\varepsilon)\right)\boldsymbol{\nu} = -p_{\mathcal{I}}\boldsymbol{\nu} & \text{on } \Gamma_{\mathcal{I},T}, \\ \left(\mathbb{E}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\mathbf{u}^\varepsilon) + \mathbb{V}^\varepsilon(b^\varepsilon, x)\partial_t\mathbf{e}(\mathbf{u}^\varepsilon)\right)\boldsymbol{\nu} = \mathbf{f} & \text{on } \Gamma_{\mathcal{EU},T}, \\ \mathbf{u}^\varepsilon & a_3\text{-periodic in } x_3, \\ \mathbf{u}^\varepsilon(0,x) = \mathbf{u}_0(x) & \text{in } \Omega, \end{cases} \tag{2.1}$$

and

$$\begin{aligned} \partial_t\mathbf{p}^\varepsilon &= \operatorname{div}(D_p\nabla\mathbf{p}^\varepsilon) - \mathbf{F}_p(\mathbf{p}^\varepsilon) && \text{in } \Omega_{M,T}^\varepsilon, \\ \partial_t\mathbf{n}^\varepsilon &= \operatorname{div}(D_n\nabla\mathbf{n}^\varepsilon) + \mathbf{F}_n(\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon) + \mathbf{R}_n(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))) && \text{in } \Omega_{M,T}^\varepsilon, \\ \partial_t b^\varepsilon &= R_b(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))) && \text{in } \Omega_{M,T}^\varepsilon, \end{aligned} \tag{2.2}$$

where $\mathbf{p}^\varepsilon = (p_1^\varepsilon, p_2^\varepsilon)^T$, $\mathbf{n}^\varepsilon = (n_1^\varepsilon, n_2^\varepsilon)^T$, and $\operatorname{div}(D_p \nabla \mathbf{p}^\varepsilon) = (\operatorname{div}(D_p^1 \nabla p_1^\varepsilon), \operatorname{div}(D_p^2 \nabla p_2^\varepsilon))^T$, and $\operatorname{div}(D_n \nabla \mathbf{n}^\varepsilon) = (\operatorname{div}(D_n^1 \nabla n_1^\varepsilon), \operatorname{div}(D_n^2 \nabla n_2^\varepsilon))^T$, together with the initial and boundary conditions

$$
\begin{aligned}
&D_p \nabla \mathbf{p}^\varepsilon \boldsymbol{\nu} = \mathbf{J}_p(\mathbf{p}^\varepsilon) && \text{on } \Gamma_{\mathcal{I},T},\\
&D_p \nabla \mathbf{p}^\varepsilon \boldsymbol{\nu} = -\gamma_p \mathbf{p}^\varepsilon && \text{on } \Gamma_{\mathcal{E},T},\\
&D_n \nabla \mathbf{n}^\varepsilon \boldsymbol{\nu} = \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))\mathbf{G}(\mathbf{n}^\varepsilon) && \text{on } \Gamma_{\mathcal{I},T},\\
&D_n \nabla \mathbf{n}^\varepsilon \boldsymbol{\nu} = \mathbf{J}_n(\mathbf{n}^\varepsilon) && \text{on } \Gamma_{\mathcal{E},T},\\
&D_p \nabla \mathbf{p}^\varepsilon \boldsymbol{\nu} = \mathbf{0}, \quad D_n \nabla \mathbf{n}^\varepsilon \boldsymbol{\nu} = \mathbf{0} && \text{on } \Gamma_T^\varepsilon \text{ and } \Gamma_{\mathcal{U},T},\\
&\mathbf{p}^\varepsilon, \ \ \mathbf{n}^\varepsilon && a_3\text{-periodic in } x_3,\\
&\mathbf{p}^\varepsilon(0,x) = \mathbf{p}_0(x), \quad \mathbf{n}^\varepsilon(0,x) = \mathbf{n}_0(x), \quad b^\varepsilon(0,x) = b_0(x) && \text{in } \Omega_M^\varepsilon.
\end{aligned}
\tag{2.3}
$$

Here $\mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))$, defined as

$$
\mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon)) = \left(\fint_{B_\delta(x)\cap\Omega} \operatorname{tr}\left(\mathbb{E}^\varepsilon(b^\varepsilon, \tilde{x})\mathbf{e}(\mathbf{u}^\varepsilon)\right) d\tilde{x} \right)^+ \quad \text{in } (0,T)\times\overline{\Omega}, \quad \text{for } \delta > 0,
\tag{2.4}
$$

represents the nonlocal impact of mechanical stresses on the calcium-pectin cross-links chemistry, where $B_\delta(x)$ is a ball of a fixed radius $\delta > 0$ at $x \in \overline{\Omega}$. From a biological point of view the nonlocal dependence of the chemical reactions on the displacement gradient is motivated by the fact that pectins are long molecules and hence cell wall mechanics has a nonlocal impact on the chemical processes. The positive part in the definition of $\mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))$ reflects the fact that extension rather than compression causes the breakage of cross-links. The boundary condition $(2.3)_3$ reflects the fact that the flow of calcium ions between the interior of the cell and the cell wall depends on the displacement gradient, which corresponds to the stress-dependent opening of calcium channels in the plasma membrane [28].

The elasticity and viscosity tensors are defined as $\mathbb{E}^\varepsilon(\xi, x) = \mathbb{E}(\xi, \hat{x}/\varepsilon)$ and $\mathbb{V}^\varepsilon(\xi, x) = \mathbb{V}(\xi, \hat{x}/\varepsilon)$, where the $\hat{Y}$-periodic in y functions $\mathbb{E}$ and $\mathbb{V}$ are given by $\mathbb{E}(\xi, y) = \mathbb{E}_M(\xi)\chi_{\hat{Y}_M}(y) + \mathbb{E}_F \chi_{\hat{Y}_F}(y)$ and $\mathbb{V}(\xi, y) = \mathbb{V}_M(\xi)\chi_{\hat{Y}_M}(y)$.

For a given measurable set $\mathcal{A}$ we use the notation $\langle \phi_1, \phi_2 \rangle_{\mathcal{A}} = \int_{\mathcal{A}} \phi_1 \phi_2 \, dx$, where the product of ϕ_1 and ϕ_2 is the scalar-product if they are vector valued. By $\langle \psi_1, \psi_2 \rangle_{\mathcal{V},\mathcal{V}'}$ we denote the dual product between $\psi_1 \in L^2(0,T;\mathcal{V}(\Omega_M^\varepsilon))$ and $\psi_2 \in L^2(0,T;\mathcal{V}(\Omega_M^\varepsilon)')$. We also denote $\mathcal{I}_\mu^k = (-\mu, +\infty)^k$, for an arbitrary fixed $\mu > 0$ and $k \in \mathbb{N}$.

Throughout the text we shall use boldface letters, either upper or lower case, to denote vectors. However, matrices are not denoted with bold letters. Blackboard bold characters, with the exception of the standard symbols for the real numbers and the integers, denote fourth-order tensors.

Assumption 2.1.

1. $D_\alpha^j \in \mathbb{R}^{3\times 3}$ is symmetric, with $(D_\alpha^j \boldsymbol{\xi}, \boldsymbol{\xi}) \geq d_\alpha |\boldsymbol{\xi}|^2$ for all $\boldsymbol{\xi} \in \mathbb{R}^3$ and some $d_\alpha > 0$, where $\alpha = p, n$, $j = 1, 2$, and $\gamma_p \geq 0$.
2. $\mathbf{F}_p : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in $\mathcal{I}_\mu^2$, with $F_{p,1}(0,\eta) = 0$, $F_{p,2}(\xi, 0) = 0$, $F_{p,1}(\xi,\eta) \geq 0$, and $|F_{p,2}(\xi,\eta)| \leq g_1(\xi)(1+\eta)$ for all $\xi, \eta \in \mathbb{R}_+$ and some $g_1 \in C^1(\mathbb{R}_+;\mathbb{R}_+)$.
3. $\mathbf{J}_p : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in $\mathcal{I}_\mu^2$, with $J_{p,1}(0,\eta) \geq 0$, $J_{p,2}(\xi,0) \geq 0$, $|J_{p,1}(\xi,\eta)| \leq \gamma_J(1+\xi)$, and $|J_{p,2}(\xi,\eta)| \leq g(\xi)(1+\eta)$ for all $\xi, \eta \in \mathbb{R}_+$ and some $\gamma_J > 0$ and $g \in C^1(\mathbb{R}_+;\mathbb{R}_+)$.
4. $\mathbf{F}_n : \mathbb{R}^4 \to \mathbb{R}^2$ is continuously differentiable in $\mathcal{I}_\mu^4$, with $F_{n,1}(\boldsymbol{\xi}, 0, \eta_2) \geq 0$, $F_{n,2}(\boldsymbol{\xi}, \eta_1, 0) \geq 0$, and

$$
|F_{n,1}(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq \gamma_F^1(1 + g_2(\boldsymbol{\xi}) + |\boldsymbol{\eta}|), \quad |F_{n,2}(\boldsymbol{\xi}, \boldsymbol{\eta})| \leq \gamma_F^2(1 + g_2(\boldsymbol{\xi}) + |\boldsymbol{\eta}|),
$$

for all $\boldsymbol{\xi} = (\xi_1, \xi_2)^T$, $\boldsymbol{\eta} = (\eta_1, \eta_2)^T \in \mathbb{R}_+^2$ and some $\gamma_F^1, \gamma_F^2 > 0$, and $g_2 \in C^1(\mathbb{R}_+^2;\mathbb{R}_+)$.

5. $\mathbf{R}_n : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}^2$ and $R_b : \mathbb{R}^3 \times \mathbb{R}_+ \to \mathbb{R}$ are continuously differentiable in $\mathcal{I}_\mu^3 \times \mathbb{R}_+$ and satisfy

$$\begin{aligned} &R_{n,1}(0,\xi_2,\eta,\zeta) \geq 0, && |R_{n,1}(\boldsymbol{\xi},\eta,\zeta)| \leq \beta_1(1+|\boldsymbol{\xi}|+\eta)(1+\zeta), \\ &R_{n,2}(\xi_1,0,\eta,\zeta) \geq 0, && |R_{n,2}(\boldsymbol{\xi},\eta,\zeta)| \leq \beta_2(1+|\boldsymbol{\xi}|+\eta)(1+\zeta), \\ &R_b(\boldsymbol{\xi},0,\zeta) \geq 0, && |R_b(\boldsymbol{\xi},\eta,\zeta)| \leq \beta_3(1+|\boldsymbol{\xi}|+\eta)(1+\zeta), \qquad (R_b(\boldsymbol{\xi},\eta,\zeta))^+ \leq \beta_4 \end{aligned}$$

   for some $\beta_j > 0$, $j = 1,\dots,4$, and all $\boldsymbol{\xi} = (\xi_1,\xi_2)^T \in \mathbb{R}^2_+$, $\eta,\zeta \in \mathbb{R}_+$.
6. $\mathbf{J}_n : \mathbb{R}^2 \to \mathbb{R}^2$ is continuously differentiable in $\mathcal{I}_\mu^2$, with $J_{n,1}(0,\eta) \geq 0$, $J_{n,2}(\xi,0) \geq 0$, $|J_{n,1}(\xi,\eta)| \leq \gamma_n^1(1+\xi)$, and $|J_{n,2}(\xi,\eta)| \leq \gamma_n^2(1+\xi+\eta)$ for all $\xi,\eta \in \mathbb{R}_+$ and some $\gamma_n^1, \gamma_n^2 > 0$.
7. $\mathbf{G}(\xi,\eta) : \mathbb{R}^2 \to \mathbb{R}^2$, with $\mathbf{G}(\xi,\eta) = (0, \gamma_1 - \gamma_2\eta)^T$ for $\eta \in \mathbb{R}$ and some $\gamma_1,\gamma_2 \geq 0$.
8. $\mathbb{V}_M \in C^1(\mathbb{R})$ possesses major and minor symmetries, *i.e.* $\mathbb{V}_{M,ijkl} = \mathbb{V}_{M,klij} = \mathbb{V}_{M,jikl} = \mathbb{V}_{M,ijlk}$, and there exists $\omega_V > 0$ such that $\mathbb{V}_M(\xi)A \cdot A \geq \omega_V|A|^2$ for all symmetric $A \in \mathbb{R}^{3\times3}$ and $\xi \in \mathbb{R}_+$.
9. $\mathbb{E}_M \in C^1(\mathbb{R})$, $\mathbb{E}_F$, $\mathbb{E}_M$ possess major and minor symmetries, *i.e.* $\mathbb{E}_{L,ijkl} = \mathbb{E}_{L,klij} = \mathbb{E}_{L,jikl} = \mathbb{E}_{L,ijlk}$, for $L = F, M$, and there exists $\omega_E > 0$ such that $\mathbb{E}_F A \cdot A \geq \omega_E|A|^2$, $\mathbb{E}_M(\xi)A \cdot A \geq \omega_E|A|^2$, and $\mathbb{E}'_M(\xi)A \cdot A \geq 0$ for all symmetric $A \in \mathbb{R}^{3\times3}$ and $\xi \in \mathbb{R}_+$. There exists $\gamma_M > 0$ such that $|\mathbb{E}_M(\xi)| \leq \gamma_M$ for all $\xi \in \mathbb{R}_+$.
10. The initial conditions $\mathbf{p}_0 = (p_{0,1}, p_{0,2})^T, \mathbf{n}_0 = (n_{0,1}, n_{0,2})^T \in L^\infty(\Omega)^2$, $b_0 \in H^1(\Omega) \cap L^\infty(\Omega)$ are non-negative, and $\mathbf{u}_0 \in \mathcal{W}(\Omega)$.
11. $\mathbf{f} \in H^1(0,T; L^2(\Gamma_\mathcal{E} \cup \Gamma_\mathcal{U}))^3$ and $p_\mathcal{I} \in H^1(0,T; L^2(\Gamma_\mathcal{I}))$.

Remark 2.2. Notice that Assumption 2.1.9 is not restrictive from a physical point of view, since every biological material will have a maximal possible stiffness. Also, in contrast to [26], we assume that $(R_b(\boldsymbol{\xi},\eta,\zeta))^+$ is bounded, see Assumption 2.1.5. This assumption is used to derive *a priori* estimates for solutions of the equations of linear viscoelasticity, independent of b^ε, and to prove the global in time existence of a weak solution of (2.1) and (2.3) for arbitrary initial data and boundary conditions satisfying Assumptions 2.1.10 and 2.1.11. The local in time existence of a weak solution or the existence of a weak solution for small data can be shown by considering the same assumptions as in [26], *i.e.* without the assumption of the boundedness of $(R_b(\boldsymbol{\xi},\eta,\zeta))^+$. Notice that possible biologically relevant forms for reaction terms in (2.2) are given by $\mathbf{F}_p(\mathbf{p}) = (R_{eE}(\mathbf{p}), 0)^T$, $\mathbf{F}_n(\mathbf{p},\mathbf{n}) = (R_{eE}(\mathbf{p}) - 2R_{dc}(\mathbf{n}) - R_d n_1, -R_{dc}(\mathbf{n}))^T$, $\mathbf{R}_n(\mathbf{n}, b, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}))) = (2R_{bb}(b)\mathcal{N}_\delta(\mathbf{e}(\mathbf{u})), R_{bb}(b)\mathcal{N}_\delta(\mathbf{e}(\mathbf{u})))^T$, and $R_b(\mathbf{n}, b, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}))) = R_{dc}(\mathbf{n}) - R_{bb}(b)\mathcal{N}_\delta(\mathbf{e}(\mathbf{u}))$. Then the boundedness of $(R_b(\boldsymbol{\xi},\eta,\zeta))^+$, assumed in Assumption 2.1.5, is ensured if $(R_{dc}(\boldsymbol{\xi}))^+$ is bounded for nonnegative ξ_1 and ξ_2, *e.g.* R_{dc} is a Hill function.

A weak solution of (2.1)–(2.3) is defined in the following way.

Definition 2.3. A weak solution of the microscopic model (2.1)–(2.3) is a tuple $(\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon, b^\varepsilon, \mathbf{u}^\varepsilon)$, such that $b^\varepsilon \in H^1(0,T; L^2(\Omega_M^\varepsilon))$, $\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon \in L^2(0,T; \mathcal{V}(\Omega_M^\varepsilon))^2$, $\partial_t\mathbf{p}^\varepsilon, \partial_t\mathbf{n}^\varepsilon \in L^2(0,T; \mathcal{V}(\Omega_M^\varepsilon)')^2$ and satisfy the equations

$$\begin{aligned} \langle \partial_t \mathbf{p}^\varepsilon, \boldsymbol{\phi}_p\rangle_{\mathcal{V},\mathcal{V}'} + \langle D_p \nabla\mathbf{p}^\varepsilon, \nabla\boldsymbol{\phi}_p\rangle_{\Omega^\varepsilon_{M,T}} &= -\langle \mathbf{F}_p(\mathbf{p}^\varepsilon), \boldsymbol{\phi}_p\rangle_{\Omega^\varepsilon_{M,T}} + \langle \mathbf{J}_p(\mathbf{p}^\varepsilon), \boldsymbol{\phi}_p\rangle_{\Gamma_{\mathcal{I},T}} - \langle \gamma_p\mathbf{p}^\varepsilon, \boldsymbol{\phi}_p\rangle_{\Gamma_{\mathcal{E},T}}, \\ \langle \partial_t \mathbf{n}^\varepsilon, \boldsymbol{\phi}_n\rangle_{\mathcal{V},\mathcal{V}'} + \langle D_n \nabla\mathbf{n}^\varepsilon, \nabla\boldsymbol{\phi}_n\rangle_{\Omega^\varepsilon_{M,T}} &= \big\langle \mathbf{F}_n(\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon) + \mathbf{R}_n(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))), \boldsymbol{\phi}_n\big\rangle_{\Omega^\varepsilon_{M,T}} \\ &\quad + \big\langle \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))\mathbf{G}(\mathbf{n}^\varepsilon), \boldsymbol{\phi}_n\big\rangle_{\Gamma_{\mathcal{I},T}} + \langle \mathbf{J}_n(\mathbf{n}^\varepsilon), \boldsymbol{\phi}_n\rangle_{\Gamma_{\mathcal{E},T}} \end{aligned} \tag{2.5}$$

for all $\boldsymbol{\phi}_p, \boldsymbol{\phi}_n \in L^2(0,T; \mathcal{V}(\Omega_M^\varepsilon))^2$,

$$\partial_t b^\varepsilon = R_b(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))) \quad \text{a.e. in } \Omega^\varepsilon_{M,T}, \tag{2.6}$$

and $\mathbf{u}^\varepsilon \in L^2(0,T; \mathcal{W}(\Omega))$, with $\partial_t\mathbf{e}(\mathbf{u}^\varepsilon) \in L^2((0,T)\times\Omega_M^\varepsilon)^3$, satisfies

$$\big\langle \mathbb{E}^\varepsilon(b^\varepsilon, x)\mathbf{e}(\mathbf{u}^\varepsilon) + \mathbb{V}^\varepsilon(b^\varepsilon, x)\partial_t\mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{e}(\boldsymbol{\psi})\big\rangle_{\Omega_T} = \langle \mathbf{f}, \boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{EU},T}} - \langle p_\mathcal{I}\boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{I},T}} \tag{2.7}$$

for all $\boldsymbol{\psi} \in L^2(0,T; \mathcal{W}(\Omega))$. Furthermore, $\mathbf{p}^\varepsilon$, $\mathbf{n}^\varepsilon$, b^ε satisfy the initial conditions in $L^2(\Omega_M^\varepsilon)$ and $\mathbf{u}^\varepsilon$ satisfies the initial condition in $\mathcal{W}(\Omega)$, *i.e.* $\mathbf{u}^\varepsilon(t,\cdot) \to \mathbf{u}_0$ in $\mathcal{W}(\Omega)$, $\mathbf{p}^\varepsilon(t,\cdot) \to \mathbf{p}_0$, $\mathbf{n}^\varepsilon(t,\cdot) \to \mathbf{n}_0$ in $L^2(\Omega_M^\varepsilon)^2$, and $b^\varepsilon(t,\cdot) \to b_0$ in $L^2(\Omega_M^\varepsilon)$ as $t \to 0$.

3. Main results

The main result of this paper is the derivation of the macroscopic equations for the microscopic viscoelastic model for plant cell wall biomechanics. The main difference between the homogenization results presented here and those in [26] is due to the presence of a degenerate viscous term in the equations for the mechanical deformations of a cell wall. The fact that only the cell wall matrix is viscoelastic and the dependence of the viscosity tensor on the time variable, *via* the dependence on the cross-links density b^ε, make the multiscale analysis nonclassical and complex.

First we formulate the well-posedness result for the model (2.1)–(2.3).

Theorem 3.1. *Under Assumption 2.1 there exists a unique weak solution of (2.1)–(2.3) satisfying the* a priori *estimates*

$$\|b^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} + \|(\partial_t b^\varepsilon)^+\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} \leq C_1, \tag{3.1}$$

where the constant C_1 is independent of ε and δ,

$$\|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(\Omega))} + \|\partial_t \mathbf{e}(\mathbf{u}^\varepsilon)\|_{L^2((0,T)\times\Omega^\varepsilon_M)} \leq C_2, \tag{3.2}$$

where the constant C_2 is independent of ε, and

$$\begin{aligned}\|\mathbf{p}^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} + \|\nabla\mathbf{p}^\varepsilon\|_{L^2(\Omega^\varepsilon_{M,T})} + \|\mathbf{n}^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} + \|\nabla\mathbf{n}^\varepsilon\|_{L^2(\Omega^\varepsilon_{M,T})} &\leq C_3,\\ \|\partial_t b^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} &\leq C_3,\\ \|\theta_h\mathbf{p}^\varepsilon - \mathbf{p}^\varepsilon\|_{L^2(\Omega^\varepsilon_{M,T-h})} + \|\theta_h\mathbf{n}^\varepsilon - \mathbf{n}^\varepsilon\|_{L^2(\Omega^\varepsilon_{M,T-h})} &\leq C_3 h^{1/4}\end{aligned} \tag{3.3}$$

for any $h > 0$, where $\theta_h v(t,x) = v(t+h,x)$ for $(t,x) \in \Omega^\varepsilon_{M,T-h}$, with $h \in (0,T)$, and the constant C_3 is independent of ε and h.

The proof of Theorem 3.1 is similar to the proof of the corresponding existence and uniqueness results in [26]. Thus here we will only sketch the main ideas of the proof and emphasise the steps that are different from those of the proof in [26].

To formulate the macroscopic equations for the microscopic model (2.1)–(2.3), first we define the macroscopic coefficients which will be obtained in the derivation of the limit equations. The macroscopic coefficients coming from the elasticity tensor are given by

$$\begin{aligned}\widetilde{\mathbb{E}}_{\mathrm{hom},ijkl}(b) &= \fint_{\hat{Y}} \left[\mathbb{E}_{ijkl}(b,y) + \left(\mathbb{E}(b,y)\,\hat{\mathbf{e}}_y(\mathbf{w}^{ij})\right)_{kl}\right]\mathrm{d}y,\\ \widetilde{\mathbb{K}}_{ijkl}(t,s,b) &= \fint_{\hat{Y}} \left(\mathbb{E}(b(t+s),y)\,\hat{\mathbf{e}}_y(\mathbf{v}^{ij}(t,s))\right)_{kl}\,\mathrm{d}y,\end{aligned} \tag{3.4}$$

and the macroscopic elasticity and viscosity tensors and the memory kernel read:

$$\begin{aligned}\mathbb{E}_{\mathrm{hom},ijkl}(b) &= \widetilde{\mathbb{E}}_{\mathrm{hom},ijkl}(b) + \frac{1}{|\hat{Y}|}\int_{\hat{Y}_M} \left(\mathbb{V}_M(b)\,\partial_t\hat{\mathbf{e}}_y(\mathbf{w}^{ij})\right)_{kl}\,\mathrm{d}y,\\ \mathbb{V}_{\mathrm{hom},ijkl}(b) &= \frac{1}{|\hat{Y}|}\int_{\hat{Y}_M} \left[\mathbb{V}_{M,ijkl}(b) + \left(\mathbb{V}_M(b)\,\hat{\mathbf{e}}_y(\boldsymbol{\chi}^{ij}_{\mathbb{V}})\right)_{kl}\right]\mathrm{d}y,\\ \mathbb{K}_{ijkl}(t,s,b) &= \widetilde{\mathbb{K}}_{ijkl}(t,s,b) + \frac{1}{|\hat{Y}|}\int_{\hat{Y}_M} \left(\mathbb{V}_M(b(t+s))\,\partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij}(t,s))\right)_{kl}\,\mathrm{d}y,\end{aligned} \tag{3.5}$$

where $\mathbf{w}^{ij}$, $\boldsymbol{\chi}_{\mathbb{V}}^{ij}$, and $\mathbf{v}^{ij}$, with $i,j=1,2,3$, are solutions of the 'unit cell' problems

$$\begin{aligned}
\hat{\operatorname{div}}_y\left(\mathbb{E}(b,y)(\hat{\mathbf{e}}_y(\mathbf{w}^{ij})+\mathbf{b}_{ij})+\mathbb{V}(b,y)\partial_t\hat{\mathbf{e}}_y(\mathbf{w}^{ij})\right)&=\mathbf{0} &&\text{in } \hat{Y}_T,\\
\mathbf{w}^{ij}(0,x,y)&=\mathbf{0} &&\text{in } \hat{Y},\\
\hat{\operatorname{div}}_y\left(\mathbb{V}_M(b)(\hat{\mathbf{e}}_y(\boldsymbol{\chi}_{\mathbb{V}}^{ij})+\mathbf{b}_{ij})\right)&=\mathbf{0} &&\text{in } \hat{Y}_M,\\
\mathbb{V}_M(b)(\hat{\mathbf{e}}_y(\boldsymbol{\chi}_{\mathbb{V}}^{ij})+\mathbf{b}_{ij})\boldsymbol{\nu}&=\mathbf{0} &&\text{on } \hat{\Gamma},\\
\int_{\hat{Y}}\mathbf{w}^{ij}\mathrm{d}y=\mathbf{0},\quad \int_{\hat{Y}_M}\boldsymbol{\chi}_{\mathbb{V}}^{ij}\mathrm{d}y=\mathbf{0},\qquad &\mathbf{w}^{ij},\ \boldsymbol{\chi}_{\mathbb{V}}^{ij} &&\hat{Y}\text{-periodic,}
\end{aligned}\tag{3.6}$$

where $\mathbf{b}_{jk}=\frac{1}{2}(\mathbf{b}_j\otimes\mathbf{b}_k+\mathbf{b}_k\otimes\mathbf{b}_j)$, with $\{\mathbf{b}_j\}_{1\le j\le 3}$ being the canonical basis of $\mathbb{R}^3$, and

$$\begin{aligned}
&\hat{\operatorname{div}}_y\left(\mathbb{E}(b(t+s,x),y)\hat{\mathbf{e}}_y(\mathbf{v}^{ij})+\mathbb{V}(b(t+s,x),y)\partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij})\right)=\mathbf{0} &&\text{in } \hat{Y}_{T-s},\\
&\mathbf{v}^{ij}(0,s,x,y)=\overline{\boldsymbol{\chi}}_{\mathbb{V}}^{ij}(s,x,y)-\mathbf{w}^{ij}(s,x,y) &&\text{in } \hat{Y},\\
&\int_{\hat{Y}}\mathbf{v}^{ij}\mathrm{d}y=\int_{\hat{Y}}\overline{\boldsymbol{\chi}}_{\mathbb{V}}^{ij}\mathrm{d}y, \qquad\qquad \mathbf{v}^{ij} &&\hat{Y}\text{-periodic,}
\end{aligned}\tag{3.7}$$

for $x\in\Omega$ and $s\in[0,T]$, where $\overline{\boldsymbol{\chi}}_{\mathbb{V}}^{ij}$ is an extension of $\boldsymbol{\chi}_{\mathbb{V}}^{ij}$ from $\hat{Y}_M$ into $\hat{Y}$. Here for a vector function $\mathbf{v}=(v_1,v_2,v_3)^T$ we denote $\hat{\operatorname{div}}_y\mathbf{v}=\partial_{y_1}v_1+\partial_{y_2}v_2$ and $\hat{\mathbf{e}}_y(\mathbf{v})$ is defined in the following way: $\hat{\mathbf{e}}_y(\mathbf{v})_{33}=0$, $\hat{\mathbf{e}}_y(\mathbf{v})_{3j}=\hat{\mathbf{e}}_y(\mathbf{v})_{j3}=\frac{1}{2}\partial_{y_j}v_3$ for $j=1,2$, and $\hat{\mathbf{e}}_y(\mathbf{v})_{ij}=\frac{1}{2}(\partial_{y_i}v_j+\partial_{y_j}v_i)$ for $i,j=1,2$.

The macroscopic diffusion coefficients are defined by

$$\mathcal{D}_{\alpha,ij}^l=⨍_{\hat{Y}_M}\left[D_{\alpha,ij}^l+(D_\alpha^l\hat{\nabla}_y v_{\alpha,l}^j)_i\right]\mathrm{d}y \quad \text{for}\quad i,j=1,2,3,\quad \alpha=p,n,\ \ l=1,2,\tag{3.8}$$

where $\hat{\nabla}_y v_{\alpha,l}^j=(\partial_{y_1}v_{\alpha,l}^j,\partial_{y_2}v_{\alpha,l}^j,0)^T$ and the functions $v_{\alpha,l}^j$ are solutions of the 'unit cell' problems

$$\begin{aligned}
&\operatorname{div}_{\hat{y}}(\hat{D}_\alpha^l\nabla_{\hat{y}}v_{\alpha,l}^j)=0 &&\text{in } \hat{Y}_M,\quad j=1,2,3,\\
&(\hat{D}_\alpha^l\nabla_{\hat{y}}v_{\alpha,l}^j+\tilde{D}_\alpha^l\mathbf{b}_j)\cdot\boldsymbol{\nu}=0 &&\text{on } \hat{\Gamma},\quad v_{\alpha,l}^j\quad \hat{Y}-\text{ periodic},\quad \int_{\hat{Y}_M}v_{\alpha,l}^j\,\mathrm{d}y=0,
\end{aligned}\tag{3.9}$$

where $\nabla_{\hat{y}}=(\partial_{y_1},\partial_{y_2})^T$, $\hat{D}_\alpha^l=(D_{\alpha,ik}^l)_{i,k=1,2}$ and $\tilde{D}_\alpha^l=(D_{\alpha,ik}^l)_{i=1,2,k=1,2,3}$, with $l=1,2$ and $\alpha=p,n$.

Applying techniques of periodic homogenization we obtain the macroscopic equations for plant cell wall biomechanics.

Theorem 3.2. *A sequence of solutions of the microscopic model* (2.1)–(2.3) *converges to a solution of the macroscopic equations*

$$\begin{aligned}
\partial_t\mathbf{p}&=\operatorname{div}(\mathcal{D}_p\nabla\mathbf{p})-\mathbf{F}_p(\mathbf{p}) &&\textit{in } \Omega_T,\\
\partial_t\mathbf{n}&=\operatorname{div}(\mathcal{D}_n\nabla\mathbf{n})+\mathbf{F}_n(\mathbf{p},\mathbf{n})+\mathbf{R}_n(\mathbf{n},b,\mathcal{N}_\delta^{\text{eff}}(\mathbf{e}(\mathbf{u}))) &&\textit{in } \Omega_T,\\
\partial_t b&=R_b(\mathbf{n},b,\mathcal{N}_\delta^{\text{eff}}(\mathbf{e}(\mathbf{u}))) &&\textit{in } \Omega_T,
\end{aligned}\tag{3.10}$$

together with the initial and boundary conditions

$$\begin{aligned}
&\mathcal{D}_p\nabla\mathbf{p}\,\boldsymbol{\nu}=\theta_M^{-1}\mathbf{J}_p(\mathbf{p}), &&\mathcal{D}_n\nabla\mathbf{n}\,\boldsymbol{\nu}=\theta_M^{-1}\mathbf{G}(\mathbf{n})\mathcal{N}_\delta^{\text{eff}}(\mathbf{e}(\mathbf{u})) &&\textit{on } \Gamma_{\mathcal{I},T},\\
&\mathcal{D}_p\nabla\mathbf{p}\,\boldsymbol{\nu}=-\theta_M^{-1}\gamma_p\,\mathbf{p}, &&\mathcal{D}_n\nabla\mathbf{n}\,\boldsymbol{\nu}=\theta_M^{-1}\mathbf{J}_n(\mathbf{n}) &&\textit{on } \Gamma_{\mathcal{E},T},\\
&\mathcal{D}_p\nabla\mathbf{p}\,\boldsymbol{\nu}=\mathbf{0}, &&\mathcal{D}_n\nabla\mathbf{n}\,\boldsymbol{\nu}=\mathbf{0} &&\textit{on } \Gamma_{\mathcal{U},T},\\
&\mathbf{p},\quad \mathbf{n} &&a_3\textit{-periodic in } x_3, &&\\
&\mathbf{p}(0)=\mathbf{p}_0,\quad \mathbf{n}(0)=\mathbf{n}_0, &&b(0)=b_0 &&\textit{in } \Omega,
\end{aligned}\tag{3.11}$$

where $\theta_M = |\hat{Y}_M|/|\hat{Y}|$, *and the macroscopic equations of linear viscoelasticity*

$$\begin{aligned}
\operatorname{div}\Big(\mathbb{E}_{\rm hom}(b)\mathbf{e}(\mathbf{u}) + \mathbb{V}_{\rm hom}(b)\partial_t \mathbf{e}(\mathbf{u}) + \int_0^t \mathbb{K}(t-s,s,b)\partial_s \mathbf{e}(\mathbf{u})\,\mathrm{d}s\Big) &= \mathbf{0} && \text{in } \Omega_T,\\
\Big(\mathbb{E}_{\rm hom}(b)\mathbf{e}(\mathbf{u}) + \mathbb{V}_{\rm hom}(b)\partial_t \mathbf{e}(\mathbf{u}) + \int_0^t \mathbb{K}(t-s,s,b)\partial_s \mathbf{e}(\mathbf{u})\,\mathrm{d}s\Big)\boldsymbol{\nu} &= \mathbf{f} && \text{on } \Gamma_{\mathcal{EU},T},\\
\Big(\mathbb{E}_{\rm hom}(b)\mathbf{e}(\mathbf{u}) + \mathbb{V}_{\rm hom}(b)\partial_t \mathbf{e}(\mathbf{u}) + \int_0^t \mathbb{K}(t-s,s,b)\partial_s \mathbf{e}(\mathbf{u})\,\mathrm{d}s\Big)\boldsymbol{\nu} &= -p_{\mathcal{I}}\boldsymbol{\nu} && \text{on } \Gamma_{\mathcal{I},T},\\
\mathbf{u} & && a_3\text{-periodic in } x_3,\\
\mathbf{u}(0) &= \mathbf{u}_0 && \text{in } \Omega.
\end{aligned} \tag{3.12}$$

Here

$$\mathcal{N}_\delta^{\rm eff}(\mathbf{e}(\mathbf{u})) = \left(\fint_{B_\delta(x)\cap\Omega} \operatorname{tr}\Big[\widetilde{\mathbb{E}}_{\rm hom}(b)\mathbf{e}(\mathbf{u}) + \int_0^t \widetilde{\mathbb{K}}(t-s,s,b)\partial_s \mathbf{e}(\mathbf{u})\mathrm{d}s\Big]\mathrm{d}\tilde{x}\right)^+ \quad \text{for } (t,x)\in(0,T)\times\overline{\Omega}. \tag{3.13}$$

4. Existence of a unique weak solution of the microscopic problem (2.1)–(2.3). *A priori* estimates

In the derivation of *a priori* estimates for solutions of the microscopic problem (2.1)–(2.3) we shall use an extension of a function defined on a connected perforated domain Ω_M^ε to Ω. Applying classical extension results [2, 9, 15, 22], we obtain the following lemma.

Lemma 4.1. *There exists an extension* $\overline{v}^\varepsilon$ *of* v^ε *from* $W^{1,p}(\Omega_M^\varepsilon)$ *into* $W^{1,p}(\Omega)$, *with* $1 \le p < \infty$, *such that*

$$\|\overline{v}^\varepsilon\|_{L^p(\Omega)} \le \mu_1 \|v^\varepsilon\|_{L^p(\Omega_M^\varepsilon)} \quad \text{and} \quad \|\nabla\overline{v}^\varepsilon\|_{L^p(\Omega)} \le \mu_1 \|\nabla v^\varepsilon\|_{L^p(\Omega_M^\varepsilon)},$$

where the constant μ_1 *depends only on* Y *and* Y_M, *and* $Y_M \subset Y$ *is connected.*

There exists an extension $\overline{\mathbf{w}}^\varepsilon$ *of* $\mathbf{w}^\varepsilon$ *from* $H^1(\Omega_M^\varepsilon)^3$ *into* $H^1(\Omega)^3$ *such that*

$$\|\overline{\mathbf{w}}^\varepsilon\|_{L^p(\Omega)} \le \mu_2\|\mathbf{w}^\varepsilon\|_{L^p(\Omega_M^\varepsilon)}, \qquad \|\nabla\overline{\mathbf{w}}^\varepsilon\|_{L^p(\Omega)} \le \mu_2\|\nabla\mathbf{w}^\varepsilon\|_{L^p(\Omega_M^\varepsilon)}, \qquad \|\mathbf{e}(\overline{\mathbf{w}}^\varepsilon)\|_{L^p(\Omega)} \le \mu_2\|\mathbf{e}(\mathbf{w}^\varepsilon)\|_{L^p(\Omega_M^\varepsilon)},$$

where the constant μ_2 *does not depend on* $\mathbf{w}^\varepsilon$ *and* ε.

Remark 4.2. Notice that the microfibrils do not intersect the boundaries $\Gamma_{\mathcal{I}}$, $\Gamma_{\mathcal{U}}$, and $\Gamma_{\mathcal{E}}$, and near the boundaries $\Gamma_{\mathcal{P}} = \partial\Omega \setminus (\Gamma_{\mathcal{I}} \cup \Gamma_{\mathcal{E}} \cup \Gamma_{\mathcal{U}})$ it is sufficient to extend v^ε and $\mathbf{w}^\varepsilon$ by reflection in the directions normal to the microfibrils and parallel to the boundary. Thus, classical extension results [2, 9, 15, 22, 25] apply to Ω_M^ε.

In the sequel, we identify $\mathbf{p}^\varepsilon$ and $\mathbf{n}^\varepsilon$ with their extensions.

First we show the well-posedness and *a priori* estimates for equations (2.2) and (2.3) for a given $\mathbf{u}^\varepsilon \in L^\infty(0,T;\mathcal{W}(\Omega))$. Next for a given b^ε we show the existence of a unique solution of the viscoelastic problem (2.1). Then using the fact that the estimates for b^ε can be obtained independently of $\mathbf{u}^\varepsilon$ and applying a fixed point argument we show the well-posedness of the coupled system.

Lemma 4.3. *Under Assumption 2.1 and for* $\mathbf{u}^\varepsilon \in L^\infty(0,T;\mathcal{W}(\Omega))$ *such that*

$$\|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(\Omega))} \le C, \tag{4.1}$$

where the constant C *is independent of* ε, *there exists a unique weak solution* $(\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon, b^\varepsilon)$ *of the microscopic problem* (2.2) *and* (2.3), *with* $\mathbf{p}^\varepsilon = (p_1^\varepsilon, p_2^\varepsilon)^T$ *and* $\mathbf{n}^\varepsilon = (n_1^\varepsilon, n_2^\varepsilon)^T$, *satisfying*

$$p_j^\varepsilon(t,x) \ge 0, \quad n_j^\varepsilon(t,x) \ge 0, \quad b^\varepsilon(t,x) \ge 0 \qquad \text{for } (t,x)\in(0,T)\times\Omega_M^\varepsilon, \quad j=1,2,$$

and the a priori *estimates* (3.1) *and* (3.3).

Proof. The proof of this lemma follows along the same lines as the proof of Theorem 3.1 in [26]. The only difference is in the derivation of the estimates for b^ε. Using the non-negativity of n_1^ε, n_2^ε, b^ε, and Assumption 2.1.5 we obtain from the equation for b^ε

$$\begin{aligned} &0 \le b^\varepsilon(t,x) \le \|b_0\|_{L^\infty(\Omega)} + T\|(R_b(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))))^+\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))} \le C \quad \text{for } (t,x) \in \Omega^\varepsilon_{M,T}, \\ &(\partial_t b^\varepsilon(t,x))^+ \le \|(R_b(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))))^+\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))} \le \beta_4 \quad \text{for } (t,x) \in \Omega^\varepsilon_{M,T}. \end{aligned} \tag{4.2}$$

Hence, the bounds for b^ε and $(\partial_t b^\varepsilon)^+$ are independent of the bound for $\|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(\Omega))}$. This fact is important for the derivation of *a priori* estimates for $\mathbf{u}^\varepsilon$ and for the fixed point argument in the proof of the existence of a global weak solution for the coupled system.

Using the equation for b^ε, the definition of $\mathcal{N}_\delta$, and the estimates for $\|\mathbf{n}^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))}$, $\|b^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))}$, and $\|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(\Omega))}$ we obtain the estimate for $\|\partial_t b^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))}$ uniformly in ε.

Similar to [26], integrating the equations for $\mathbf{p}^\varepsilon$ and $\mathbf{n}^\varepsilon$ over $(t, t+h)$, with $h \in (0,T)$, and considering $\boldsymbol{\phi}_p = \theta_h \mathbf{p}^\varepsilon - \mathbf{p}^\varepsilon$ and $\boldsymbol{\phi}_n = \theta_h \mathbf{n}^\varepsilon - \mathbf{n}^\varepsilon$ as test functions, respectively, we obtain the last estimate in (3.3). □

Next we prove the existence, uniqueness and *a priori* estimates for a solution of the viscoelastic equations for a given $b^\varepsilon \in L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))$.

Lemma 4.4. *Under Assumption 2.1 for a given $b^\varepsilon \in L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))$, satisfying*

$$\|b^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))} + \|(\partial_t b^\varepsilon)^+\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))} \le B, \tag{4.3}$$

where the constant B is independent of ε, there exists a weak solution of the degenerate viscoelastic equations (2.1) satisfying the a priori *estimate (3.2).*

Proof. Using the estimates for $\mathbf{u}^\varepsilon$ and $\partial_t \mathbf{u}^\varepsilon$, similar to those in (4.5), along with the positive definiteness of $\mathbb{E}$ and $\mathbb{V}$, and applying the Galerkin method, yield the existence of a weak solution of the problem (2.1).

Since $\partial_t \mathbf{e}(\mathbf{u}^\varepsilon)$ is only defined in Ω_M^ε, to derive *a priori* estimates we first consider an approximation of $\partial_t \mathbf{u}^\varepsilon$

$$\partial_t \mathbf{u}^{\varepsilon,\zeta}(t,x) = \frac{1}{\zeta}\int_{t-\zeta}^{t} \partial_s \frac{1}{\zeta}\int_s^{s+\zeta} \mathbf{u}^\varepsilon(\sigma, x)\, \mathrm{d}\sigma\, \mathrm{d}s \tag{4.4}$$

as a test function in (2.7), then integrate by parts in the elastic term and take the limit as $\zeta \to 0$. Using the assumptions on $\mathbb{E}$ and $\mathbb{V}$, together with the non-negativity of b^ε, the boundedness of b^ε and $(\partial_t b^\varepsilon)^+$, independent of ε and $\mathbf{u}^\varepsilon$, and the trace and Korn inequalities, we obtain

$$\begin{aligned} \frac{\omega_E}{2}\|\mathbf{e}(\mathbf{u}^\varepsilon)(\tau)\|^2_{L^2(\Omega)} + \omega_V \|\partial_t \mathbf{e}(\mathbf{u}^\varepsilon)\|^2_{L^2(\Omega^\varepsilon_{M,\tau})} &\le \frac{1}{2}\langle (\partial_t b^\varepsilon)^+ \mathbb{E}'_M(b^\varepsilon)\mathbf{e}(\mathbf{u}^\varepsilon), \mathbf{e}(\mathbf{u}^\varepsilon)\rangle_{\Omega^\varepsilon_{M,\tau}} + C_1\|\mathbf{e}(\mathbf{u}_0)\|^2_{L^2(\Omega)} \\ +\langle \mathbf{f}, \partial_t \mathbf{u}^\varepsilon\rangle_{\Gamma_{\mathcal{EU},\tau}} - \langle p_{\mathcal{I}}\boldsymbol{\nu}, \partial_t\mathbf{u}^\varepsilon\rangle_{\Gamma_{\mathcal{I},\tau}} &\le C_2\|\mathbf{e}(\mathbf{u}^\varepsilon)\|^2_{L^2(\Omega_\tau)} + \sigma\|\mathbf{e}(\mathbf{u}^\varepsilon)(\tau)\|^2_{L^2(\Omega)} + C_\sigma\Big[\|\partial_t \mathbf{f}\|^2_{L^2(\Gamma_{\mathcal{EU},\tau})} \\ +\|\partial_t p_{\mathcal{I}}\|^2_{L^2(\Gamma_{\mathcal{I},\tau})} + \|\mathbf{f}(\tau)\|^2_{L^2(\Gamma_{\mathcal{EU}})} &+ \|p_{\mathcal{I}}(\tau)\|^2_{L^2(\Gamma_{\mathcal{I}})} + \|\mathbf{f}(0)\|^2_{L^2(\Gamma_{\mathcal{EU}})} + \|p_{\mathcal{I}}(0)\|^2_{L^2(\Gamma_{\mathcal{I}})}\Big] + C_3 \end{aligned}$$

for $\tau \in (0,T]$. Choosing σ sufficiently small and applying the Gronwall inequality imply

$$\|\mathbf{e}(\mathbf{u}^\varepsilon)\|_{L^\infty(0,T;L^2(\Omega))} + \|\partial_t \mathbf{e}(\mathbf{u}^\varepsilon)\|_{L^2(\Omega^\varepsilon_{M,T})} \le C, \tag{4.5}$$

with a constant C independent of ε. Then the second Korn inequality yields (3.2). □

Now applying a fixed point argument and using the results in Lemmas 4.3 and 4.4 we obtain the well-posedness of the coupled system (2.1)–(2.3).

Proof of Theorem 3.1. For a given $\widetilde{\mathbf{u}}^\varepsilon \in L^\infty(0,T;\mathcal{W}(\Omega))$, with $\|\widetilde{\mathbf{u}}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(\Omega))} \leq C$, Lemma 4.3 implies the existence of a non-negative weak solution $(\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon, b^\varepsilon)$ of the problem (2.2) and (2.3), where the estimates for $\|b^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))}$ and $\|(\partial_t b^\varepsilon)^+\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))}$ are independent of $\widetilde{\mathbf{u}}^\varepsilon$ and ε. Thus b^ε satisfies (4.3) from Lemma 4.4 and we have a solution $\mathbf{u}^\varepsilon$ of (2.1).

We define $\mathcal{K} : L^\infty(0,T;\mathcal{W}(\Omega)) \to L^\infty(0,T;\mathcal{W}(\Omega))$ by $\mathcal{K}(\widetilde{\mathbf{u}}^\varepsilon) = \mathbf{u}^\varepsilon$, where $\mathbf{u}^\varepsilon$ is a solution of (2.1) for b^ε given as a solution of (2.2) and (2.3) with $\widetilde{\mathbf{u}}^\varepsilon$ instead of $\mathbf{u}^\varepsilon$, and show that for sufficiently small $\widetilde{T} \in (0,T]$, the operator $\mathcal{K} : L^\infty(0,\widetilde{T};\mathcal{W}(\Omega)) \to L^\infty(0,\widetilde{T};\mathcal{W}(\Omega))$ is a contraction, *i.e.* there is a $\gamma \in (0,1)$ such that

$$\|\mathcal{K}(\widetilde{\mathbf{u}}^{\varepsilon,1}) - \mathcal{K}(\widetilde{\mathbf{u}}^{\varepsilon,2})\|_{L^\infty(0,\widetilde{T};\mathcal{W}(\Omega))} \leq \gamma \|\widetilde{\mathbf{u}}^{\varepsilon,1} - \widetilde{\mathbf{u}}^{\varepsilon,2}\|_{L^\infty(0,\widetilde{T};\mathcal{W}(\Omega))} \qquad \text{for} \quad \widetilde{\mathbf{u}}^{\varepsilon,1}, \widetilde{\mathbf{u}}^{\varepsilon,2} \in L^\infty(0,\widetilde{T};\mathcal{W}(\Omega)).$$

Considering the difference of equation (2.7) for $(\mathbf{u}^{\varepsilon,1}, b^{\varepsilon,1})$ and $(\mathbf{u}^{\varepsilon,2}, b^{\varepsilon,2})$, and taking the approximation of $\partial_t(\mathbf{u}^{\varepsilon,1} - \mathbf{u}^{\varepsilon,2})$, defined as in (4.4), as a test function, in the same way as in the proof of Lemma 4.4, yield

$$\begin{aligned}
&\frac{1}{2}\langle \mathbb{E}^\varepsilon(b^{\varepsilon,1}, x)\mathbf{e}(\mathbf{u}^{\varepsilon,1}(\tau) - \mathbf{u}^{\varepsilon,2}(\tau)), \mathbf{e}(\mathbf{u}^{\varepsilon,1}(\tau) - \mathbf{u}^{\varepsilon,2}(\tau))\rangle_\Omega + \langle \mathbb{V}^\varepsilon(b^{\varepsilon,1}, x)\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,1} - \mathbf{u}^{\varepsilon,2}), \partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,1} - \mathbf{u}^{\varepsilon,2})\rangle_{\Omega_\tau} \\
&-\frac{1}{2}\langle \partial_t b^{\varepsilon,1}\mathbb{E}'_M(b^{\varepsilon,1})\mathbf{e}(\mathbf{u}^{\varepsilon,1} - \mathbf{u}^{\varepsilon,2}), \mathbf{e}(\mathbf{u}^{\varepsilon,1} - \mathbf{u}^{\varepsilon,2})\rangle_{\Omega^\varepsilon_{M,\tau}} = \langle (\mathbb{E}_M(b^{\varepsilon,2}) - \mathbb{E}_M(b^{\varepsilon,1}))\mathbf{e}(\mathbf{u}^{\varepsilon,2}), \partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,1} - \mathbf{u}^{\varepsilon,2})\rangle_{\Omega^\varepsilon_{M,\tau}} \\
&\qquad + \langle (\mathbb{V}_M(b^{\varepsilon,2}) - \mathbb{V}_M(b^{\varepsilon,1}))\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,2}), \partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,1} - \mathbf{u}^{\varepsilon,2})\rangle_{\Omega^\varepsilon_{M,\tau}}
\end{aligned}$$

for $\tau \in (0,T]$. By the assumptions on $\mathbb{E}^\varepsilon$ and $\mathbb{V}^\varepsilon$ and the boundedness of $b^{\varepsilon,1}$ and $b^{\varepsilon,2}$, we have

$$\begin{aligned}
\|\mathbf{e}(\mathbf{u}^{\varepsilon,1}(\tau)) - \mathbf{e}(\mathbf{u}^{\varepsilon,2}(\tau))\|^2_{L^2(\Omega)} &\leq C_1 \|(\partial_t b^{\varepsilon,1})^+\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))} \int_0^\tau \|\mathbf{e}(\mathbf{u}^{\varepsilon,1} - \mathbf{u}^{\varepsilon,2})\|^2_{L^2(\Omega_M^\varepsilon)} \mathrm{d}\tau \\
&\quad + C_2 \|\mathbf{e}(\mathbf{u}^{\varepsilon,2})\|^2_{H^1(0,T;L^2(\Omega_M^\varepsilon))} \|b^{\varepsilon,1} - b^{\varepsilon,2}\|^2_{L^\infty(0,\tau;L^\infty(\Omega_M^\varepsilon))}.
\end{aligned}$$

Applying the Gronwall inequality and the estimates for $(\partial_t b^{\varepsilon,1})^+$ and $\mathbf{e}(\mathbf{u}^{\varepsilon,2})$ implies

$$\|\mathbf{e}(\mathbf{u}^{\varepsilon,1}) - \mathbf{e}(\mathbf{u}^{\varepsilon,2})\|^2_{L^\infty(0,\widetilde{T};L^2(\Omega))} \leq C_3 \|b^{\varepsilon,1} - b^{\varepsilon,2}\|^2_{L^\infty(0,\widetilde{T};L^\infty(\Omega_M^\varepsilon))} \tag{4.6}$$

for $\widetilde{T} \in (0,T]$.

Now we shall estimate $\|b^{\varepsilon,1} - b^{\varepsilon,2}\|^2_{L^\infty(0,\widetilde{T};L^\infty(\Omega_M^\varepsilon))}$ in terms of $\widetilde{T}\|\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}) - \mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2})\|^2_{L^\infty(0,\widetilde{T};L^2(\Omega))}$ for any $\widetilde{T} \in (0,T]$. Following the same calculations as in [26], we first consider equation $(2.5)_2$ for $\mathbf{n}^{\varepsilon,1}$ and $\mathbf{n}^{\varepsilon,2}$, take $\boldsymbol{\phi}_n = (|\overline{n}_1^\varepsilon|^{q-2}\overline{n}_1^\varepsilon, |\overline{n}_2^\varepsilon|^{q-2}\overline{n}_2^\varepsilon)^T$, where $\overline{n}_j^\varepsilon = n_j^{\varepsilon,1} - n_j^{\varepsilon,2}$ with $j = 1,2$ and $q \geq 2$, and subtract the resulting equations. Using the definition of $\mathcal{N}_\delta$, the assumptions on $\mathbf{G}$ and $\mathbf{J}_n$, and the trace inequality, the boundary terms are estimated in the following way

$$\begin{aligned}
&\big\langle \mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}))\big[\mathbf{G}(\mathbf{n}^{\varepsilon,1}) - \mathbf{G}(\mathbf{n}^{\varepsilon,2})\big], (|\overline{n}_1^\varepsilon|^{q-2}\overline{n}_1^\varepsilon, |\overline{n}_2^\varepsilon|^{q-2}\overline{n}_2^\varepsilon)^T\big\rangle_{\Gamma_\mathcal{I}} \leq 0, \\
&\qquad \big\langle \mathbf{J}_n(\mathbf{n}^{\varepsilon,1}) - \mathbf{J}_n(\mathbf{n}^{\varepsilon,2}), (|\overline{n}_1^\varepsilon|^{q-2}\overline{n}_1^\varepsilon, |\overline{n}_2^\varepsilon|^{q-2}\overline{n}_2^\varepsilon)^T\big\rangle_{\Gamma_\mathcal{E}} \leq C_\sigma q\, \|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}\|^q_{L^q(\Omega_M^\varepsilon)} \\
&\qquad\qquad + \sigma(q-1)/q^2\, \|\nabla|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}|^{\frac{q}{2}}\|^2_{L^2(\Omega_M^\varepsilon)}
\end{aligned}$$

and

$$\begin{aligned}
&\Big|\big\langle \mathbf{G}(\mathbf{n}^{\varepsilon,2})\big[\mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1})) - \mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2}))\big], (|\overline{n}_1^\varepsilon|^{q-2}\overline{n}_1^\varepsilon, |\overline{n}_2^\varepsilon|^{q-2}\overline{n}_2^\varepsilon)^T\big\rangle_{\Gamma_\mathcal{I}}\Big| \leq C_\sigma(q-1)\|n_2^{\varepsilon,1} - n_2^{\varepsilon,2}\|^q_{L^q(\Omega_M^\varepsilon)} \\
&\qquad + \sigma(q-1)/q^2 \|\nabla|n_2^{\varepsilon,1} - n_2^{\varepsilon,2}|^{\frac{q}{2}}\|^2_{L^2(\Omega_M^\varepsilon)} + (C/q)\|\mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1})) - \mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2}))\|^q_{L^q(\Omega)},
\end{aligned}$$

with an arbitrary $\sigma > 0$. Using the assumptions on $\mathbf{F}_n$ and $\mathbf{R}_n$ and the uniform boundedness of $\mathbf{p}^\varepsilon$, $\mathbf{n}^{\varepsilon,j}$, and $b^{\varepsilon,j}$, with $j = 1,2$, we obtain

$$\begin{aligned}
&\langle \mathbf{F}_n(\mathbf{p}^\varepsilon, \mathbf{n}^{\varepsilon,1}) - \mathbf{F}_n(\mathbf{p}^\varepsilon, \mathbf{n}^{\varepsilon,2}), (|\overline{n}_1^\varepsilon|^{q-2}\overline{n}_1^\varepsilon, |\overline{n}_2^\varepsilon|^{q-2}\overline{n}_2^\varepsilon)^T\rangle_{\Omega_M^\varepsilon} \leq C_1\|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}\|^q_{L^q(\Omega_M^\varepsilon)}, \\
&\langle \mathbf{R}_n(\mathbf{n}^{\varepsilon,1}, b^{\varepsilon,1}, \mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}))) - \mathbf{R}_n(\mathbf{n}^{\varepsilon,2}, b^{\varepsilon,2}, \mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2}))), (|\overline{n}_1^\varepsilon|^{q-2}\overline{n}_1^\varepsilon, |\overline{n}_2^\varepsilon|^{q-2}\overline{n}_2^\varepsilon)^T\rangle_{\Omega_M^\varepsilon} \leq C_2\big[\|\mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}))\|_{L^\infty(\Omega)} \\
&+ \|\mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2}))\|_{L^\infty(\Omega)} + 1\big]\Big[\|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}\|^q_{L^q(\Omega_M^\varepsilon)} + \frac{1}{q}\|b^{\varepsilon,1} - b^{\varepsilon,2}\|^q_{L^q(\Omega_M^\varepsilon)} + \frac{1}{q}\|\mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1})) - \mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2}))\|^q_{L^q(\Omega)}\Big].
\end{aligned}$$

Then, the Gagliardo–Nirenberg inequality applied to $|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}|^{q/2}$, the definition of $\mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,j}))$, the *a priori* estimates for $\widetilde{\mathbf{u}}^{\varepsilon,1}$ and $\widetilde{\mathbf{u}}^{\varepsilon,2}$, together with the estimate

$$\|\mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1})) - \mathcal{N}_\delta(\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2}))\|^q_{L^q(\Omega)} \leq C^q \delta^{-\frac{3q}{2}} \big[\|\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}) - \mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2})\|^q_{L^2(\Omega)} + \|b^{\varepsilon,1} - b^{\varepsilon,2}\|^q_{L^q(\Omega^\varepsilon_M)}\big],$$

ensure

$$\begin{aligned}\partial_t \|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}\|^q_{L^q(\Omega^\varepsilon_M)} + 2\frac{q-1}{q}\|\nabla|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}|^{\frac{q}{2}}\|^2_{L^2(\Omega^\varepsilon_M)} \leq C\Big[q^5 \|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}\|^q_{L^{q/2}(\Omega^\varepsilon_M)} \\ +(C^q_\delta + 1)\big[\|b^{\varepsilon,1} - b^{\varepsilon,2}\|^q_{L^q(\Omega^\varepsilon_M)} + \|\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}) - \mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2})\|^q_{L^2(\Omega)}\big]\Big].\end{aligned}$$

Here we use the notation $|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}|^\alpha = |\mathbf{n}^{\varepsilon,1}_1 - \mathbf{n}^{\varepsilon,2}_1|^\alpha + |\mathbf{n}^{\varepsilon,1}_2 - \mathbf{n}^{\varepsilon,2}_2|^\alpha$. Considering iterations in q as in ([3], Lem. 3.2) with $q = 2^\kappa$ and $\kappa = 2, 3, \ldots$, we obtain

$$\|\mathbf{n}^{\varepsilon,1}(\tau) - \mathbf{n}^{\varepsilon,2}(\tau)\|^q_{L^q(\Omega^\varepsilon_M)} \leq C^q_\delta 2^{10q} 2^{2(q-1)} \big[\|\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}) - \mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2})\|^q_{L^\infty(0,\tau;L^2(\Omega))} + \|b^{\varepsilon,1} - b^{\varepsilon,2}\|^q_{L^\infty(0,\tau;L^q(\Omega^\varepsilon_M))}\big]$$

for $\tau \in (0, T]$ and $C_\delta \geq 1$. Taking the qth root, and considering $q \to \infty$ yield

$$\|\mathbf{n}^{\varepsilon,1} - \mathbf{n}^{\varepsilon,2}\|_{L^\infty(0,\tau;L^\infty(\Omega^\varepsilon_M))} \leq C_\delta \big[\|\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}) - \mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2})\|_{L^\infty(0,\tau;L^2(\Omega))} + \|b^{\varepsilon,1} - b^{\varepsilon,2}\|_{L^\infty(0,\tau;L^\infty(\Omega^\varepsilon_M))}\big]. \tag{4.7}$$

Considering the difference of equations (2.6) for $b^{\varepsilon,1}$ and $b^{\varepsilon,2}$, multiplying by $b^{\varepsilon,1} - b^{\varepsilon,2}$, and using the assumptions on R_b and estimate (4.7) we obtain the following estimate for $b^{\varepsilon,1} - b^{\varepsilon,2}$

$$\|b^{\varepsilon,1} - b^{\varepsilon,2}\|^2_{L^\infty(0,\tau;L^\infty(\Omega^\varepsilon_M))} \leq C_\delta \tau \big[\|\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1}) - \mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,2})\|^2_{L^\infty(0,\tau;L^2(\Omega))} + \|b^{\varepsilon,1} - b^{\varepsilon,2}\|^2_{L^\infty(0,\tau;L^\infty(\Omega^\varepsilon_M))}\big]$$

for $\tau \in (0, T]$. Then, the iteration over time intervals of length $1/(2C_\delta)$ ensures

$$\|b^{\varepsilon,1} - b^{\varepsilon,2}\|^2_{L^\infty(0,\widetilde{T};L^\infty(\Omega^\varepsilon_M))} \leq C\widetilde{T}\|\mathbf{e}(\widetilde{\mathbf{u}}^{\varepsilon,1} - \widetilde{\mathbf{u}}^{\varepsilon,2})\|^2_{L^\infty(0,\widetilde{T};L^2(\Omega))} \tag{4.8}$$

for $\widetilde{T} \in (0, T]$. Thus, combining (4.6) and (4.8) we have that the operator $\mathcal{K} : L^\infty(0, \widetilde{T}; \mathcal{W}(\Omega)) \to L^\infty(0, \widetilde{T}; \mathcal{W}(\Omega))$, defined by $\mathcal{K}(\widetilde{\mathbf{u}}^\varepsilon) = \mathbf{u}^\varepsilon$, where $\mathbf{u}^\varepsilon$ is a weak solution of (2.1), is a contraction for sufficiently small $\widetilde{T}$, where $\widetilde{T}$ depends on the coefficients in the microscopic equations and is independent of $(\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon, b^\varepsilon, \mathbf{u}^\varepsilon)$ and $\widetilde{\mathbf{u}}^\varepsilon$. Hence, using the Banach fixed point theorem and iterating over time intervals, we obtain the existence of a unique weak solution of the microscopic problem (2.1)–(2.3). □

Remark 4.5. Without the assumption that $(R_b(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))))^+$ is bounded we can prove a local in time existence of a weak solution of the microscopic problem using a cut-off method. First we assume that

$$(R_b(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))))^+ \leq \beta_3(1 + \|\mathbf{n}^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} + \|b^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))})(1 + C_\delta \|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(\Omega))}) \leq \tilde{\beta}.$$

Then we have that b^ε satisfies (4.3) and obtain $\|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(\Omega))} \leq C_1 \mathrm{e}^{T(B(\tilde{\beta})+C_2)}$. The derivation of the estimates for $\mathbf{n}^\varepsilon$ and b^ε yields

$$\|b^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} + \|\mathbf{n}^\varepsilon\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} \leq C(2^{C_1 T(1+\|\mathbf{u}^\varepsilon\|_{L^\infty(0,T;\mathcal{W}(\Omega))})} + 1) \leq C(2^{C_2 T(\mathrm{e}^{T(B(\tilde{\beta})+C_3)}+1)} + 1).$$

Then for sufficient small T and an appropriate choice of $\tilde{\beta}$ we obtain that $(R_b(\mathbf{n}^\varepsilon, b^\varepsilon, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^\varepsilon))))^+ \leq \tilde{\beta}$.

5. Derivation of the macroscopic equations of the problem (2.1)–(2.3): Proof of Theorem 3.2

Due to the fact that the viscous term is positive definite in the cell wall matrix and is zero for the cell wall microfibrils, to derive macroscopic equations for the microscopic problem (2.1)–(2.3) we consider a perturbed problem by adding the inertial term $\vartheta\, \partial^2_t \mathbf{u}^{\varepsilon,\vartheta} \chi_{\Omega^\varepsilon_M}$, where $\vartheta > 0$ is a small perturbation parameter,

$$\vartheta \chi_{\Omega^\varepsilon_M} \partial^2_t \mathbf{u}^{\varepsilon,\vartheta} = \operatorname{div}\big(\mathbb{E}^\varepsilon(b^{\varepsilon,\vartheta}, x)\, \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}) + \mathbb{V}^\varepsilon(b^{\varepsilon,\vartheta}, x)\, \partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\big) \qquad \text{in } \Omega_T, \tag{5.1}$$

and the additional initial condition

$$\partial_t \mathbf{u}^{\varepsilon,\vartheta}(0,x) = \mathbf{0} \quad \text{in} \quad \Omega. \tag{5.2}$$

We split the proof of Theorem 3.2 into three steps. First we derive the macroscopic equations for the perturbed system. Then letting the perturbation parameter ϑ go to zero we obtain the macroscopic equations (3.10)–(3.12). To verify that (3.10)–(3.12) are the macroscopic equations for the microscopic problem (2.1)–(2.3), we show that the macroscopic two-scale problem is the same for the original microscopic model and for the perturbed microscopic model when the perturbation parameter $\vartheta \to 0$.

Lemma 5.1. *There exists a unique weak solution $(\mathbf{p}^{\varepsilon,\vartheta}, \mathbf{n}^{\varepsilon,\vartheta}, b^{\varepsilon,\vartheta}, \mathbf{u}^{\varepsilon,\vartheta})$ of the perturbed microscopic problem (2.2), (2.3) and (5.1), together with the initial and boundary conditions in (2.1) and (5.2), satisfying the* a priori *estimates*

$$\vartheta^{\frac{1}{2}} \|\partial_t \mathbf{u}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^2(\Omega^\varepsilon_M))} + \|\mathbf{u}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;\mathcal{W}(\Omega))} + \|\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\|_{L^2(\Omega^\varepsilon_{M,T})} \le C, \tag{5.3}$$

and

$$\begin{aligned}
&\|\mathbf{p}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} + \|\nabla \mathbf{p}^{\varepsilon,\vartheta}\|_{L^2(\Omega^\varepsilon_{M,T})} + \|\mathbf{n}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} + \|\nabla \mathbf{n}^{\varepsilon,\vartheta}\|_{L^2(\Omega^\varepsilon_{M,T})} \le C,\\
&\|b^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} + \|\partial_t b^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} \le C,\\
&\|\theta_h \mathbf{p}^{\varepsilon,\vartheta} - \mathbf{p}^{\varepsilon,\vartheta}\|_{L^2(\Omega^\varepsilon_{M,T-h})} + \|\theta_h \mathbf{n}^{\varepsilon,\vartheta} - \mathbf{n}^{\varepsilon,\vartheta}\|_{L^2(\Omega^\varepsilon_{M,T-h})} \le C h^{1/4},
\end{aligned} \tag{5.4}$$

where $\theta_h v(t,x) = v(t+h,x)$ for $(t,x) \in \Omega^\varepsilon_{M,T-h}$ and $h \in (0,T)$, and the constant C is independent of ε, ϑ, and h.

Proof. For a given $\mathbf{u}^{\varepsilon,\vartheta} \in L^\infty(0,T;\mathcal{W}(\Omega))$, with $\|\mathbf{u}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;\mathcal{W}(\Omega))} \le C$, in the same way as in Lemma 4.3 we obtain the existence of a unique solution of the problem (2.2) and (2.3), satisfying the *a priori* estimates (5.4). Notice that the estimates for $b^{\varepsilon,\vartheta}$ and $(\partial_t b^{\varepsilon,\vartheta})^+$ are independent of $\mathbf{u}^{\varepsilon,\vartheta}$, ε, and ϑ.

Then for $b^{\varepsilon,\vartheta} \in L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))$, with $\|b^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} \le C$ and $\|(\partial_t b^{\varepsilon,\vartheta})^+\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} \le C$, similar to Lemma 4.4, we obtain the existence of a weak solution of the perturbed equations (5.1) with initial and boundary conditions in (2.1) and (5.2), satisfying estimate (5.3).

Similar to the proof of Theorem 3.1, considering the difference of equation (5.1) for $(\mathbf{u}^{\varepsilon,\vartheta,j}, b^{\varepsilon,\vartheta,j})$, with $j = 1,2$, and taking the approximation of $\partial_t(\mathbf{u}^{\varepsilon,\vartheta,1} - \mathbf{u}^{\varepsilon,\vartheta,2})$, as in (4.4), as a test function yield

$$\begin{aligned}
\frac{1}{2}\vartheta \|\partial_t \mathbf{u}^{\varepsilon,\vartheta,1}(\tau) - \partial_t \mathbf{u}^{\varepsilon,\vartheta,2}(\tau)\|^2_{L^2(\Omega^\varepsilon_M)} &+ \frac{1}{2}\big\langle \mathbb{E}^\varepsilon(b^{\varepsilon,\vartheta,1},x)\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,1} - \mathbf{u}^{\varepsilon,\vartheta,2})(\tau), \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,1} - \mathbf{u}^{\varepsilon,\vartheta,2})(\tau)\big\rangle_\Omega\\
&- \frac{1}{2}\big\langle \partial_t b^\varepsilon\, \mathbb{E}'_M(b^{\varepsilon,\vartheta,1})\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,1} - \mathbf{u}^{\varepsilon,\vartheta,2}), \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,1} - \mathbf{u}^{\varepsilon,\vartheta,2})\big\rangle_{\Omega^\varepsilon_{M,\tau}}\\
&+ \big\langle \mathbb{V}^\varepsilon(b^{\varepsilon,\vartheta,1},x)\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,1} - \mathbf{u}^{\varepsilon,\vartheta,2}), \partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,1} - \mathbf{u}^{\varepsilon,\vartheta,2})\big\rangle_{\Omega_\tau}\\
= \big\langle (\mathbb{E}_M(b^{\varepsilon,\vartheta,2}) - \mathbb{E}_M(b^{\varepsilon,\vartheta,1}))\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,2}) &+ (\mathbb{V}_M(b^{\varepsilon,\vartheta,2}) - \mathbb{V}_M(b^{\varepsilon,\vartheta,1}))\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,2}), \partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,1} - \mathbf{u}^{\varepsilon,\vartheta,2})\big\rangle_{\Omega^\varepsilon_{M,\tau}}
\end{aligned}$$

for $\tau \in (0,T]$. By the assumptions on $\mathbb{E}^\varepsilon$ and $\mathbb{V}^\varepsilon$, using the estimates for $(\partial_t b^{\varepsilon,\vartheta,1})^+$, $\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,2})$, and $\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,2})$, together with the boundedness of $b^{\varepsilon,\vartheta,1}$ and $b^{\varepsilon,\vartheta,2}$, and applying the Gronwall inequality we obtain

$$\|\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,1}) - \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta,2})\|^2_{L^\infty(0,\widetilde{T};L^2(\Omega))} \le C\|b^{\varepsilon,\vartheta,1} - b^{\varepsilon,\vartheta,2}\|^2_{L^\infty(0,\widetilde{T};L^\infty(\Omega^\varepsilon_M))} \tag{5.5}$$

for all $\widetilde{T} \in (0,T]$. Then, using the estimates (4.8) and (5.5), together with the *a priori* estimates for $\mathbf{u}^{\varepsilon,\vartheta}$, $\mathbf{p}^{\varepsilon,\vartheta}$, $\mathbf{n}^{\varepsilon,\vartheta}$, and $b^{\varepsilon,\vartheta}$, in the same way as in the proof of Theorem 3.1 we obtain the existence of a unique weak solution of the perturbed problem (2.2) and (5.1), with the initial and boundary conditions in (2.1), (2.3), and (5.2). □

To verify a relation between the perturbed and original microscopic problems, we show that a sequence of weak solutions of the perturbed problem (2.2), (2.3), and (5.1), with initial and boundary conditions in (2.1) and (5.2), converges as $\vartheta \to 0$ to a weak solution of the original problem (2.1)–(2.3).

Lemma 5.2. *A sequence of solutions* $\{\mathbf{p}^{\varepsilon,\vartheta}, \mathbf{n}^{\varepsilon,\vartheta}, b^{\varepsilon,\vartheta}, \mathbf{u}^{\varepsilon,\vartheta}\}$ *of the problem* (2.2), (2.3) *and* (5.1), *together with the initial and boundary conditions in* (2.1) *and* (5.2), *converges as* $\vartheta \to 0$ *to a unique solution* $(\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon, b^\varepsilon, \mathbf{u}^\varepsilon)$ *of the microscopic problem* (2.1)–(2.3).

Proof. Estimates (5.3) and (5.4) ensure that there exist functions $\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon \in L^2(0,T;\mathcal{V}(\Omega_M^\varepsilon))^2 \cap L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))^2$, $b^\varepsilon \in W^{1,\infty}(0,T;L^\infty(\Omega_M^\varepsilon))$, $\mathbf{u}^\varepsilon \in L^2(0,T;\mathcal{W}(\Omega))$, $\Lambda^\varepsilon \in L^2(\Omega_{M,T}^\varepsilon)^{3\times 3}$, and $\boldsymbol{\eta}^\varepsilon \in L^\infty(0,T;L^2(\Omega_M^\varepsilon))^3$ such that, up to a subsequence,

$$
\begin{aligned}
&\mathbf{p}^{\varepsilon,\vartheta} \rightharpoonup \mathbf{p}^\varepsilon, \quad \mathbf{n}^{\varepsilon,\vartheta} \rightharpoonup \mathbf{n}^\varepsilon && \text{weakly in } L^2(0,T;H^1(\Omega_M^\varepsilon))^2,\\
&\mathbf{p}^{\varepsilon,\vartheta} \to \mathbf{p}^\varepsilon, \quad \mathbf{n}^{\varepsilon,\vartheta} \to \mathbf{n}^\varepsilon && \text{strongly in } L^2(\Omega_{M,T}^\varepsilon)^2,\\
&b^{\varepsilon,\vartheta} \rightharpoonup b^\varepsilon, \quad \partial_t b^{\varepsilon,\vartheta} \rightharpoonup \partial_t b^\varepsilon && \text{weakly in } L^p(\Omega_{M,T}^\varepsilon), \quad p\in[2,\infty),\\
&\mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \mathbf{u}^\varepsilon && \text{weakly in } L^2(0,T;\mathcal{W}(\Omega)),\\
&\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}) \rightharpoonup \Lambda^\varepsilon && \text{weakly in } L^2(\Omega_{M,T}^\varepsilon)^{3\times 3},\\
&\vartheta^{1/2}\partial_t \mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \boldsymbol{\eta}^\varepsilon && \text{weakly in } L^2(\Omega_{M,T}^\varepsilon)^3,
\end{aligned}
\tag{5.6}
$$

as $\vartheta \to 0$. Using the weak convergence of $\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})$ we obtain that $\Lambda^\varepsilon = \partial_t \mathbf{e}(\mathbf{u}^\varepsilon)$ a.e. in $L^2(\Omega_{M,T}^\varepsilon)$. Considering the equation for $b^{\varepsilon,\vartheta}$ at (t,x) and $(t, x+\mathbf{h}_j)$ and using the assumptions on R_b yield

$$
\begin{aligned}
&\|b^{\varepsilon,\vartheta}(\tau,\cdot+\mathbf{h}_j) - b^{\varepsilon,\vartheta}(\tau,\cdot)\|^2_{L^2(\Omega^\varepsilon_{M,h})} \le \|b_0(\cdot+\mathbf{h}_j) - b_0(\cdot)\|^2_{L^2(\Omega^\varepsilon_{M,h})} + C_1\int_0^\tau \|b^{\varepsilon,\vartheta}(t,\cdot+\mathbf{h}_j) - b^{\varepsilon,\vartheta}(t,\cdot)\|^2_{L^2(\Omega^\varepsilon_{M,h})}\mathrm{d}t\\
&+ C_2\int_0^\tau \Big[\|\mathbf{n}^{\varepsilon,\vartheta}(t,\cdot+\mathbf{h}_j) - \mathbf{n}^{\varepsilon,\vartheta}(t,\cdot)\|^2_{L^2(\Omega^\varepsilon_{M,h})} + \delta^{-6}\Big\| \int_{B_{\delta,h}(x)\cap\Omega} \mathrm{tr}\,\big(\mathbb{E}^\varepsilon(b^{\varepsilon,\vartheta},\tilde x)\,\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t,\tilde x))\big)\,\mathrm{d}\tilde x\Big\|^2_{L^2(\Omega^\varepsilon_{M,h})}\Big]\mathrm{d}t
\end{aligned}
$$

for $\tau \in (0,T]$, where $\mathbf{h}_j = h\mathbf{b}_j$, with $\{\mathbf{b}_j\}_{j=1,2,3}$ being the canonical basis in $\mathbb{R}^3$ and $h>0$, $\Omega^\varepsilon_{M,h} = \{x \in \Omega^\varepsilon_M \mid \mathrm{dist}(x,\partial\Omega^\varepsilon_M) > 2h\}$, $B_{\delta,h}(x) = \big[B_\delta(x+\mathbf{h}_j)\setminus B_\delta(x)\big] \cup \big[B_\delta(x)\setminus B_\delta(x+\mathbf{h}_j)\big]$, and the constants C_1, C_2 are independent of ϑ and h. Using the regularity of b_0, the estimates for $\nabla\mathbf{n}^{\varepsilon,\vartheta}$ and $\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})$, the boundedness of $b^{\varepsilon,\vartheta}$, and the fact that $|B_{\delta,h}(x)\cap\Omega| \le C\delta^2 h$ for all $x\in\overline{\Omega}$, and applying the Gronwall inequality we obtain

$$
\sup_{t\in(0,T)} \|b^{\varepsilon,\vartheta}(t,\cdot+\mathbf{h}_j) - b^{\varepsilon,\vartheta}(t,\cdot)\|^2_{L^2(\Omega^\varepsilon_{M,h})} \le Ch. \tag{5.7}
$$

The estimate for $\partial_t b^{\varepsilon,\vartheta}$ ensures

$$
\|b^{\varepsilon,\vartheta}(\cdot+h,\cdot) - b^{\varepsilon,\vartheta}(\cdot,\cdot)\|^2_{L^2((0,T-h)\times\Omega^\varepsilon_M)} \le C_1 h^2 \|\partial_t b^{\varepsilon,\vartheta}\|^2_{L^2(\Omega^\varepsilon_{M,T})} \le C_2 h^2, \tag{5.8}
$$

where C_2 is independent of ϑ and h. Combining (5.7) and (5.8), using the uniform boundedness of $b^{\varepsilon,\vartheta}$, and applying the Kolmogorov compactness theorem, see *e.g.* [5,20], yield the strong convergence of $b^{\varepsilon,\vartheta}$ in $L^2(\Omega^\varepsilon_{M,T})$ as $\vartheta\to 0$. Using arguments similar to those in the proof of Theorem 5.4 we obtain

$$
\begin{aligned}
&\|\theta_h \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}) - \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\|^2_{L^2((0,T-h)\times\Omega)} \le Ch^{1/2},\\
&\|\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\|^2_{L^2((T-h,T)\times\Omega)} \le Ch,
\end{aligned}
\tag{5.9}
$$

with a constant C independent of ϑ and h. The last estimates, together with the strong convergence of $b^{\varepsilon,\vartheta}$, the continuity of $\mathbb{E}_M$, and Lebesgue's dominated convergence theorem, ensure the following strong convergences

$$
\begin{aligned}
\int_\Omega \mathbb{E}(b^{\varepsilon,\vartheta}, x/\varepsilon)\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\mathrm{d}x &\to \int_\Omega \mathbb{E}(b^\varepsilon, x/\varepsilon)\mathbf{e}(\mathbf{u}^\varepsilon)\mathrm{d}x && \text{in } L^2(0,T),\\
\int_{B_\delta(x)\cap\Omega} \mathbb{E}(b^{\varepsilon,\vartheta}, \tilde x/\varepsilon)\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\mathrm{d}\tilde x &\to \int_{B_\delta(x)\cap\Omega} \mathbb{E}(b^\varepsilon, \tilde x/\varepsilon)\mathbf{e}(\mathbf{u}^\varepsilon)\mathrm{d}\tilde x && \text{in } L^2(\Omega_T) \text{ and } L^2(\Gamma_{\mathcal{I},T}),
\end{aligned}
$$

as $\vartheta \to 0$. Hence we can pass to the limit as $\vartheta \to 0$ in the weak formulation of equations (2.2) and (5.1), with the initial and boundary conditions in (2.1), (2.3), and (5.2), and obtain that the limit functions $(\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon, b^\varepsilon, \mathbf{u}^\varepsilon)$ satisfy the microscopic problem (2.1)–(2.3). The uniqueness of a weak solution of (2.1)–(2.3) ensures the convergence of the whole sequence of weak solutions of the perturbed microscopic problem. $\square$

Next we consider the convergence of a sequence of solutions $\{\mathbf{p}^{\varepsilon,\vartheta}, \mathbf{n}^{\varepsilon,\vartheta}, b^{\varepsilon,\vartheta}, \mathbf{u}^{\varepsilon,\vartheta}\}$ of the perturbed microscopic problem as $\varepsilon \to 0$.

Lemma 5.3. *There exist functions* $\mathbf{p}^\vartheta, \mathbf{n}^\vartheta \in L^2(0,T;\mathcal{V}(\Omega))^2 \cap L^\infty(0,T;L^\infty(\Omega))^2$, $\hat{\mathbf{p}}^\vartheta, \hat{\mathbf{n}}^\vartheta \in L^2(\Omega_T; H^1_{\text{per}}(\hat{Y})/\mathbb{R})^2$ *and* $b^\vartheta \in W^{1,\infty}(0,T;L^\infty(\Omega))$, $\mathbf{u}^\vartheta \in H^1(0,T;\mathcal{W}(\Omega))$, $\hat{\mathbf{u}}^\vartheta \in L^2(\Omega_T; H^1_{\text{per}}(\hat{Y})/\mathbb{R})^3$, $\partial_t \hat{\mathbf{u}}^\vartheta \in L^2(\Omega_T; H^1_{\text{per}}(\hat{Y}_M)/\mathbb{R})^3$ *such that for a subsequence of solutions* $(\mathbf{p}^{\varepsilon,\vartheta}, \mathbf{n}^{\varepsilon,\vartheta}, b^{\varepsilon,\vartheta}, \mathbf{u}^{\varepsilon,\vartheta})$ *of the perturbed microscopic problem* (2.2) *and* (5.1)*, with initial and boundary conditions in* (2.1)*,* (2.3) *and* (5.2)*,* (*denoted again by* $(\mathbf{p}^{\varepsilon,\vartheta}, \mathbf{n}^{\varepsilon,\vartheta}, b^{\varepsilon,\vartheta}, \mathbf{u}^{\varepsilon,\vartheta})$) *we have the following convergence results:*

$$
\begin{aligned}
&\mathbf{p}^{\varepsilon,\vartheta} \rightharpoonup \mathbf{p}^\vartheta, \quad \mathbf{n}^{\varepsilon,\vartheta} \rightharpoonup \mathbf{n}^\vartheta && \textit{weakly in } L^2(0,T;H^1(\Omega))^2,\\
&\mathbf{p}^{\varepsilon,\vartheta} \to \mathbf{p}^\vartheta, \quad \mathbf{n}^{\varepsilon,\vartheta} \to \mathbf{n}^\vartheta && \textit{strongly in } L^2(\Omega_T)^2,\\
&\nabla \mathbf{p}^{\varepsilon,\vartheta} \rightharpoonup \nabla \mathbf{p}^\vartheta + \hat{\nabla}_y \hat{\mathbf{p}}^\vartheta, \quad \nabla \mathbf{n}^{\varepsilon,\vartheta} \rightharpoonup \nabla \mathbf{n}^\vartheta + \hat{\nabla}_y \hat{\mathbf{n}}^\vartheta && \textit{two-scale},\\
&b^{\varepsilon,\vartheta} \rightharpoonup b^\vartheta, \quad \partial_t b^{\varepsilon,\vartheta} \rightharpoonup \partial_t b^\vartheta && \textit{two-scale},\\
&\mathcal{T}^*_\varepsilon(b^{\varepsilon,\vartheta}) \to b^\vartheta && \textit{strongly in } L^2(\Omega_T \times \hat{Y}_M),\\
&\mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \mathbf{u}^\vartheta && \textit{weakly in } L^2(0,T;\mathcal{W}(\Omega)),\\
&\nabla \mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \nabla \mathbf{u}^\vartheta + \hat{\nabla}_y \hat{\mathbf{u}}^\vartheta && \textit{two-scale},\\
&\partial_t \mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \partial_t \mathbf{u}^\vartheta && \textit{weakly in } L^2(\Omega_T)^3 \textit{ and two-scale},\\
&\chi_{\Omega^\varepsilon_M} \nabla \partial_t \mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \chi_{\hat{Y}_M} (\nabla \partial_t \mathbf{u}^\vartheta + \hat{\nabla}_y \partial_t \hat{\mathbf{u}}^\vartheta) && \textit{two-scale},
\end{aligned}
\tag{5.10}
$$

as $\varepsilon \to 0$*, where* $\partial_t \mathbf{u}^{\varepsilon,\vartheta}$ *and* $\nabla \partial_t \mathbf{u}^{\varepsilon,\vartheta}$ *are extended by zero from* Ω^ε_M *into* Ω *and* $\hat{\nabla}_y \partial_t \hat{\mathbf{u}}^\vartheta$ *is extended by zero from* $\hat{Y}_M$ *into* $\hat{Y}$*.*

Here $\mathcal{T}^*_\varepsilon : L^p(\Omega^\varepsilon_{M,T}) \to L^p(\Omega_T \times \hat{Y}_M)$ is the unfolding operator defined as $\mathcal{T}^*_\varepsilon(\phi)(t,x,y) = \phi(t, \varepsilon[\hat{x}/\varepsilon]_{\hat{Y}_M} + \varepsilon y, x_3)$ for $(t,x) \in \Omega_T$ and $y \in \hat{Y}_M$, where $\hat{x} = (x_1, x_2)$ and $[\hat{x}/\varepsilon]_{\hat{Y}_M}$ is the unique integer combination of the periods such that $\hat{x}/\varepsilon - [\hat{x}/\varepsilon]_{\hat{Y}_M} \in \hat{Y}_M$, see *e.g.* [8].

Proof. The *a priori* estimates in (5.4) imply the weak and two-scale convergences of $\mathbf{p}^{\varepsilon,\vartheta}$, $\mathbf{n}^{\varepsilon,\vartheta}$, $b^{\varepsilon,\vartheta}$, and $\partial_t b^{\varepsilon,\vartheta}$. Using the estimates for $\theta_h \mathbf{p}^{\varepsilon,\vartheta} - \mathbf{p}^{\varepsilon,\vartheta}$, $\theta_h \mathbf{n}^{\varepsilon,\vartheta} - \mathbf{n}^{\varepsilon,\vartheta}$, $\nabla \mathbf{n}^{\varepsilon,\vartheta}$, and $\nabla \mathbf{p}^{\varepsilon,\vartheta}$ in (5.4), together with the properties of the extension of $\mathbf{n}^{\varepsilon,\vartheta}$ and $\mathbf{p}^{\varepsilon,\vartheta}$ from Ω^ε_M into Ω, see Lemma 4.1, and applying the Kolmogorov theorem, see *e.g.* [5,20], we obtain the strong convergence of $\mathbf{n}^{\varepsilon,\vartheta}$ and $\mathbf{p}^{\varepsilon,\vartheta}$ in $L^2(\Omega_T)$.

In the same way as in [26] we show the strong convergence $\mathcal{T}^*_\varepsilon(b^{\varepsilon,\vartheta}) \to b^\vartheta$ in $L^2(\Omega_T \times \hat{Y}_M)$ as $\varepsilon \to 0$. Here we present only a sketch of the calculations. Using the extension of $\mathbf{n}^{\varepsilon,\vartheta}$ from Ω^ε_M into Ω, see Lemma 4.1, we define the extension of $b^{\varepsilon,\vartheta}$ from Ω^ε_M into Ω as a solution of the ordinary differential equation

$$
\begin{aligned}
&\partial_t b^{\varepsilon,\vartheta} = R_b(\mathbf{n}^{\varepsilon,\vartheta}, b^{\varepsilon,\vartheta}, \mathcal{N}_\delta(\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}))) && \text{in } (0,T) \times \Omega,\\
&b^{\varepsilon,\vartheta}(0) = b_0 && \text{in } \Omega.
\end{aligned}
\tag{5.11}
$$

The construction of the extension for $\mathbf{n}^{\varepsilon,\vartheta}$ and the uniform boundedness of $n_1^{\varepsilon,\vartheta}$ and $n_2^{\varepsilon,\vartheta}$ in $\Omega^\varepsilon_{M,T}$, see (5.4), ensure

$$
\|\mathbf{n}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega))} \le C_1 \|\mathbf{n}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega^\varepsilon_M))} \le C,
$$

with the constant C independent of ε and ϑ. Notice that we identify $\mathbf{n}^{\varepsilon,\vartheta}$ with its extension. Hence from (5.11), using estimate (5.3) for $\mathbf{u}^{\varepsilon,\vartheta}$, we obtain the boundedness of $b^{\varepsilon,\vartheta}$ and $\partial_t b^{\varepsilon,\vartheta}$. We show the strong convergence of $b^{\varepsilon,\vartheta}$

using arguments similar to those found in the proof of Lemma 5.2 by applying the Kolmogorov theorem [5,20]. Considering equation (5.11) at $(t, x + \mathbf{h}_j)$ and (t, x), for $j = 1, 2, 3$, taking $b^{\varepsilon,\vartheta}(t, x + \mathbf{h}_j) - b^{\varepsilon,\vartheta}(t, x)$ as a test function and using the Lipschitz continuity of R_b yield

$$\|b^{\varepsilon,\vartheta}(\tau, \cdot + \mathbf{h}_j) - b^{\varepsilon,\vartheta}(\tau, \cdot)\|^2_{L^2(\Omega_{2h})} \le \|b_0(\cdot + \mathbf{h}_j) - b_0(\cdot)\|^2_{L^2(\Omega_{2h})} + C_1 \int_0^\tau \|b^{\varepsilon,\vartheta}(t, \cdot + \mathbf{h}_j) - b^{\varepsilon,\vartheta}(t, \cdot)\|^2_{L^2(\Omega_{2h})} \mathrm{d}t$$
$$+ C_2 \int_0^\tau \left(\|\mathbf{n}^{\varepsilon,\vartheta}(t, \cdot + \mathbf{h}_j) - \mathbf{n}^{\varepsilon,\vartheta}(t, \cdot)\|^2_{L^2(\Omega_{2h})} + \delta^{-6} \Big\| \int_{B_{\delta,h}(x)\cap\Omega} \operatorname{tr} \mathbb{E}^\varepsilon(b^{\varepsilon,\vartheta}, \tilde{x}) \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t, \tilde{x})) \mathrm{d}\tilde{x} \Big\|^2_{L^2(\Omega_{2h})} \right) \mathrm{d}t$$

for $\tau \in (0, T]$, where $\Omega_{2h} = \{x \in \Omega \mid \operatorname{dist}(x, \partial\Omega) > 2h\}$, $B_{\delta,h}(x)$ is defined as in Lemma 5.2 with $|B_{\delta,h}(x) \cap \Omega| \le C\delta^2 h$ for all $x \in \overline{\Omega}$, and the constants C_1, C_2 are independent of ε, ϑ, and h. Using the regularity of the initial condition $b_0 \in H^1(\Omega)$ and the *a priori* estimates for $\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})$ and $\nabla \mathbf{n}^{\varepsilon,\vartheta}$, and applying the Gronwall inequality we obtain

$$\sup_{t\in(0,T)} \|b^{\varepsilon,\vartheta}(t, \cdot + \mathbf{h}_j) - b^{\varepsilon,\vartheta}(t, \cdot)\|^2_{L^2(\Omega_{2h})} \le C_\delta h. \tag{5.12}$$

Extending $b^{\varepsilon,\vartheta}$ by zero from Ω_T into $\mathbb{R}_+ \times \mathbb{R}^3$ and using the uniform boundedness of $b^{\varepsilon,\vartheta}$ imply

$$\|b^{\varepsilon,\vartheta}\|^2_{L^\infty(0,T;L^2(\widetilde{\Omega}_{3h}))} + \|b^{\varepsilon,\vartheta}\|^2_{L^2((T-2h,T+2h)\times\Omega)} \le Ch, \tag{5.13}$$

where $\widetilde{\Omega}_{3h} = \{x \in \mathbb{R}^3 \mid \operatorname{dist}(x, \partial\Omega) < 3h\}$ and the constant C is independent of ε, ϑ, and h. The estimate for $\partial_t b^{\varepsilon,\vartheta}$ ensures that

$$\|b^{\varepsilon,\vartheta}(\cdot + h, \cdot) - b^{\varepsilon,\vartheta}(\cdot, \cdot)\|^2_{L^2((0,T-h)\times\Omega)} \le C_1 h^2 \|\partial_t b^{\varepsilon,\vartheta}\|^2_{L^2(\Omega_T)} \le C_2 h^2, \tag{5.14}$$

where C_1 and C_2 are independent of ε, ϑ, and h. Combining (5.12)–(5.14) and applying the Kolmogorov theorem yield the strong convergence of $b^{\varepsilon,\vartheta}$ to $\widetilde{b}^\vartheta$ in $L^2(\Omega_T)$ as $\varepsilon \to 0$. The definition of two-scale convergence implies that $\widetilde{b}^\vartheta = b^\vartheta$ and, hence, the two-scale limit of $b^{\varepsilon,\vartheta}$ is independent of y. Then using the properties of the unfolding operator, see *e.g.* [7,8], we obtain the strong convergence of $\mathcal{T}^*_\varepsilon(b^{\varepsilon,\vartheta})$.

Considering an extension $\partial_t \overline{\mathbf{u}}^{\varepsilon,\vartheta}$ of $\partial_t \mathbf{u}^{\varepsilon,\vartheta}$ from Ω^ε_M into Ω, see Lemma 4.1, and applying the Korn inequality, see *e.g.* [22], yield

$$\begin{aligned} \|\partial_t \mathbf{u}^{\varepsilon,\vartheta}\|_{L^2(0,T;H^1(\Omega^\varepsilon_M))} &\le \|\partial_t \overline{\mathbf{u}}^{\varepsilon,\vartheta}\|_{L^2(0,T;H^1(\Omega))} \le C_1 \big[\|\partial_t \overline{\mathbf{u}}^{\varepsilon,\vartheta}\|_{L^2(\Omega_T)} + \|\mathbf{e}(\partial_t \overline{\mathbf{u}}^{\varepsilon,\vartheta})\|_{L^2(\Omega_T)} \big] \\ &\le C_2 \big[\|\partial_t \mathbf{u}^{\varepsilon,\vartheta}\|_{L^2(\Omega^\varepsilon_{M,T})} + \|\mathbf{e}(\partial_t \mathbf{u}^{\varepsilon,\vartheta})\|_{L^2(\Omega^\varepsilon_{M,T})} \big] \le C_3(1 + \vartheta^{-\frac{1}{2}}), \end{aligned} \tag{5.15}$$

where the constant C_3 is independent of ε and ϑ.

The estimates (5.3) and (5.15) ensure the existence of functions $\mathbf{u}^\vartheta \in L^2(0, T; \mathcal{W}(\Omega))$, $\hat{\mathbf{u}}^\vartheta \in L^2(\Omega_T; H^1_{\rm per}(\hat{Y})/\mathbb{R})^3$, $\boldsymbol{\xi}^\vartheta \in L^2(0, T; H^1(\Omega))^3$, and $\hat{\boldsymbol{\xi}}^\vartheta \in L^2(\Omega_T; H^1_{\rm per}(\hat{Y}_M)/\mathbb{R})^3$ such that

$$\begin{aligned} &\mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \mathbf{u}^\vartheta, && \nabla \mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \nabla \mathbf{u}^\vartheta + \hat{\nabla}_y \hat{\mathbf{u}}^\vartheta && \text{two-scale,} \\ &\chi_{\Omega^\varepsilon_M} \partial_t \mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \chi_{\hat{Y}_M} \boldsymbol{\xi}^\vartheta, && \chi_{\Omega^\varepsilon_M} \nabla \partial_t \mathbf{u}^{\varepsilon,\vartheta} \rightharpoonup \chi_{\hat{Y}_M} (\nabla \boldsymbol{\xi}^\vartheta + \hat{\nabla}_y \hat{\boldsymbol{\xi}}^\vartheta) && \text{two-scale,} \end{aligned}$$

as $\varepsilon \to 0$, see *e.g.* [4]. Considering the two-scale convergence of $\mathbf{u}^{\varepsilon,\vartheta}$ and $\partial_t \mathbf{u}^{\varepsilon,\vartheta}$, we obtain

$$\frac{|\hat{Y}_M|}{|\hat{Y}|} \langle \boldsymbol{\xi}^\vartheta, \phi \rangle_{\Omega_T} = \lim_{\varepsilon\to 0} \langle \partial_t \mathbf{u}^{\varepsilon,\vartheta}, \phi \rangle_{\Omega^\varepsilon_{M,T}} = -\lim_{\varepsilon\to 0} \langle \mathbf{u}^{\varepsilon,\vartheta}, \partial_t \phi \rangle_{\Omega^\varepsilon_{M,T}} = -\frac{|\hat{Y}_M|}{|\hat{Y}|} \langle \mathbf{u}^\vartheta, \partial_t \phi \rangle_{\Omega_T}$$

for all $\phi \in C_0^\infty(\Omega_T)$. Hence, $\partial_t \mathbf{u}^\vartheta \in L^2(\Omega_T)^3$ and $\boldsymbol{\xi}^\vartheta = \partial_t \mathbf{u}^\vartheta$ a.e. in Ω_T. Thus $\partial_t \mathbf{u}^\vartheta \in L^2(0,T;\mathcal{W}(\Omega))$. The two-scale convergence of $\nabla \mathbf{u}^{\varepsilon,\vartheta}$ and $\partial_t \nabla \mathbf{u}^{\varepsilon,\vartheta}$ implies

$$
\begin{aligned}
|\hat{Y}|^{-1}\langle \partial_t \nabla \mathbf{u}^\vartheta + \hat{\nabla}_y \hat{\boldsymbol{\xi}}^\vartheta, \phi\rangle_{\Omega_T\times\hat{Y}_M} &= \lim_{\varepsilon\to 0}\langle \partial_t \nabla \mathbf{u}^{\varepsilon,\vartheta}, \phi\rangle_{\Omega^\varepsilon_{M,T}} \\
&= -\lim_{\varepsilon\to 0}\langle \nabla \mathbf{u}^{\varepsilon,\vartheta}, \partial_t\phi\rangle_{\Omega^\varepsilon_{M,T}} = -|\hat{Y}|^{-1}\langle \nabla \mathbf{u}^\vartheta + \hat{\nabla}_y \hat{\mathbf{u}}^\vartheta, \partial_t \phi\rangle_{\Omega_T\times\hat{Y}_M}
\end{aligned}
$$

for all $\phi \in C_0^\infty(\Omega_T; C^\infty_{\rm per}(\hat{Y}))$. Thus, $\partial_t \hat{\nabla}_y \hat{\mathbf{u}}^\vartheta \in L^2(\Omega_T\times\hat{Y}_M)^{3\times 3}$ and $\hat{\nabla}_y \hat{\boldsymbol{\xi}}^\vartheta = \partial_t \hat{\nabla}_y\hat{\mathbf{u}}^\vartheta$ a.e. in $\Omega_T\times\hat{Y}_M$. Therefore, $\mathbf{u}^\vartheta \in H^1(0,T;\mathcal{W}(\Omega))$, $\partial_t \hat{\mathbf{u}}^\vartheta \in L^2(\Omega_T; H^1_{\rm per}(\hat{Y}_M)/\mathbb{R})^3$ and $\chi_{\Omega^\varepsilon_M}\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}) \rightharpoonup \chi_{\hat{Y}_M}\left(\partial_t \mathbf{e}(\mathbf{u}^\vartheta) + \partial_t \hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)\right)$ two-scale. $\square$

To derive macroscopic equations for the microscopic problem (2.1)–(2.3), we first derive the macroscopic equations for the perturbed system (2.2) and (5.1), with the initial and boundary conditions in (2.1), (2.3) and (5.2). Then letting the perturbation parameter go to zero we derive the macroscopic equations (3.10)–(3.12). By showing that the macroscopic two-scale problem is the same for the original microscopic equations (2.1)–(2.3) and for the perturbed microscopic problem as the perturbation parameter ϑ goes to zero, we conclude that (3.10)–(3.12) are the macroscopic equations for (2.1)–(2.3).

Theorem 5.4. *A sequence of solutions $(\mathbf{p}^{\varepsilon,\vartheta}, \mathbf{n}^{\varepsilon,\vartheta}, b^{\varepsilon,\vartheta}, \mathbf{u}^{\varepsilon,\vartheta})$, of the perturbed microscopic equations (2.2) and (5.1), with the initial and boundary conditions in (2.1), (2.3) and (5.2), converges, as $\varepsilon \to 0$, to a solution $(\mathbf{p}^\vartheta, \mathbf{n}^\vartheta, b^\vartheta, \mathbf{u}^\vartheta)$ of the perturbed macroscopic problem*

$$
\begin{aligned}
&\vartheta\partial_t^2 \mathbf{u}^\vartheta - \operatorname{div}\left(\mathbb{E}^\vartheta_{\rm hom}(b^\vartheta)\,\mathbf{e}(\mathbf{u}^\vartheta) + \mathbb{V}^\vartheta_{\rm hom}(b^\vartheta)\,\partial_t\mathbf{e}(\mathbf{u}^\vartheta) + \int_0^t \mathbb{K}^\vartheta(t-s,s,b^\vartheta)\,\partial_s \mathbf{e}(\mathbf{u}^\vartheta)\,ds\right) = \mathbf{0} && \text{in } \Omega_T,\\
&\left(\mathbb{E}^\vartheta_{\rm hom}(b^\vartheta)\,\mathbf{e}(\mathbf{u}^\vartheta) + \mathbb{V}^\vartheta_{\rm hom}(b^\vartheta)\,\partial_t\mathbf{e}(\mathbf{u}^\vartheta) + \int_0^t \mathbb{K}^\vartheta(t-s,s,b^\vartheta)\,\partial_s \mathbf{e}(\mathbf{u}^\vartheta)\,ds\right)\boldsymbol{\nu} = \mathbf{f} && \text{on } \Gamma_{\mathcal{E}\mathcal{U},T},\\
&\left(\mathbb{E}^\vartheta_{\rm hom}(b^\vartheta)\,\mathbf{e}(\mathbf{u}^\vartheta) + \mathbb{V}^\vartheta_{\rm hom}(b^\vartheta)\,\partial_t\mathbf{e}(\mathbf{u}^\vartheta) + \int_0^t \mathbb{K}^\vartheta(t-s,s,b^\vartheta)\,\partial_s \mathbf{e}(\mathbf{u}^\vartheta)\,ds\right)\boldsymbol{\nu} = -p_{\mathcal{I}}\boldsymbol{\nu} && \text{on } \Gamma_{\mathcal{I},T},\\
&\mathbf{u}^\vartheta && a_3\text{-periodic in } x_3,\\
&\mathbf{u}^\vartheta(0) = \mathbf{u}_0, \qquad \partial_t \mathbf{u}^\vartheta(0) = \mathbf{0} && \text{in } \Omega,
\end{aligned}
\tag{5.16}
$$

and

$$
\begin{aligned}
\partial_t \mathbf{p}^\vartheta &= \operatorname{div}(\mathcal{D}_p\nabla\mathbf{p}^\vartheta) - \mathbf{F}_p(\mathbf{p}^\vartheta) && \text{in } \Omega_T,\\
\partial_t \mathbf{n}^\vartheta &= \operatorname{div}(\mathcal{D}_n\nabla\mathbf{n}^\vartheta) + \mathbf{F}_n(\mathbf{p}^\vartheta,\mathbf{n}^\vartheta) + \mathbf{R}_n(\mathbf{n}^\vartheta, b^\vartheta, \mathcal{N}^{\rm eff}_\delta(\mathbf{e}(\mathbf{u}^\vartheta))) && \text{in } \Omega_T,\\
\partial_t b^\vartheta &= R_b(\mathbf{n}^\vartheta, b^\vartheta, \mathcal{N}^{\rm eff}_\delta(\mathbf{e}(\mathbf{u}^\vartheta))) && \text{in } \Omega_T,
\end{aligned}
\tag{5.17}
$$

together with the initial and boundary conditions

$$
\begin{aligned}
&\mathcal{D}_p\nabla\mathbf{p}^\vartheta\,\boldsymbol{\nu} = \theta_M^{-1}\mathbf{J}_p(\mathbf{p}^\vartheta), && \mathcal{D}_n\nabla\mathbf{n}^\vartheta\,\boldsymbol{\nu} = \theta_M^{-1}\mathbf{G}(\mathbf{n}^\vartheta)\mathcal{N}^{\rm eff}_\delta(\mathbf{e}(\mathbf{u}^\vartheta)) && \text{on } \Gamma_{\mathcal{I},T},\\
&\mathcal{D}_p\nabla\mathbf{p}^\vartheta\,\boldsymbol{\nu} = -\theta_M^{-1}\gamma_p\,\mathbf{p}^\vartheta, && \mathcal{D}_n\nabla\mathbf{n}^\vartheta\,\boldsymbol{\nu} = \theta_M^{-1}\mathbf{J}_n(\mathbf{n}^\vartheta) && \text{on } \Gamma_{\mathcal{E},T},\\
&\mathcal{D}_p\nabla\mathbf{p}^\vartheta\,\boldsymbol{\nu} = \mathbf{0}, && \mathcal{D}_n\nabla\mathbf{n}^\vartheta\,\boldsymbol{\nu} = \mathbf{0} && \text{on } \Gamma_{\mathcal{U},T},\\
&\mathbf{p}^\vartheta, \quad \mathbf{n}^\vartheta && a_3\text{-periodic in } x_3, &&\\
&\mathbf{p}^\vartheta(0) = \mathbf{p}_0, \quad \mathbf{n}^\vartheta(0) = \mathbf{n}_0, && b(0) = b_0 && \text{in } \Omega,
\end{aligned}
\tag{5.18}
$$

where $\mathbb{E}^\vartheta_{\rm hom}$, $\mathbb{V}^\vartheta_{\rm hom}$, and $\mathbb{K}^\vartheta$ are defined as in (3.4) and (3.5), with b^ϑ, $\mathbf{w}^{ij}_\vartheta$, $\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta}$, and $\mathbf{v}^{ij}_\vartheta$ instead of b, $\mathbf{w}^{ij}$, $\boldsymbol{\chi}^{ij}_{\mathbb{V}}$, and $\mathbf{v}^{ij}$, where $\mathbf{w}^{ij}_\vartheta$, $\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta}$, and $\mathbf{v}^{ij}_\vartheta$ are solutions of the 'unit cell' problems (3.6) and (3.7) with b^ϑ instead of b, for $i,j = 1,2,3$. The macroscopic diffusion matrices $\mathcal{D}^l_\alpha$, with $\alpha = n,p$ and $l = 1,2$, are defined as in (3.8) and $\mathcal{N}^{\rm eff}_\delta$ is defined as in (3.13) with b^ϑ and $\mathbf{u}^\vartheta$ instead of b and $\mathbf{u}$.

Proof. To pass to the limit in the equations for $\mathbf{n}^{\varepsilon,\vartheta}$ and $b^{\varepsilon,\vartheta}$, we shall first prove the strong convergence of $\int_\Omega \mathbb{E}(b^\vartheta, x/\varepsilon)\, \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\mathrm{d}x$ in $L^2(0,T)$, as $\varepsilon \to 0$.

Considering the difference of (5.1) for t and $t+h$ and taking $\delta^h \mathbf{u}^{\varepsilon,\vartheta}(t,x) = \mathbf{u}^{\varepsilon,\vartheta}(t+h,x) - \mathbf{u}^{\varepsilon,\vartheta}(t,x)$ as a test function yield

$$\begin{aligned}
&\int_0^{T-h} \Big[\big\langle \mathbb{E}^\varepsilon(b^{\varepsilon,\vartheta}(t+h),x)\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t+h)) - \mathbb{E}^\varepsilon(b^{\varepsilon,\vartheta}(t),x)\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t)), \mathbf{e}(\delta^h\mathbf{u}^{\varepsilon,\vartheta})\big\rangle_\Omega \\
&\quad + \big\langle \mathbb{V}_M(b^{\varepsilon,\vartheta}(t+h))\partial_t \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t+h)) - \mathbb{V}_M(b^{\varepsilon,\vartheta}(t))\partial_t\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t)), \mathbf{e}(\delta^h\mathbf{u}^{\varepsilon,\vartheta})\big\rangle_{\Omega_M^\varepsilon}\Big]\mathrm{d}t \\
&\quad + \vartheta\big\langle \delta^h\partial_t\mathbf{u}^{\varepsilon,\vartheta}(T-h), \delta^h\mathbf{u}^{\varepsilon,\vartheta}(T-h)\big\rangle_{\Omega_M^\varepsilon} - \vartheta\big\langle\delta^h\partial_t\mathbf{u}^{\varepsilon,\vartheta}(0),\delta^h\mathbf{u}^{\varepsilon,\vartheta}(0)\big\rangle_{\Omega_M^\varepsilon} \\
&\qquad = \int_0^{T-h}\Big[\vartheta\|\delta^h\partial_t\mathbf{u}^{\varepsilon,\vartheta}\|^2_{L^2(\Omega_M^\varepsilon)} + \big\langle\delta^h\mathbf{f}, \delta^h\mathbf{u}^{\varepsilon,\vartheta}\big\rangle_{L^2(\Gamma_{\mathcal{E}\mathcal{U}})} - \big\langle\delta^h p_{\mathcal{I}}\boldsymbol{\nu}, \delta^h\mathbf{u}^{\varepsilon,\vartheta}\big\rangle_{L^2(\Gamma_{\mathcal{I}})}\Big]\mathrm{d}t.
\end{aligned} \tag{5.19}$$

To estimate the first term on the right-hand side we integrate (5.1) over $(t,t+h)$ and take $\partial_t\overline{\mathbf{u}}^{\varepsilon,\vartheta}(t+h,x) - \partial_t\overline{\mathbf{u}}^{\varepsilon,\vartheta}(t,x)$ as a test function, with $\partial_t\overline{\mathbf{u}}^{\varepsilon,\vartheta}$ being an extension of $\partial_t\mathbf{u}^{\varepsilon,\vartheta}$ from Ω_M^ε into Ω as in Lemma 4.1, to obtain

$$\begin{aligned}
&\vartheta\|\delta^h\partial_t\mathbf{u}^{\varepsilon,\vartheta}\|^2_{L^2((0,T-h)\times\Omega_M^\varepsilon)} \le hC_1\big[\|p_{\mathcal{I}}\|^2_{H^1(0,T;L^2(\Gamma_{\mathcal{I}}))} + \|\mathbf{f}\|^2_{H^1(0,T;L^2(\Gamma_{\mathcal{E}\mathcal{U}}))}\big] \\
&+ h^{\frac12}C_2\big[\|\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{e}(\partial_t\overline{\mathbf{u}}^{\varepsilon,\vartheta})\|^2_{L^2(\Omega_T)} + \|\mathbf{e}(\partial_t\mathbf{u}^{\varepsilon,\vartheta})\|^2_{L^2(\Omega^\varepsilon_{M,T})}\big] \le C(h^{1/2}+h),
\end{aligned} \tag{5.20}$$

where the constant C is independent of ε, ϑ, and h, and $h\in(0,T)$. Here we used estimate (5.3), the equality $\delta^h\partial_t\mathbf{u}^{\varepsilon,\vartheta}(t,x) = \partial_t\mathbf{u}^{\varepsilon,\vartheta}(t+h,x) - \partial_t\mathbf{u}^{\varepsilon,\vartheta}(t,x) = \int_t^{t+h}\partial^2_\tau\mathbf{u}^{\varepsilon,\vartheta}(\tau,x)\mathrm{d}\tau$, and the property of the extension of $\partial_t\mathbf{u}^{\varepsilon,\vartheta}$ from Ω_M^ε into Ω, *i.e.* $\|\mathbf{e}(\partial_t\overline{\mathbf{u}}^{\varepsilon,\vartheta})\|_{L^2(\Omega_T)} \le C\|\mathbf{e}(\partial_t\mathbf{u}^{\varepsilon,\vartheta})\|_{L^2(\Omega^\varepsilon_{M,T})}$, with a constant C independent of ε and ϑ, see *e.g.* [22] or Lemma 4.1.

Using the estimate for $\vartheta^{1/2}\|\partial_t\mathbf{u}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^2(\Omega_M^\varepsilon))}$ in (5.3) we obtain

$$\begin{aligned}
\vartheta\langle\delta^h\partial_t\mathbf{u}^{\varepsilon,\vartheta}(T-h), \delta^h\mathbf{u}^{\varepsilon,\vartheta}(T-h)\rangle_{\Omega_M^\varepsilon} &\le 2\vartheta\|\partial_t\mathbf{u}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^2(\Omega_M^\varepsilon))}\|\delta^h\mathbf{u}^{\varepsilon,\vartheta}(T-h)\|_{L^2(\Omega_M^\varepsilon)} \\
&\le C_1\vartheta^{1/2}\Big\|\int_{T-h}^T\partial_t\mathbf{u}^{\varepsilon,\vartheta}\mathrm{d}t\Big\|_{L^2(\Omega_M^\varepsilon)} \le C_2h\,\vartheta^{1/2}\|\partial_t\mathbf{u}^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^2(\Omega_M^\varepsilon))} \le Ch.
\end{aligned} \tag{5.21}$$

In the same way we also have

$$\vartheta\langle\delta^h\partial_t\mathbf{u}^{\varepsilon,\vartheta}(0), \delta^h\mathbf{u}^{\varepsilon,\vartheta}(0)\rangle_{\Omega_M^\varepsilon} \le Ch, \tag{5.22}$$

where C is independent of ε, ϑ, and h. To estimate the first two terms on the left-hand side of (5.19) we use the uniform boundedness of $b^{\varepsilon,\vartheta}$ and $\partial_t b^{\varepsilon,\vartheta}$, the equality $\delta^h\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t,x)) = h\int_0^1\partial_t\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t+hs,x))\mathrm{d}s$, and estimate (5.3):

$$\begin{aligned}
&\int_0^{T-h}\langle(\mathbb{E}^\varepsilon(b^{\varepsilon,\vartheta}(t+h),x) - \mathbb{E}^\varepsilon(b^{\varepsilon,\vartheta}(t),x))\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t)), \mathbf{e}(\delta^h\mathbf{u}^{\varepsilon,\vartheta}(t))\rangle_\Omega\mathrm{d}t \\
&\qquad\qquad \le hC_1\|\partial_t b^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))}\|\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\|^2_{L^2(\Omega_T)} \le C_2h, \\
&\int_0^{T-h}\langle\mathbb{V}_M(b^{\varepsilon,\vartheta}(t+h))\partial_t\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t+h)) - \mathbb{V}_M(b^{\varepsilon,\vartheta}(t))\partial_t\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t)), \mathbf{e}(\delta^h\mathbf{u}^{\varepsilon,\vartheta}(t))\rangle_{\Omega_M^\varepsilon}\mathrm{d}t \\
&\qquad\qquad \le hC_3\|b^{\varepsilon,\vartheta}\|_{L^\infty(0,T;L^\infty(\Omega_M^\varepsilon))}\|\partial_t\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\|^2_{L^2(\Omega^\varepsilon_{M,T})} \le C_4h,
\end{aligned} \tag{5.23}$$

with the constants C_j, for $j=1,2,3,4$, independent of ε, ϑ, and h. Then, the assumptions on $\mathbb{E}$, $\mathbf{f}$, and $p_{\mathcal{I}}$, the boundedness of $b^{\varepsilon,\vartheta}$, and estimates (5.3) and (5.20)–(5.23) ensure

$$\begin{aligned}
&\|\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t+h)) - \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta}(t))\|^2_{L^2((0,T-h)\times\Omega)} \le C(h^{1/2}+h), \\
&\|\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\|^2_{L^2((T-h,T)\times\Omega)} \le h\|\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})\|^2_{L^\infty(0,T;L^2(\Omega))} \le Ch,
\end{aligned} \tag{5.24}$$

with a constant C independent of ε, ϑ, and h. Thus, estimates (5.24), the Kolmogorov theorem, and the two-scale convergence of $\mathbf{u}^{\varepsilon,\vartheta}$, yield the strong convergences, up to a subsequence,

$$\int_\Omega \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})dx \to \int_\Omega \fint_{\hat Y} [\mathbf{e}(\mathbf{u}^\vartheta) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)]dydx \quad \text{in } L^2(0,T),$$
$$\int_\Omega \mathbb{E}(b^\vartheta, x/\varepsilon)\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})dx \to \int_\Omega \fint_{\hat Y} \mathbb{E}(b^\vartheta, y)(\mathbf{e}(\mathbf{u}^\vartheta) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta))dydx \quad \text{in } L^2(0,T), \quad \text{as } \varepsilon \to 0.$$

Then the Lebesgue dominated convergence theorem ensures the strong convergence in $L^2(\Omega_T)$ and $L^2(\Gamma_{\mathcal{I},T})$ of $\int_{B_\delta(x)\cap\Omega} \mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})d\tilde x$ and $\int_{B_\delta(x)\cap\Omega} \mathbb{E}(b^\vartheta, \tilde x/\varepsilon)\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})d\tilde x$, as $\varepsilon \to 0$.

Now we can pass to the limit as $\varepsilon \to 0$ in the microscopic equations (2.2) and (5.1), with initial and boundary conditions in (2.1), (2.3), and (5.2). Considering $\boldsymbol{\phi}_\alpha(t,x) = \boldsymbol{\varphi}_\alpha(t,x) + \varepsilon\boldsymbol{\psi}_\alpha(t,x,\hat x/\varepsilon)$ as a test function in (2.5), where $\boldsymbol{\varphi}_\alpha \in C_0^1(0,T; C^1(\overline{\Omega}))^2$ and a_3-periodic in x_3, and $\boldsymbol{\psi}_\alpha \in C_0^1(\Omega_T; C^1_{\text{per}}(\hat Y))^2$, for $\alpha = p, n$, applying the two-scale convergence and using the strong convergence of $\mathcal{T}^*_\varepsilon(b^{\varepsilon,\vartheta})$ and $\mathbf{p}^{\varepsilon,\vartheta}$, $\mathbf{n}^{\varepsilon,\vartheta}$, see Lemma 5.3, along with the strong convergence of $\int_{B_\delta(x)\cap\Omega} \mathbb{E}(b^\vartheta, \tilde x/\varepsilon)\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})d\tilde x$, we obtain the macroscopic equations (5.17) and (5.18) for $\mathbf{p}^\vartheta$, $\mathbf{n}^\vartheta$, and b^ϑ in the same way as in [26].

The strong convergence of $\mathcal{T}^*_\varepsilon(b^{\varepsilon,\vartheta})$, along with the two-scale convergence of $\mathbf{u}^{\varepsilon,\vartheta}$, $\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})$, $\partial_t\mathbf{u}^{\varepsilon,\vartheta}$, and $\partial_t\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})$, as $\varepsilon \to 0$, yields the macroscopic equation

$$\begin{aligned} &\langle \mathbb{E}(b^\vartheta, y)(\mathbf{e}(\mathbf{u}^\vartheta) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)) + \mathbb{V}(b^\vartheta, y)\partial_t(\mathbf{e}(\mathbf{u}^\vartheta) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)), \mathbf{e}(\boldsymbol{\psi}) + \hat{\mathbf{e}}_y(\boldsymbol{\psi}_1)\rangle_{\Omega_T\times\hat Y} \\ &\qquad -\vartheta|\hat Y_M|\langle\partial_t\mathbf{u}^\vartheta, \partial_t\boldsymbol{\psi}\rangle_{\Omega_T} = |\hat Y|\big[\langle \mathbf{f}, \boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{E}\mathcal{U},T}} - \langle p_{\mathcal{I}}\boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{I},T}}\big] \end{aligned} \tag{5.25}$$

for $\boldsymbol{\psi} \in C_0^1(0,T; C^1(\overline{\Omega}))^3$, with $\boldsymbol{\psi}$ being a_3-periodic in x_3, and $\boldsymbol{\psi}_1 \in C_0^1(\Omega_T; C^1_{\text{per}}(\hat Y))^3$.

Taking $\boldsymbol{\psi} \equiv \mathbf{0}$ we obtain

$$\langle \mathbb{E}(b^\vartheta, y)(\mathbf{e}(\mathbf{u}^\vartheta) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)) + \mathbb{V}(b^\vartheta, y)\partial_t(\mathbf{e}(\mathbf{u}^\vartheta) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)), \hat{\mathbf{e}}_y(\boldsymbol{\psi}_1)\rangle_{\Omega_T\times\hat Y} = 0. \tag{5.26}$$

Considering the structure of (5.26) and taking into account the fact that $\mathbb{E}(b^\vartheta, \cdot)$ and $\mathbb{V}(b^\vartheta, \cdot)$ depend on t, we seek $\hat{\mathbf{u}}^\vartheta$ in the form

$$\hat{\mathbf{u}}^\vartheta(t,x,y) = \sum_{i,j=1}^{3}\Big[\mathbf{e}(\mathbf{u}^\vartheta(t,x))_{ij}\mathbf{w}^{ij}_\vartheta(t,x,y) + \int_0^t \partial_s\mathbf{e}(\mathbf{u}^\vartheta(s,x))_{ij}\mathbf{v}^{ij}_\vartheta(t-s,s,x,y)ds\Big]$$

and rewrite equation (5.26) as

$$\begin{aligned} &\Big\langle \mathbb{E}(b^\vartheta, y)\Big(\mathbf{e}(\mathbf{u}^\vartheta) + \sum_{i,j=1}^{3}\Big[\mathbf{e}(\mathbf{u}^\vartheta)_{ij}\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta) + \int_0^t \partial_s\mathbf{e}(\mathbf{u}^\vartheta)_{ij}\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta)ds\Big]\Big), \hat{\mathbf{e}}_y(\boldsymbol{\psi}_1)\Big\rangle_{\Omega_T\times\hat Y} \\ &\quad + \Big\langle \mathbb{V}_M(b^\vartheta)\Big(\partial_t\mathbf{e}(\mathbf{u}^\vartheta) + \sum_{i,j=1}^{3}\Big[\partial_t\mathbf{e}(\mathbf{u}^\vartheta)_{ij}\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta) + \mathbf{e}(\mathbf{u}^\vartheta)_{ij}\partial_t\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta) \\ &\quad + \partial_t\mathbf{e}(\mathbf{u}^\vartheta)_{ij}\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(0,t,x,y)) + \int_0^t \partial_s\mathbf{e}(\mathbf{u}^\vartheta)_{ij}\partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta)ds\Big]\Big), \hat{\mathbf{e}}_y(\boldsymbol{\psi}_1)\Big\rangle_{\Omega_T\times\hat Y_M} = 0. \end{aligned} \tag{5.27}$$

Considering the terms with $\mathbf{e}(\mathbf{u}^\vartheta)$ and $\partial_t\mathbf{e}(\mathbf{u}^\vartheta)$, respectively, we obtain that $\mathbf{v}^{ij}_\vartheta(0,t,x,y) = \boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta}(t,x,y) - \mathbf{w}^{ij}_\vartheta(t,x,y)$ a.e. in $\Omega_T \times \hat Y_M$, where $\mathbf{w}^{ij}_\vartheta$ and $\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta}$ are solutions of the 'unit cell' problems (3.6) with b^ϑ instead of b. Using this in (5.27) implies that $\mathbf{v}^{ij}_\vartheta$ satisfies (3.7) with b^ϑ instead of b. Then, taking $\boldsymbol{\psi}_1 \equiv \mathbf{0}$ in (5.25) yields the macroscopic equations (5.16) for $\mathbf{u}^\vartheta$.

Notice that the assumptions on $\mathbb{E}$ and $\mathbb{V}$ and the boundedness of b^ϑ and $\partial_t b^\vartheta$ ensure the existence of weak solutions $\mathbf{w}^{ij}_\vartheta$, $\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta}$, and $\mathbf{v}^{ij}_\vartheta$, with $i,j=1,2,3$, of the 'unit cell' problems (3.6) and (3.7), with b^ϑ instead of b.

In the same way as for the macroscopic elasticity tensor for the equations of linear elasticity, see *e.g.* [16,22], we obtain that $\mathbb{V}^\vartheta_{\rm hom}$ is positive-definite and possesses major and minor symmetries, as in Assumption 2.1.8. The assumptions on $\mathbb{E}$ and $\mathbb{V}$ and the uniform boundedness of b^ϑ ensure $\widetilde{\mathbb{E}}^\vartheta_{\rm hom} \in L^\infty(0,T;L^\infty(\Omega))^{3^4}$, $\mathbb{E}^\vartheta_{\rm hom} \in L^2(0,T;L^\infty(\Omega))^{3^4}$, $\mathbb{V}^\vartheta_{\rm hom} \in L^\infty(0,T;L^\infty(\Omega))^{3^4}$, $\widetilde{\mathbb{K}}^\vartheta(t-s,s) \in L^\infty(0,T;L^\infty(0,t;L^\infty(\Omega)))^{3^4}$, and $\mathbb{K}^\vartheta(t-s,s) \in L^2(0,T;L^\infty(0,t;L^\infty(\Omega)))^{3^4}$. Notice that the positive-definiteness and symmetry properties of $\mathbb{V}^\vartheta_{\rm hom}$, together with the boundedness of $\mathbb{E}^\vartheta_{\rm hom}$, $\mathbb{V}^\vartheta_{\rm hom}$, and $\mathbb{K}^\vartheta$, ensure the well-posedness of the viscoelastic equations (5.16). □

Now we can complete the proof of the main result of the paper.

Proof of Theorem 3.2. To complete the proof of Theorem 3.2, we have to prove that the sequence $\{\mathbf{p}^\vartheta, \mathbf{n}^\vartheta, b^\vartheta, \mathbf{u}^\vartheta\}$ converges to a solution of the macroscopic problem (3.10)–(3.13) and to show that the limit problem as $\vartheta \to 0$ of (5.25), together with the corresponding equations for $(\mathbf{p}^\vartheta, \mathbf{n}^\vartheta, b^\vartheta)$ in (5.17) and (5.18), is the same as the two-scale macroscopic problem for the original microscopic equations (2.1)–(2.3).

Using the fact that estimates (5.3) and (5.24) for $\mathbf{u}^{\varepsilon,\vartheta}$ are independent of ϑ and ε and applying the weak and two-scale convergence of $\mathbf{u}^{\varepsilon,\vartheta}$, together with the lower semicontinuity of a norm, yield

$$\begin{aligned} \|\mathbf{u}^\vartheta\|^2_{L^\infty(0,T;\mathcal{W}(\Omega))} + \|\mathbf{e}(\mathbf{u}^\vartheta) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)\|^2_{L^\infty(0,T;L^2(\Omega\times\hat{Y}))} \leq C, \\ \|\mathbf{e}(\mathbf{u}^\vartheta(\cdot+h,\cdot)) - \mathbf{e}(\mathbf{u}^\vartheta)\|^2_{L^2((0,T-h)\times\Omega)} + \|\mathbf{e}(\mathbf{u}^\vartheta)\|^2_{L^2((T-h,T)\times\Omega)} \leq C(h+h^{1/2}), \end{aligned} \tag{5.28}$$

with a constant C independent of ϑ and h.

Similar to the proof of Lemma 4.3, using (5.28) we obtain the estimates for $\mathbf{p}^\vartheta$ and $\mathbf{n}^\vartheta$ in $L^2(0,T;\mathcal{V}(\Omega))^2 \cap L^\infty(0,T;L^\infty(\Omega))^2$ and b^ϑ in $W^{1,\infty}(0,T;L^\infty(\Omega))$, uniformly in ϑ. In a similar way as in the proof of Lemma 5.2, we obtain

$$\|b^\vartheta(\cdot,\cdot+\mathbf{h}_k) - b^\vartheta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|b^\vartheta(\cdot+h,\cdot) - b^\vartheta\|^2_{L^2(\Omega_T)} + \|b^\vartheta(\cdot+h,\cdot) - b^\vartheta\|_{L^\infty(0,T-h;L^\infty(\Omega))} \leq Ch, \tag{5.29}$$

where b^ϑ is extended by zero from Ω_T into $\mathbb{R}_+\times\mathbb{R}^3$ and $\mathbf{h}_k = h\mathbf{b}_k$, with $h\in(0,T)$ and $k=1,2,3$. Then, applying the Kolmogorov theorem we obtain the strong convergence in $L^r(0,T;L^2(\Omega))$, for $2\leq r\leq\infty$ of a subsequence of b^ϑ, as $\vartheta\to 0$.

In a similar way as in the proof of Lemma 4.4, considering the assumptions on $\mathbb{E}$ and $\mathbb{V}$, together with the boundedness of b^ϑ and $\partial_t b^\vartheta$, uniformly in ϑ, we obtain that the weak solutions of the 'unit cell' problems (3.6), with b^ϑ instead of b, satisfy

$$\begin{aligned} &\|\mathbf{w}^{ij}_\vartheta\|_{L^\infty(0,T;H^1_{\rm per}(\hat{Y}))} + \|\partial_t \hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|_{L^2(0,T;L^2(\hat{Y}_M))} \leq C \quad && \text{for } x\in\Omega, \\ &\|\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta}\|_{H^1_{\rm per}(\hat{Y}_M)} \leq C && \text{for } (t,x)\in\Omega_T, \end{aligned} \tag{5.30}$$

where the constant C is independent of ϑ. The estimates (5.30) and boundedness of b^ϑ and $\partial_t b^\vartheta$ ensure the uniform in ϑ estimate for the weak solutions of the 'unit cell' problems (3.7), with b^ϑ instead of b, *i.e.*

$$\|\mathbf{v}^{ij}_\vartheta\|_{L^\infty(0,T-s;H^1_{\rm per}(\hat{Y}))} + \|\partial_t \hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta)\|_{L^2(0,T-s;L^2(\hat{Y}_M))} \leq C \tag{5.31}$$

for $x\in\Omega$ and $s\in[0,T]$.

Using the assumptions on $\mathbb{V}_M$, we obtain the symmetry properties and strong ellipticity of $\mathbb{V}^\vartheta_{\rm hom}$, see *e.g.* [22,27], with an ellipticity constant independent of ϑ. The assumptions on $\mathbb{E}$ and $\mathbb{V}_M$, the uniform boundedness of b^ϑ, and the estimates (5.30) and (5.31) ensure

$$\|\mathbb{E}^\vartheta_{\rm hom}(b^\vartheta)\|_{L^2(0,T;L^\infty(\Omega))} + \|\mathbb{V}^\vartheta_{\rm hom}(b^\vartheta)\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\mathbb{K}^\vartheta(t-s,s,b^\vartheta)\|_{L^2(0,T;L^\infty(0,t;L^\infty(\Omega)))} \leq C, \tag{5.32}$$

with a constant C independent of ϑ.

Taking $\partial_t \mathbf{u}^\vartheta$ as a test function in the weak formulation of (5.16), using the strong ellipticity of $\mathbb{V}^\vartheta_{\text{hom}}$, together with estimates (5.28) and (5.32), and applying the second Korn inequality for $\mathbf{u}^\vartheta(t) \in \mathcal{W}(\Omega)$ yield

$$\vartheta \|\partial_t \mathbf{u}^\vartheta\|^2_{L^\infty(0,T;L^2(\Omega))} + \|\mathbf{u}^\vartheta\|^2_{H^1(0,T;\mathcal{W}(\Omega))} \leq C, \tag{5.33}$$

with a constant C independent of ϑ. Hence we have the weak convergence, up to a subsequence, of $\mathbf{u}^\vartheta$ in $H^1(0,T;\mathcal{W}(\Omega))$ and weak-$*$ convergence of $\vartheta^{1/2}\partial_t \mathbf{u}^\vartheta$ in $L^\infty(0,T;L^2(\Omega))$, as $\vartheta \to 0$.

To pass to the limit as $\vartheta \to 0$ in the macroscopic equations (5.16) we have to show the strong convergence of $\mathbb{E}^\vartheta_{\text{hom}}$, $\mathbb{V}^\vartheta_{\text{hom}}$, and $\mathbb{K}^\vartheta$.

Considering the first equation in (3.6) for $t+h$ and t, with $h \in (0,T)$ and b^ϑ instead of b, taking $\delta^h \mathbf{w}^{ij}_\vartheta(t,x,y) = \mathbf{w}^{ij}_\vartheta(t+h,x,y) - \mathbf{w}^{ij}_\vartheta(t,x,y)$ as a test function, and using $\delta^h \hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta(t)) = h\int_0^1 \partial_t \hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta(t+h\tau))d\tau$, we obtain

$$\begin{aligned}\|\delta^h \hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|^2_{L^2((0,T-h)\times\hat{Y})} \leq & C_1 h\big[\|b^\vartheta\|_{L^\infty(0,T;L^\infty(\Omega))}\|\partial_t \hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|^2_{L^2(\hat{Y}_{M,T})} \\ & + \|\partial_t b^\vartheta\|_{L^2(0,T;L^\infty(\Omega))}(\|\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|^2_{L^\infty(0,T;L^2(\hat{Y}))} + \|\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|_{L^2(\hat{Y}_T)})\big] \leq C_2 h\end{aligned} \tag{5.34}$$

for $x \in \Omega$ and the constants C_1 and C_2 are independent of ϑ and h. Taking an extension $\delta^h \partial_t \overline{\mathbf{w}}^{ij}_\vartheta$ of $\delta^h \partial_t \mathbf{w}^{ij}_\vartheta$ from $\hat{Y}_M$ into $\hat{Y}$ as a test function in the weak formulation of $(3.6)_1$, with b^ϑ instead of b, yields

$$\begin{aligned}\|\delta^h \hat{\mathbf{e}}_y(\partial_t \mathbf{w}^{ij}_\vartheta)\|^2_{L^2((0,T-h)\times\hat{Y}_M)} \leq & C_1\|b^\vartheta\|_{L^\infty(0,T;L^\infty(\Omega))}\|\delta^h \hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|_{L^2(\hat{Y}_{T-h})}\|\hat{\mathbf{e}}_y(\delta^h\partial_t \overline{\mathbf{w}}^{ij}_\vartheta)\|_{L^2((0,T-h)\times\hat{Y})} \\ & + C_2\big[1 + \|\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|^2_{L^2(\hat{Y}_T)} + \|\hat{\mathbf{e}}_y(\partial_t \mathbf{w}^{ij}_\vartheta)\|^2_{L^2(\hat{Y}_{M,T})}\big]\|\delta^h b^\vartheta\|^2_{L^\infty(0,T-h;L^\infty(\Omega))} \leq C_3(h^{1/2}+h)\end{aligned} \tag{5.35}$$

for $x \in \Omega$ and the constants C_1, C_2, and C_3 are independent of ϑ and h. Here, we used estimate (5.34) and the fact that due to the periodicity of $\mathbf{w}^{ij}_\vartheta$ and the second Korn inequality we have

$$\|\delta^h \partial_t \mathbf{w}^{ij}_\vartheta\|_{L^2(0,T-h;H^1(\hat{Y}_M))} \leq C\|\delta^h \hat{\mathbf{e}}_y(\partial_t \mathbf{w}^{ij}_\vartheta)\|_{L^2((0,T-h)\times\hat{Y}_M)},$$

for $x \in \Omega$, and the property of the extension, *i.e.* $\|\hat{\mathbf{e}}_y(\delta^h\partial_t \overline{\mathbf{w}}^{ij}_\vartheta)\|_{L^2((0,T-h)\times\hat{Y})} \leq C\|\hat{\mathbf{e}}_y(\delta^h \partial_t \mathbf{w}^{ij}_\vartheta)\|_{L^2((0,T-h)\times\hat{Y}_M)}$, where the constant C is independent of $\partial_t \mathbf{w}^{ij}_\vartheta$, ϑ, and h. Estimates (5.30) ensure

$$\|\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|^2_{L^2((T-h,T)\times\hat{Y})} \leq Ch \quad \text{and} \quad \|\hat{\mathbf{e}}_y(\partial_t \mathbf{w}^{ij}_\vartheta)\|^q_{L^q(T-h,T;L^2(\hat{Y}_M))} \leq Ch^{\frac{2-q}{2}} \quad \text{for } 1<q<2.$$

Considering $(3.6)_1$, with b^ϑ instead of b, for $x + \mathbf{h}_k$ and x, where $\mathbf{h}_k = h\mathbf{b}_k$, for $k = 1,2,3$, and using (5.29) imply

$$\|\delta^{\mathbf{h}_k} \hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|^2_{L^2(\Omega_T\times\hat{Y})} + \|\delta^{\mathbf{h}_k}\partial_t \hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\|^2_{L^2(\Omega_T\times\hat{Y}_M)} \leq Ch, \tag{5.36}$$

where $\delta^{\mathbf{h}_k}\mathbf{w}^{ij}_\vartheta(t,x,y) = \mathbf{w}^{ij}_\vartheta(t,x+\mathbf{h}_k,y) - \mathbf{w}^{ij}_\vartheta(t,x,y)$, the function b^ϑ is extended by zero from Ω_T into $\mathbb{R}_+\times\mathbb{R}^3$, $\mathbf{w}^{ij}_\vartheta$ is extended by zero from Ω into $\mathbb{R}^3$, and C is independent of ϑ and h. In the same manner we obtain

$$\|\delta^h \hat{\mathbf{e}}_y(\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta})\|^2_{L^2(\Omega_T\times\hat{Y}_M)} + \|\delta^{\mathbf{h}_k}\hat{\mathbf{e}}_y(\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta})\|^2_{L^2(\Omega_T\times\hat{Y}_M)} \leq Ch, \tag{5.37}$$

where b^ϑ and $\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta}$ are extended by zero from Ω_T into $\mathbb{R}_+\times\mathbb{R}^3$, and

$$\|\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s+h,s)) - \hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s))\|^2_{L^2(0,T-h;L^2(\Omega_t\times\hat{Y}))} + \|\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s))\|^2_{L^2(T-h,T;L^2(\Omega_t\times\hat{Y}))} \leq Ch. \tag{5.38}$$

Considering the difference of the equations in (3.7), with b^ϑ instead of b, for $s+h$ and s and for $x+\mathbf{h}_k$ and x, taking $\mathbf{v}^{ij}_\vartheta(t,s+h,x,y) - \mathbf{v}^{ij}_\vartheta(t,s,x,y)$, $\delta^{\mathbf{h}_k}\mathbf{v}^{ij}_\vartheta$, and extensions of $\partial_t\left(\mathbf{v}^{ij}_\vartheta(t,s+h,x,y) - \mathbf{v}^{ij}_\vartheta(t,s,x,y)\right)$ and

$\delta^{\mathbf{h}_k}\partial_t \mathbf{v}^{ij}_\vartheta$ from $\hat{Y}_M$ into $\hat{Y}$, as test functions, respectively, and using estimates (5.34)–(5.37) yield

$$\begin{aligned}
\|\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s+h)) - \hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s))\|^2_{L^2(0,T-h;L^2(\Omega_t\times\hat{Y}))} &\le Ch,\\
\|\delta^{\mathbf{h}_k}\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s))\|^2_{L^2(0,T;L^2(\Omega_t\times\hat{Y}))} &\le Ch,\\
\|\partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s+h)) - \partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s))\|^2_{L^2(0,T-h;L^2(\Omega_t\times\hat{Y}_M))} &\le C(h^{1/2}+h),\\
\|\delta^{\mathbf{h}_k}\partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s))\|^2_{L^2(0,T;L^2(\Omega_t\times\hat{Y}_M))} &\le C(h^{1/2}+h),\\
\Big\|\int_0^{T-s}\int_{\hat{Y}_M}\partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t,\cdot,\cdot,y))\mathrm{d}y\mathrm{d}t\Big\|^2_{L^2((T-h,T)\times\Omega)} &\le Ch,
\end{aligned} \tag{5.39}$$

for $k=1,2,3$ and the constant C is independent of h and ϑ. Thus, (5.34)–(5.39) along with the Kolmogorov theorem and the strong convergence and boundedness of b^ϑ ensure the following strong convergences

$$\begin{aligned}
&\int_{\hat{Y}}\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\mathrm{d}y \to \int_{\hat{Y}}\hat{\mathbf{e}}_y(\mathbf{w}^{ij})\mathrm{d}y, && \widetilde{\mathbb{E}}^\vartheta_{\mathrm{hom}}(b^\vartheta)\to\widetilde{\mathbb{E}}_{\mathrm{hom}}(b) && \text{in } L^2(\Omega_T),\\
&\int_{\hat{Y}_M}\partial_t\hat{\mathbf{e}}_y(\mathbf{w}^{ij}_\vartheta)\mathrm{d}y \to \int_{\hat{Y}_M}\partial_t\hat{\mathbf{e}}_y(\mathbf{w}^{ij})\mathrm{d}y, && \mathbb{E}^\vartheta_{\mathrm{hom}}(b^\vartheta)\to\mathbb{E}_{\mathrm{hom}}(b) && \text{in } L^q(0,T;L^2(\Omega)),\ 1<q<2,\\
&\int_{\hat{Y}_M}\hat{\mathbf{e}}_y(\boldsymbol{\chi}^{ij}_{\mathbb{V},\vartheta})\mathrm{d}y \to \int_{\hat{Y}_M}\hat{\mathbf{e}}_y(\boldsymbol{\chi}^{ij}_{\mathbb{V}})\mathrm{d}y, && \mathbb{V}^\vartheta_{\mathrm{hom}}(b^\vartheta)\to\mathbb{V}_{\mathrm{hom}}(b) && \text{in } L^2(\Omega_T),\\
&\int_{\hat{Y}}\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t-s,s))\mathrm{d}y \to \int_{\hat{Y}}\hat{\mathbf{e}}_y(\mathbf{v}^{ij}(t-s,s))\mathrm{d}y &&&& \text{in } L^2(0,T;L^2(\Omega_t)),\\
&\widetilde{\mathbb{K}}^\vartheta(t-s,s,b^\vartheta)\to\widetilde{\mathbb{K}}(t-s,s,b) &&&& \text{in } L^2(0,T;L^2(\Omega_t)),\\
&\int_0^{T-s}\int_{\hat{Y}_M}\partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij}_\vartheta(t,s))\mathrm{d}y\mathrm{d}t \to \int_0^{T-s}\int_{\hat{Y}_M}\partial_t\hat{\mathbf{e}}_y(\mathbf{v}^{ij}(t,s))\mathrm{d}y\mathrm{d}t &&&& \text{in } L^2(\Omega_T),\\
&\int_0^{T-s}\mathbb{K}^\vartheta(t,s,b^\vartheta)\mathrm{d}t \to \int_0^{T-s}\mathbb{K}(t,s,b)\mathrm{d}t &&&& \text{in } L^2(\Omega_T),
\end{aligned}$$

as $\vartheta\to 0$. The strong convergence of $\widetilde{\mathbb{E}}^\vartheta_{\mathrm{hom}}$ and $\widetilde{\mathbb{K}}^\vartheta$ and estimate (5.33) ensure the strong convergence

$$\mathcal{N}^{\mathrm{eff}}_\delta(\mathbf{e}(\mathbf{u}^\vartheta))\to\mathcal{N}^{\mathrm{eff}}_\delta(\mathbf{e}(\mathbf{u})) \quad \text{in } L^2(\Omega_T) \quad \text{as } \vartheta\to 0.$$

Hence, taking the limit as $\vartheta\to 0$ in the weak formulation of (5.16)–(5.18) we obtain the macroscopic equations (3.10)–(3.12). Notice that for the integral-term in (5.16) we have

$$\Big\langle\int_0^t\mathbb{K}^\vartheta(t-s,s,b^\vartheta)\partial_s\mathbf{e}(\mathbf{u}^\vartheta(s,x))\mathrm{d}s,\boldsymbol{\psi}(t,x)\Big\rangle_{\Omega_T} = \int_0^T\int_\Omega\partial_s\mathbf{e}(\mathbf{u}^\vartheta(s,x))\int_0^{T-s}\mathbb{K}^\vartheta(\tau,s,b^\vartheta)\boldsymbol{\psi}(\tau+s,x)\mathrm{d}\tau\mathrm{d}x\mathrm{d}s$$

for all $\boldsymbol{\psi}\in C^1(\overline{\Omega}_T)^3$, $\boldsymbol{\psi}$ being a_3-periodic in x_3. Thus, using the weak convergence of $\partial_s\mathbf{e}(\mathbf{u}^\vartheta)$ and the strong convergence of $\int_0^{T-s}\mathbb{K}^\vartheta(t,s,b^\vartheta)\mathrm{d}t$ we can pass to the limit in the last term in (5.16).

The assumptions on the elastic $\mathbb{E}$ and viscous $\mathbb{V}$ tensors together with the regularity and boundedness of b ensure the existence of solutions of the 'unit cell' problems (3.6) and (3.7). As before, the assumptions on $\mathbb{E}$ and $\mathbb{V}$, the boundedness of b, and the estimates (5.30) and (5.31) yield the symmetry properties and strong ellipticity of $\mathbb{V}_{\mathrm{hom}}$, see *e.g.* [22], as well as the boundedness of the macroscopic tensors, *i.e.* $\widetilde{\mathbb{E}}_{\mathrm{hom}}\in L^\infty(0,T;L^\infty(\Omega))^{3^4}$, $\mathbb{E}_{\mathrm{hom}}\in L^2(0,T;L^\infty(\Omega))^{3^4}$, $\mathbb{V}_{\mathrm{hom}}\in L^\infty(0,T;L^\infty(\Omega))^{3^4}$, $\widetilde{\mathbb{K}}(t-s,s)\in L^\infty(0,T;L^\infty(0,t;L^\infty(\Omega)))^{3^4}$, and $\mathbb{K}(t-s,s)\in L^2(0,T;L^\infty(0,t;L^\infty(\Omega)))^{3^4}$. This together with the assumptions on the coefficients and nonlinear functions in the equations for $\mathbf{p},\mathbf{n}$, and b, see Assumption 2.1, ensures the existence of a unique weak solution of the macroscopic problem (3.10)–(3.12). Thus the whole sequence $\{\mathbf{p}^\vartheta,\mathbf{n}^\vartheta,b^\vartheta,\mathbf{u}^\vartheta\}$ converges to a weak solution of (3.10)–(3.12). Estimate (5.33) implies that $\mathbf{u}\in H^1(0,T;\mathcal{W}(\Omega))$. Hence, $\mathbf{u}\in C([0,T];\mathcal{W}(\Omega))$ and $\mathbf{u}$ satisfies the initial condition $\mathbf{u}(0,x)=\mathbf{u}_0(x)$ for $x\in\Omega$. □

To complete the proof of Theorem 3.2 we have to show that a sequence of solutions of the microscopic model (2.1)–(2.3) converges as $\varepsilon \to 0$ to a solution of the macroscopic equations (3.10)–(3.12). For this, using the result of Lemma 5.2, we show that the two-scale macroscopic problem for (2.1)–(2.3) is the same as the limit problem, as $\vartheta \to 0$, of the two-scale macroscopic problem for the perturbed microscopic equations (2.2) and (5.1), with initial and boundary conditions in (2.1), (2.3), and (5.2).

Lemma 5.5. *A sequence of solutions* $\{\mathbf{p}^\varepsilon, \mathbf{n}^\varepsilon, b^\varepsilon, \mathbf{u}^\varepsilon\}$ *of the microscopic problem* (2.1)–(2.3) *and a sequence of solutions* $\{\mathbf{p}^{\varepsilon,\vartheta}, \mathbf{n}^{\varepsilon,\vartheta}, b^{\varepsilon,\vartheta}, \mathbf{u}^{\varepsilon,\vartheta}\}$ *of the perturbed microscopic equations* (2.2) *and* (5.1)*, with initial and boundary conditions in* (2.1)*,* (2.3)*, and* (5.2)*, converge as* $\varepsilon \to 0$ *and* $\vartheta \to 0$ *to a solution* $(\mathbf{p}, \mathbf{n}, b, \mathbf{u}, \hat{\mathbf{u}})$ *of the macroscopic equations* (3.10)*,* (3.11)*, and*

$$\begin{aligned}\langle \mathbb{E}(b,y)(\mathbf{e}(\mathbf{u}) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}})) + \mathbb{V}(b,y)\partial_t(\mathbf{e}(\mathbf{u}) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}})), \mathbf{e}(\boldsymbol{\psi}) + \hat{\mathbf{e}}_y(\boldsymbol{\psi}_1)\rangle_{\Omega_T \times \hat{Y}} &\\ = |\hat{Y}|\big[\langle \mathbf{f}, \boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{EU}},T} - \langle p_{\mathcal{I}}\boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{I},T}}\big] &\end{aligned} \tag{5.40}$$

together with $\mathbf{u}(0) = \mathbf{u}_0$ *in* Ω*,* $\mathbf{u}$ *is* a_3*-periodic in* x_3*, and*

$$\mathcal{N}_\delta^{\text{eff}}(\mathbf{e}(\mathbf{u})) = \left(\fint_{B_\delta(x)\cap\Omega}\fint_{\hat{Y}} \operatorname{tr}\left(\mathbb{E}(b,y)(\mathbf{e}(\mathbf{u}) + \hat{\mathbf{e}}_y(\hat{\mathbf{u}}))\right) \mathrm{d}y\, \mathrm{d}\tilde{x}\right)^+ \qquad \text{for } (t,x) \in (0,T)\times\overline{\Omega}. \tag{5.41}$$

Proof. The *a priori* estimate (3.2), together with the second Korn inequality, ensures the weak and two-scale convergence of $\mathbf{u}^\varepsilon$ and $\partial_t\mathbf{e}(\mathbf{u}^\varepsilon)$, *i.e.* there exist $\tilde{\mathbf{u}} \in L^2(0,T;\mathcal{W}(\Omega))$, $\mathbf{u}^1 \in L^2(\Omega_T; H^1_{\text{per}}(\hat{Y})/\mathbb{R})^3$, $\Lambda \in L^2(\Omega_T)^{3\times 3}$, and $\Lambda_1 \in L^2(\Omega_T \times \hat{Y}_M)^{3\times3}$ such that

$$\begin{aligned} &\mathbf{u}^\varepsilon \rightharpoonup \tilde{\mathbf{u}} && \text{weakly in } L^2(0,T;\mathcal{W}(\Omega)), && \nabla\mathbf{u}^\varepsilon \rightharpoonup \nabla\tilde{\mathbf{u}} + \hat{\nabla}_y\mathbf{u}^1 && \text{two-scale,}\\ &\partial_t\mathbf{e}(\mathbf{u}^\varepsilon) \rightharpoonup \Lambda && \text{weakly in } L^2(\Omega_T)^{3\times3}, && \chi_{\Omega_M^\varepsilon}\partial_t\mathbf{e}(\mathbf{u}^\varepsilon) \rightharpoonup \chi_{\hat{Y}_M}\Lambda_1 && \text{two-scale,}\end{aligned} \tag{5.42}$$

as $\varepsilon \to 0$, where $\partial_t\mathbf{e}(\mathbf{u}^\varepsilon)$ is extended by zero from $\Omega^\varepsilon_{M,T}$ to Ω_T. Using the two-scale convergence of $\mathbf{e}(\mathbf{u}^\varepsilon)$ we obtain that $\Lambda_1 = \partial_t(\mathbf{e}(\tilde{\mathbf{u}}) + \hat{\mathbf{e}}_y(\mathbf{u}^1))$ in $\Omega_T \times \hat{Y}_M$.

The estimates for $\mathbf{p}^\varepsilon$, $\mathbf{n}^\varepsilon$, and b^ε in (3.1) and (3.3) imply the existence of $\tilde{\mathbf{p}}, \tilde{\mathbf{n}} \in L^2(0,T;\mathcal{V}(\Omega))^2 \cap L^\infty(0,T;L^\infty(\Omega))^2$, $\tilde{b} \in W^{1,\infty}(0,T;L^\infty(\Omega\times\hat{Y}_M))$, and $\mathbf{p}^1, \mathbf{n}^1 \in L^2(\Omega_T; H^1_{\text{per}}(\hat{Y}_M)/\mathbb{R})^2$ such that

$$\begin{aligned} &\mathbf{p}^\varepsilon \rightharpoonup \tilde{\mathbf{p}}, \quad \mathbf{n}^\varepsilon \rightharpoonup \tilde{\mathbf{n}} && \text{weakly in } L^2(0,T;H^1(\Omega))^2,\\ &\mathbf{p}^\varepsilon \to \tilde{\mathbf{p}}, \quad \mathbf{n}^\varepsilon \to \tilde{\mathbf{n}} && \text{strongly in } L^2(\Omega_T)^2,\\ &\nabla\mathbf{p}^\varepsilon \rightharpoonup \nabla\tilde{\mathbf{p}} + \hat{\nabla}_y\mathbf{p}^1, \quad \nabla\mathbf{n}^\varepsilon \rightharpoonup \nabla\tilde{\mathbf{n}} + \hat{\nabla}_y\mathbf{n}^1 && \text{two-scale,}\\ &b^\varepsilon \rightharpoonup \tilde{b}, \quad \partial_t b^\varepsilon \rightharpoonup \partial_t\tilde{b} && \text{two-scale.}\end{aligned} \tag{5.43}$$

The strong convergence of $\mathbf{p}^\varepsilon$ and $\mathbf{n}^\varepsilon$ is ensured by the estimates in (3.3) and the Kolmogorov compactness theorem. Using the *a priori* estimates and the convergence results for $\mathbf{u}^\varepsilon$, $\mathbf{p}^\varepsilon$, and $\mathbf{n}^\varepsilon$, in the same way as in Lemma 5.3 (see also [26], Lem. 5.3), we show that $\tilde{b}$ is independent of $\hat{y} = (y_1, y_2)$ and

$$\mathcal{T}^*_\varepsilon(b^\varepsilon) \to \tilde{b} \qquad \text{strongly in } L^2(\Omega\times\hat{Y}_M). \tag{5.44}$$

Using similar arguments as in the proof of Theorem 5.4, we obtain the strong convergence

$$\int_{B_\delta(x)\cap\Omega} \mathbb{E}(\tilde{b}, \tilde{x}/\varepsilon)\mathbf{e}(\mathbf{u}^\varepsilon)\mathrm{d}\tilde{x} \to \int_{B_\delta(x)\cap\Omega}\fint_{\hat{Y}} \mathbb{E}(\tilde{b}, y)(\mathbf{e}(\tilde{\mathbf{u}}) + \hat{\mathbf{e}}_y(\mathbf{u}^1))\mathrm{d}y\mathrm{d}\tilde{x} \qquad \text{in } L^2(\Omega_T) \text{ and } L^2(\Gamma_{\mathcal{I},T}).$$

Then the strong convergence of $\mathcal{T}^*_\varepsilon(b^\varepsilon)$, together with the two-scale convergence of $\mathbf{u}^\varepsilon$, $\mathbf{e}(\mathbf{u}^\varepsilon)$ and $\partial_t\mathbf{e}(\mathbf{u}^\varepsilon)$, as $\varepsilon \to 0$, yields the macroscopic viscoelastic equation

$$\begin{aligned}\langle \mathbb{E}(\tilde{b}, y)(\mathbf{e}(\tilde{\mathbf{u}}) + \hat{\mathbf{e}}_y(\mathbf{u}^1)) + \mathbb{V}(\tilde{b}, y)\partial_t(\mathbf{e}(\tilde{\mathbf{u}}) + \hat{\mathbf{e}}_y(\mathbf{u}^1)), \mathbf{e}(\boldsymbol{\psi}) + \hat{\mathbf{e}}_y(\boldsymbol{\psi}_1)\rangle_{\Omega_T\times\hat{Y}} = |\hat{Y}|\big[\langle \mathbf{f}, \boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{EU},T}} &\\ - \langle p_{\mathcal{I}}\boldsymbol{\nu}, \boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{I},T}}\big] &\end{aligned} \tag{5.45}$$

for $\boldsymbol{\psi} \in C_0^1(0,T;C^1(\overline{\Omega}))^3$, with $\boldsymbol{\psi}$ being a_3-periodic in x_3, and $\boldsymbol{\psi}_1 \in C_0^1(\Omega_T;C^1_{\rm per}(\hat{Y}))^3$. Using the two-scale and strong convergence of $\mathbf{p}^\varepsilon$, $\mathbf{n}^\varepsilon$, and b^ε we obtain that $\tilde{\mathbf{p}}$, $\tilde{\mathbf{n}}$, and $\tilde{b}$ satisfy the macroscopic equations (3.10) and (3.11), where $\mathcal{N}_\delta^{\rm eff}(\mathbf{e}(\tilde{\mathbf{u}}))$ is defined as in (5.41) with $\tilde{b}$, $\tilde{\mathbf{u}}$, and $\mathbf{u}^1$ instead of b, $\mathbf{u}$, and $\hat{\mathbf{u}}$.

Now we consider equation (5.25). Using the fact that estimates (5.3) and (5.24) are independent of ϑ and ε and applying the weak and two-scale convergence of $\mathbf{u}^{\varepsilon,\vartheta}$, $\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})$ and $\partial_t\mathbf{e}(\mathbf{u}^{\varepsilon,\vartheta})$, as $\varepsilon \to 0$, together with the lower semicontinuity of a norm yield

$$\begin{aligned}\|\mathbf{u}^\vartheta\|_{L^\infty(0,T;\mathcal{W}(\Omega))} + \|\mathbf{e}(\mathbf{u}^\vartheta)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)\|_{L^\infty(0,T;L^2(\Omega\times\hat{Y}))} + \|\partial_t(\mathbf{e}(\mathbf{u}^\vartheta)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta))\|_{L^2(\Omega_T\times\hat{Y}_M)} &\le C,\\ \vartheta^{1/2}\|\partial_t\mathbf{u}^\vartheta\|_{L^2(\Omega_T\times\hat{Y}_M)} &\le C,\\ \|(\mathbf{e}(\mathbf{u}^\vartheta(\cdot+h,\cdot))-\mathbf{e}(\mathbf{u}^\vartheta))+(\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta(\cdot+h,\cdot))-\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta))\|_{L^2((0,T-h)\times\Omega\times\hat{Y})} &\le Ch^{1/4},\\ \|\mathbf{e}(\mathbf{u}^\vartheta)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)\|_{L^2((T-h,T)\times\Omega\times\hat{Y})} &\le Ch^{1/2},\end{aligned} \tag{5.46}$$

with a constant C independent of ϑ and h. Using the second Korn inequality and assuming $\int_{\hat{Y}}\hat{\mathbf{u}}^\vartheta dy = \mathbf{0}$ we obtain that

$$\|\hat{\mathbf{u}}^\vartheta\|_{L^\infty(0,T;L^2(\Omega;H^1_{\rm per}(\hat{Y})))} \le C_1\|\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)\|_{L^\infty(0,T;L^2(\Omega\times\hat{Y}))} \le C_2.$$

Hence, there exist $\mathbf{u} \in L^\infty(0,T;\mathcal{W}(\Omega))$, $\hat{\mathbf{u}} \in L^\infty(0,T;L^2(\Omega;H^1_{\rm per}(\hat{Y})/\mathbb{R}))^3$, $\Lambda_2 \in L^2(\Omega_T\times\hat{Y}_M)^{3\times3}$, and $\boldsymbol{\xi} \in L^2(\Omega_T\times\hat{Y}_M)^3$, such that $\mathbf{u}^\vartheta \rightharpoonup \mathbf{u}$ in $L^2(0,T;\mathcal{W}(\Omega))$, $\hat{\mathbf{u}}^\vartheta \rightharpoonup \hat{\mathbf{u}}$ in $L^2(\Omega_T;H^1(\hat{Y}))^3$, $\partial_t(\mathbf{e}(\mathbf{u}^\vartheta)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta)) \rightharpoonup \Lambda_2$ in $L^2(\Omega_T\times\hat{Y}_M)^{3\times3}$, and $\vartheta^{1/2}\partial_t\mathbf{u}^\vartheta \rightharpoonup \boldsymbol{\xi}$ in $L^2(\Omega_T\times\hat{Y}_M)^3$, as $\vartheta \to 0$. The convergence of $\mathbf{u}^\vartheta$ and $\hat{\mathbf{u}}^\vartheta$ implies that $\Lambda_2 = \partial_t(\mathbf{e}(\mathbf{u})+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}))$ a.e. in $\Omega_T\times\hat{Y}_M$.

Using the strong convergence of b^ϑ, shown in the proof of Theorem 3.2, together with the convergence of $\mathbf{u}^\vartheta$, $\hat{\mathbf{u}}^\vartheta$, $\partial_t(\mathbf{e}(\mathbf{u}^\vartheta)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta))$, and $\vartheta^{1/2}\partial_t\mathbf{u}^\vartheta$, and taking in (5.25) the limit as $\vartheta \to 0$ we obtain

$$\langle\mathbb{E}(b,y)(\mathbf{e}(\mathbf{u})+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}))+\mathbb{V}(b,y)\partial_t(\mathbf{e}(\mathbf{u})+\hat{\mathbf{e}}_y(\hat{\mathbf{u}})),\mathbf{e}(\boldsymbol{\psi})+\hat{\mathbf{e}}_y(\boldsymbol{\psi}_1)\rangle_{\Omega_T\times\hat{Y}} = |\hat{Y}|\big[\langle\mathbf{f},\boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{EU},T}} - \langle p_{\mathcal{I}}\boldsymbol{\nu},\boldsymbol{\psi}\rangle_{\Gamma_{\mathcal{I},T}}\big] \tag{5.47}$$

for $\boldsymbol{\psi} \in C_0^1(0,T;C^1(\overline{\Omega}))^3$, with $\boldsymbol{\psi}$ being a_3-periodic in x_3, and $\boldsymbol{\psi}_1 \in C_0^1(\Omega_T;C^1_{\rm per}(\hat{Y}))^3$.

Also using the two-scale and strong convergences of $\mathbf{p}^\vartheta$, $\mathbf{n}^\vartheta$, b^ϑ, and $\int_{B_\delta(x)\cap\Omega}⨍_{\hat{Y}}\mathbb{E}(b,y)(\mathbf{e}(\mathbf{u}^\vartheta)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^\vartheta))dyd\tilde{x}$ we obtain that $\mathbf{p}$, $\mathbf{n}$, and b satisfy the macroscopic equations (3.10) and (3.11) with $\mathcal{N}_\delta^{\rm eff}(\mathbf{e}(\mathbf{u}))$ defined in (5.41).

To show uniqueness of a solution of (5.40) with the corresponding equations for $(\mathbf{p},\mathbf{n},b)$ in (3.10) and (3.11), we first consider the equation for the difference of two solutions $(\mathbf{u}^1-\mathbf{u}^2,\hat{\mathbf{u}}^1-\hat{\mathbf{u}}^2)$ and take the approximations of $\partial_t(\mathbf{u}^1-\mathbf{u}^2)$ and $\partial_t(\hat{\mathbf{u}}^1-\hat{\mathbf{u}}^2)$, similar as in the proof of Lemma 4.4, as test functions to obtain

$$\begin{aligned}\big\|\mathbf{e}(\mathbf{u}^1-\mathbf{u}^2)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^1-\hat{\mathbf{u}}^2)\big\|^2_{L^\infty(0,\tilde{T};L^2(\Omega\times\hat{Y}))} + \big\|\partial_t\big[\mathbf{e}(\mathbf{u}^1-\mathbf{u}^2)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^1-\hat{\mathbf{u}}^2)\big]\big\|^2_{L^2((0,\tilde{T})\times\Omega\times\hat{Y}_M)}\\ \le C_1\|b^1-b^2\|^2_{L^\infty(0,\tilde{T};L^\infty(\Omega))}\end{aligned}$$

for $\tilde{T} \in (0,T]$. In the same way as for the microscopic problem we can show that

$$\begin{aligned}\|b^1-b^2\|^2_{L^\infty(0,\tilde{T};L^\infty(\Omega))} &\le C_1\tilde{T}\|\mathbf{e}(\mathbf{u}^1-\mathbf{u}^2)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^1-\hat{\mathbf{u}}^2)\|^2_{L^\infty(0,\tilde{T};L^2(\Omega\times\hat{Y}))},\\ \|\mathbf{p}^1-\mathbf{p}^2\|^2_{L^\infty(0,\tilde{T};L^2(\Omega))} + \|\mathbf{n}^1-\mathbf{n}^2\|^2_{L^\infty(0,\tilde{T};L^2(\Omega))} &\le C_2\|\mathbf{e}(\mathbf{u}^1-\mathbf{u}^2)+\hat{\mathbf{e}}_y(\hat{\mathbf{u}}^1-\hat{\mathbf{u}}^2)\|^2_{L^\infty(0,\tilde{T};L^2(\Omega\times\hat{Y}))}\end{aligned}$$

for $\tilde{T} \in (0,T]$. Considering $\tilde{T}$ sufficiently small and iterating over time intervals we obtain the uniqueness result for (5.40) with the corresponding equations for $(\mathbf{p},\mathbf{n},b)$. Hence $\tilde{\mathbf{u}} = \mathbf{u}$, $\mathbf{u}^1 = \hat{\mathbf{u}}$, $\tilde{\mathbf{p}} = \mathbf{p}$, $\tilde{\mathbf{n}} = \mathbf{n}$, and $\tilde{b} = b$ and the whole sequences $\{\mathbf{p}^\varepsilon,\mathbf{n}^\varepsilon,b^\varepsilon,\mathbf{u}^\varepsilon\}$ and $\{\mathbf{p}^\vartheta,\mathbf{n}^\vartheta,b^\vartheta,\mathbf{u}^\vartheta\}$, respectively, converge to a solution of the macroscopic two-scale problem (3.10), (3.11), and (5.40).

Using the derivation of the macroscopic equations in the proof of Theorem 5.4 and convergence results in the proof of Theorem 3.2 and Lemma 5.2 we obtain that (3.10)–(3.12) are the macroscopic equations for the original microscopic problem (2.1)–(2.3). □

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