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`On solitary wave in nonuniform shear currents*

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Abstract: In this paper, steady solutions of solitary waves in the presence of nonuniform shear currents are obtained by use of the High-Level Green-Naghdi (HLGN) model. We focus on large-amplitude solitary waves in strong opposing shear currents. The linear-type currents, quadratic-type currents and cubic-type currents are considered. In particular, the wave speed, wave profile, velocity field, particle trajectories and vorticity distribution are studied. It is demonstrated that the solitary wave and nonuniform shear current interaction changes the velocity field and vorticity field significantly.

Key words: Solitary wave, nonuniform shear current, velocity field, vorticity field

A solitary wave has been an important topic in nonlinear water wave field for many decades. Dutykh and Clamond[1] proposed an effective method to calculate the profile and the velocity field of the solitary waves ($H/d \le 0.79$, where H is the wave amplitude and d is the water depth) by solving Euler's equations. Recently, Duan et al. [2] calculated steep solitary waves, and even a limiting-amplitude solitary wave with $H/d = 0.833199$ by using the High-Level Irrotational Green-Naghdi (HLIGN) model.

Meanwhile, wave-current interaction is universal in coastal regions, and it changes the wave profile, speed and particle trajectories. Thus, it is important to study the effect of wave-current interaction of the flow field.

Steady solutions of solitary waves in the presence of linear shear currents have been studied by some researchers. Choi[3], Pak and Chow[4] used asymptotic method and third-order solution, respectively, to study the solitary-wave profile, wave speed and streamlines for a solitary wave in linear shear current. Duan et al. [5] used the High-Level Green-Naghdi (HLGN) model to obtain more accurate results for large-amplitude solitary waves in linear shear currents to compare with the results of Pak and Chow[4].

However, to our knowledge, few works considered solitary waves in the presence of nonuniform shear currents. Pak and Chow[4] studied two types of nonuniform shear currents as $u_c(z) = U\sqrt{z}$ and $u_c(z) = Uz^2$, where U is the current speed at the still-water level and $u_c(z)$ is the current profile. While Pak and Chow[4] only considered a small-amplitude solitary wave with $H/d = 0.1$. Meanwhile, particle trajectories and vorticity distribution are not considered by Pak and Chow[4].

In this paper, we use the High-Level Green-Naghdi (HLGN) model to investigate solitary waves in nonuniform shear currents. The fluid is assumed to be inviscid and incompressible. The water depth is constant.

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x is the horizontal axis and positive to the right. z is the vertical axis and positive up in this 2-dimentional study. The continuity equation is written as

$$\frac{\partial u}{\partial x} + \frac{\partial w}{\partial z} = 0, \tag{1}$$

where $u(x,z,t)$ is the horizontal component of velocity and $w(x,z,t)$ is the vertical component of velocity.

Euler's equations are written as

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + w\frac{\partial u}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x}, \tag{2a}$$

$$\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + w\frac{\partial w}{\partial z} = -\frac{1}{\rho}\left(\frac{\partial p}{\partial z} + \rho g\right), \tag{2b}$$

where ρ is the mass density, p is the pressure, g is the gravitational acceleration and t is time.

Meanwhile, the kinematic boundary conditions are written as

$$w = \frac{\partial \eta}{\partial t} + u\frac{\partial \eta}{\partial x} \qquad z = \eta(x,t), \tag{3a}$$

$$w = 0 \qquad z = -d, \tag{3b}$$

where $\eta(x,t)$ is the surface elevation.

In the HLGN model, only a single assumption on the velocity variation in vertical direction is introduced as

$$u(x,z,t) = \sum_{n=0}^{K-1} u_n(x,t) z^n, \tag{4a}$$

$$w(x,z,t) = \sum_{n=0}^{K} w_n(x,t) z^n, \tag{4b}$$

where u_n and w_n are the unknown velocity coefficients that should be solved later.

We then substitute Eq. (4) into Eq. (1) and Eq. (3b) to eliminate w_n $(n = 0,1,\ldots,K)$. We substitute Eq. (4) into Eq. (2), multiply each term by z^n and integrating from $-d$ to η and eliminate pressure terms. We finally obtain the HLGN equations. The unknowns are u_n $(n = 0,1,\ldots,K-1)$ and η. We refer the reader to Webster et al. [6] for more details.

In this paper, we consider a solitary wave with amplitude H propagating from left to right with a constant speed c. The background linear-type currents, quadratic-type currents and cubic-type currents are considered. Sketch of the physical problem is shown in

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Fig. 1.

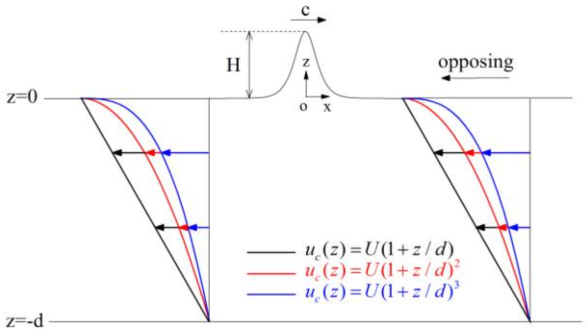

Fig.1 Sketch of a solitary wave propagating in shear currents

We note that for these three type currents, the current velocity at the bottom is $u_c(-d)=0$ and it is $u_c(0)=U$ at the still-water level. Fig.2 shows the velocity profiles of the three type currents for the case that $d=1m$ and $U=-1m/s$.

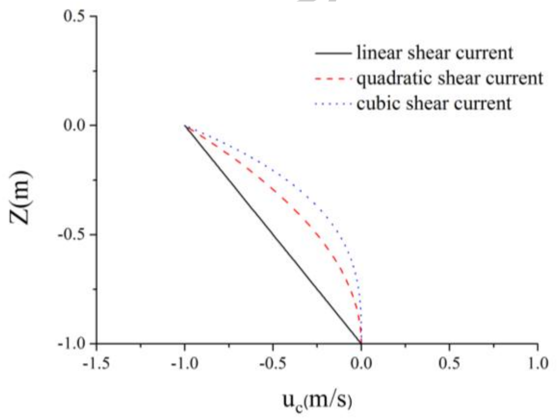

Fig.2 Velocity profile of different type shear currents, $d=1m$ and $U=-1m/s$

We solve this physical problem in wave coordinates, where $X=x-ct$ and $Z=z$. The method to solve the steady problem can be found in Duan et al. [5].

Some boundary conditions should be considered to solve u_n $(n=0,1,...,K\text{-}1)$ and η. If we set the solitary-wave crest at $X=0$, based on the symmetry characteristics, surface elevation η and velocity coefficients u_n are

$$\frac{\partial\eta}{\partial X}=0, \qquad (X=0) \tag{5a}$$

$$\frac{\partial u_n}{\partial X}=0, \qquad (n=0,1,...,K-1,\ X=0) \tag{5b}$$

When $X\to\infty$, since there is no wave-current interaction, the surface elevation $\eta=0$. Meanwhile, the boundary conditions should describe the shear currents exactly. Thus, for the quadratic shear currents discussed in the present study, we have

$$\begin{gathered} u_0=U-c,\ u_1=2U/d,\ u_2=U/d^2, \\ u_n=0 \quad (n=3,...,K-1,\ X\to\infty), \end{gathered} \tag{6}$$

and for the cubic shear currents, we have

$$\begin{gathered} u_0=U-c,\ u_1=3U/d,\ u_2=3U/d^2, \\ u_3=U/d^3,\ u_n=0 \quad (n=4,...,K-1,\ X\to\infty). \end{gathered} \tag{7}$$

The Newton-Raphson method is used to obtain the travelling solution. The solitary-wave solution with no currents of the HLGN model is used as the initial values, see Zhao et al. [7]. Then we increase U from 0 to the desired value we specify gradually.

Next, we will show the numerical results of the steady solutions of the solitary wave in nonuniform shear currents. We nondimensionalize all the parameters by g and d. The bar over the following quantities means they are dimensionless.

We firstly consider a weakly nonlinear case, where the solitary-wave amplitude is $\bar{H}=0.1$. For the quadratic shear-current case, Pak and Chow[4] obtain the third-order solution of the wave profile when $\bar{U}=-1.2$ .The comparison between the HLGN results and the third-order solution is shown in Fig. 3. We find that the two results show very good agreement. This is expected physically because the wave amplitude is not high.

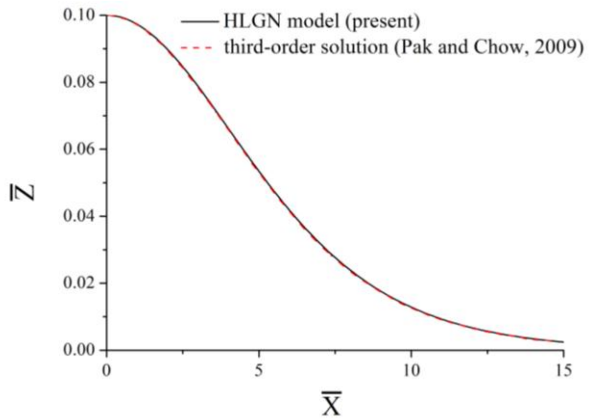

Fig.3 Solitary-wave profile for the quadratic-current case, $\bar{H}=0.1$, $\bar{U}=-1.2$

We also obtain the relationship between the wave speed $\bar{c}$ and current strength $\bar{U}$ for the quadratic-current case for the solitary wave of amplitude $\bar{H}=0.1$ by use of the HLGN model. Good agreement is found between the HLGN results and the third-order solution. Figure is not shown here for the sake of brevity.

Next, we consider a strongly nonlinear solitary wave with $\bar{H}=0.5$. Variations of the wave speed $\bar{c}$ with the current strength $\bar{U}$ is shown in Fig. 4. From Fig. 4, we find that as the opposing-current strength increases, the solitary-wave speed decreases. Meanwhile, for a fixed current strength, the solitary wave in the cubic shear currents travels faster than that for other two types of shear currents. Solitary wave in linear shear currents has the lowest wave speed.

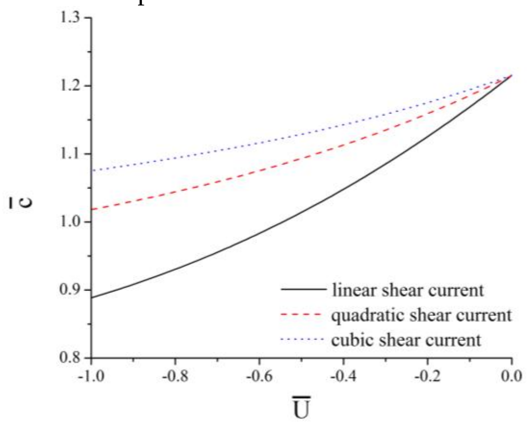

Fig.4 Relationship between the wave speed $\bar{c}$ and current strength $\bar{U}$ for the solitary wave with $\bar{H}=0.5$

In particular, we focus on solitary waves with $\bar{H}=0.5$ in the presence of strong opposing shear currents when $\bar{U}=-1$. The wave profile, velocity field, particle trajectories and vorticity distribution are studied.

The solitary-wave profiles are shown in Fig. 5. From Fig. 5, we see that the solitary-wave profiles are much wider for the opposing-current cases than that for the no-current case. Also, the solitary-wave profiles for the linear-current case, quadratic-current case and cubic-current case show very little differences in Fig. 5.

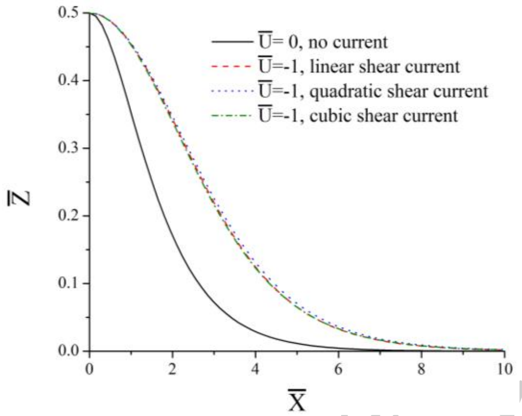

Fig.5 Solitary-wave profiles, $\bar{H}=0.5$

Next, we study the horizontal velocity along the water column at the wave crest. Results are shown in Fig. 6.

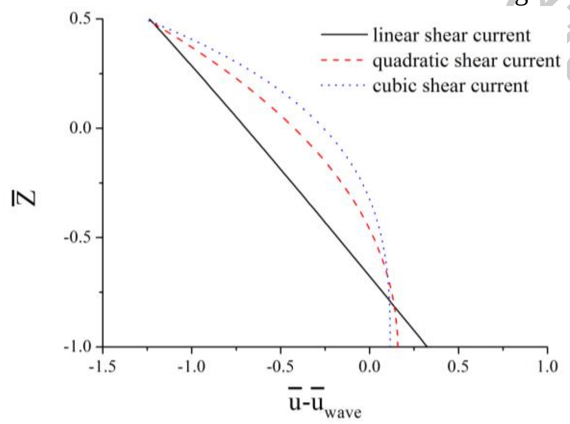

Fig. 6 Horizontal velocity along the water column at wave crest, $\bar{H}=0.5$, $\bar{U}=-1$

We mention that the initial current velocity distribution shown in Fig. 2 is $\bar{u}_{\text{current}}=-1$ at still water level. Fig. 6 shows that $\bar{u}-\bar{u}_{\text{wave}}=\bar{u}_{\text{current}}+\bar{u}_{\text{interaction}}$ at the wave crest are close to -1.2 because of the wave-current interaction. These values also show some differences at sea bottom.

The velocity field of solitary wave in shear currents are shown in Fig. 7, including the linear-current case in Fig. 7(a), quadratic-current case in Fig. 7(b) and cubic-current case in Fig. 7(c). We find an obvious vortex in each case. In the linear-current case, the vertical position of vortex is near $\bar{Z}=-0.34$, while it is near $\bar{Z}=0$ and $\bar{Z}=0.13$ for the quadratic-current case and cubic-current case. Note that there is no assumption of irrotationality in the present theory.

Results of the particle-trajectory calculations on three different vertical positions ($\bar{Z}=0$, -0.2, -0.5) are shown in Fig. 8. For each case, we have aligned the maximum displacements at the same horizontal position $\bar{X}=0$. We find that for the particle that initially lies at the free surface $\bar{Z}=0$, the trajectories are similar in these three cases. For the particle that initially lies at $\bar{Z}=-0.2$, some differences are found between these three cases, among which the shape of trajectory is steepest for the cubic-current case. When we consider a particle initially lies at half of the water depth $\bar{Z}=-0.5$, some obvious differences are found. For the quadratic-current case and cubic-current case, a closed hoop appears in the particle trajectories. Meanwhile, the hoop is wider for the cubic-current case than that for the quadratic-current case. As far as the maximum vertical displacement, it is largest for the linear-current case and it is smallest for the cubic-current case.

At last, we consider the vorticity field. Vorticity is written as

$$\omega=\frac{\partial w}{\partial x}-\frac{\partial u}{\partial z}. \tag{8}$$

The vorticity along the water column at different horizontal positions ($\bar{X}=0$, 3, 6, 40) is shown in Fig. 9. The solitary wave and linear shear current interaction will not change the vorticity field as shown in Fig. 8(a). However, the solitary wave and nonlinear shear-current interaction significantly affects the vorticity distribution. For example, at $\bar{X}=0$ in Fig. 8(b), the vorticity changes nonlinearly. As $\bar{X}$ increases, which means the spatial position is further from the wave crest and nearer to the position with no wave-current interaction, the vorticity changes more linearly. At the position $\bar{X}=40$, since there is almost no wave-current interaction, the vorticity changes linearly. For the cubic-current case shown in Fig. 9(c), we find that the vorticity changes rapidly near the wave crest due to wave-current interaction. Moreover, at all the positions, the vorticity changes from 0 to 2 for the quadratic-current case and changes from 0 to 3 for the cubic-current case.

At last, the vorticity field of the solitary wave in nonuniform shear currents is shown in Fig. 10.

In this paper, we focus on a strongly nonlinear solitary wave with $\bar{H}=0.5$ propagating in the presence of opposing nonuniform shear currents with $\bar{U}=-1$. The linear-type currents, quadratic-type currents and cubic-type currents are studied. Conclusions are reached below:

(1) The solitary wave and opposing-current interaction will significantly affects the velocity field, see Fig. 6.

(2) Different type of shear currents will significantly affects the particle trajectories, see Fig. 8.

(3) Solitary wave and nonuniform currents interaction changes the vorticity distribution, see Fig. 9 and Fig. 10.

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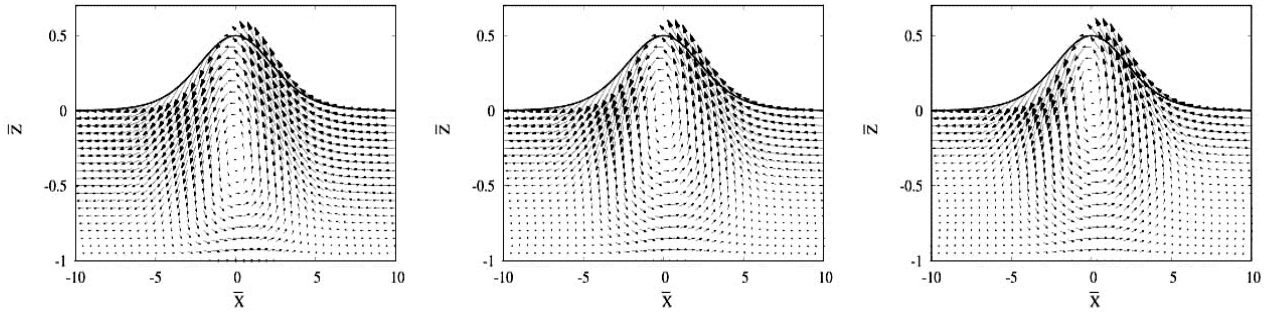
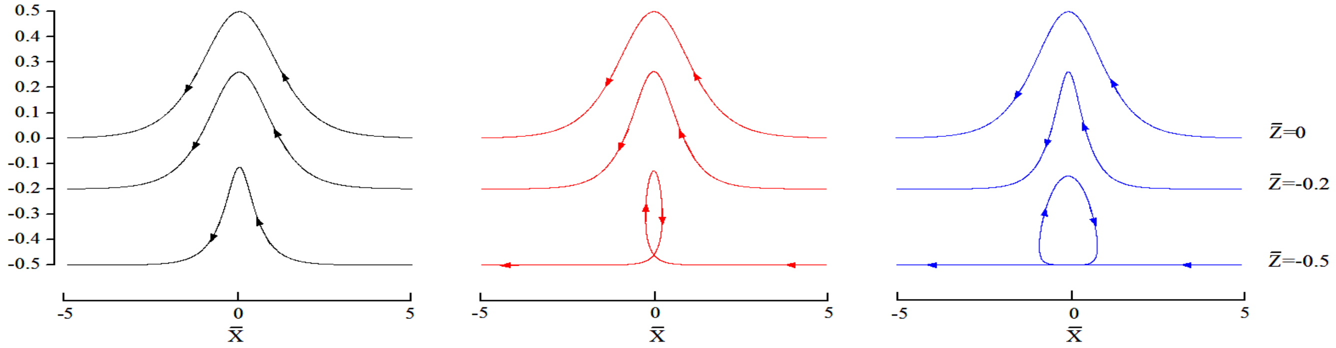


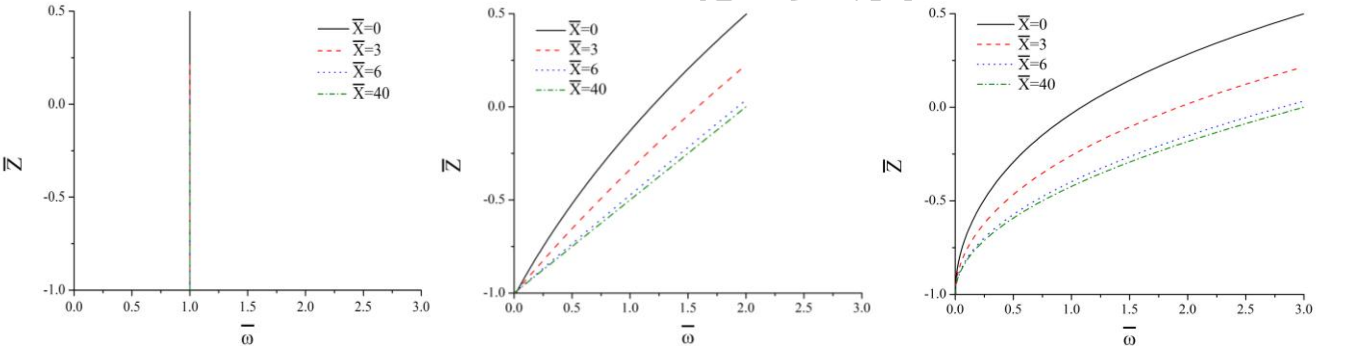
(a) linear-current case (b) quadratic-current case (c) cubic-current case

Fig. 7 Velocity field of the solitary wave in shear currents, $\bar{H}=0.5$, $\bar{U}=-1$



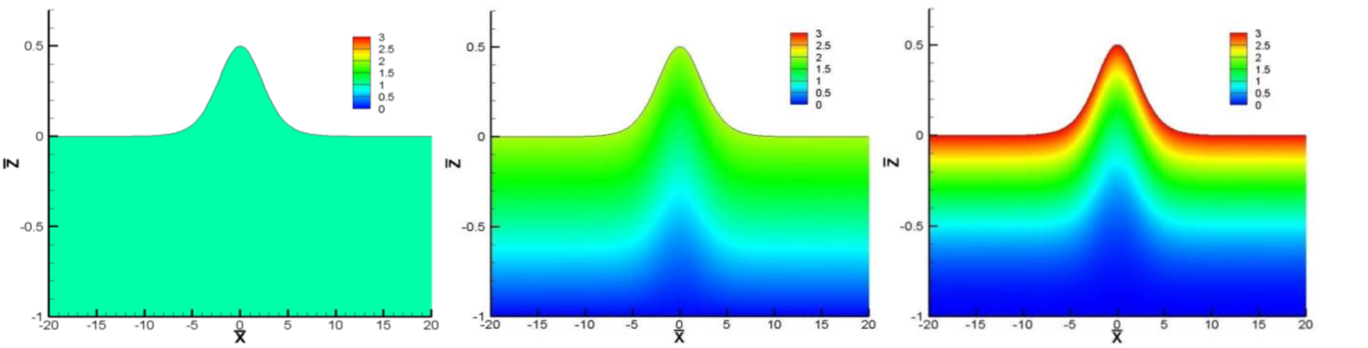
(a) linear-current case (b) quadratic-current case (c) cubic-current case

Fig. 8 Particle trajectories at different vertical positions, $\bar{H}=0.5$, $\bar{U}=-1$



(a) linear-current case (b) quadratic-current case (c) cubic-current case

Fig. 9 Vorticity distribution along the water column at different horizontal positions, $\bar{H}=0.5$, $\bar{U}=-1$



(a) linear-current case (b) quadratic-current case (c) cubic-current case

Fig. 10 Vorticity field, $\bar{H}=0.5$, $\bar{U}=-1$