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LOCALLY PERIODIC UNFOLDING METHOD AND TWO-SCALE CONVERGENCE ON SURFACES OF LOCALLY PERIODIC MICROSTRUCTURES*

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Abstract. In this paper we generalize the periodic unfolding method and the notion of two-scale convergence on surfaces of periodic microstructures to locally periodic situations. The methods that we introduce allow us to consider a wide range of nonperiodic microstructures, especially to derive macroscopic equations for problems posed in domains with perforations distributed nonperiodically. Using the methods of locally periodic two-scale convergence on oscillating surfaces and the locally periodic boundary unfolding operator, we are able to analyze differential equations defined on boundaries of nonperiodic microstructures and consider nonhomogeneous Neumann conditions on the boundaries of perforations, distributed nonperiodically.

Key words. unfolding method, two-scale convergence, locally periodic homogenization, nonperiodic microstructures, signalling processes

AMS subject classifications. 35B27, 35D30, 35Kxx

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1. Introduction.

Many natural and man-made composite materials comprise nonperiodic microscopic structures, e.g., fibrous microstructures in heart muscles [23, 45], exoskeletons [27], industrial filters [49], or space-dependent perforations in concrete [47]. An important special case of nonperiodic microstructures is that of the so-called locally periodic (l-p) microstructures, where spatial changes are observed on a scale smaller than the size of the domain under consideration, but larger than the characteristic size of the microstructure. For many l-p microstructures spatial changes cannot be represented by periodic functions depending on slow and fast variables, e.g., plywood-like structures of gradually rotated planes of parallel aligned fibers [12]. Thus, in these situations the standard two-scale convergence and periodic unfolding method cannot be applied. Hence, for a multiscale analysis of problems posed in domains with nonperiodic perforations, in this paper we extend the periodic unfolding method and two-scale convergence on oscillating surfaces to locally periodic situations (see Definitions 3.2–3.5). These generalizations are motivated by the locally periodic two-scale (l-t-s) convergence introduced in [46].

Two-scale convergence on surfaces of periodic microstructures was first introduced in [5, 41]. An extension of two-scale convergence associated with a fixed periodic Borel measure was considered in [53]. The unfolding operator maps functions defined on perforated domains, depending on small parameter ε, onto functions defined on the whole fixed domain; see [19, 18] and references therein. This helps to overcome one of the difficulties of perforated domains, which is the use of extension operators. Using the boundary unfolding operator we can prove convergence results for nonlinear equations posed on oscillating boundaries of microstructures [15, 18, 20, 34, 43]. The unfolding method is also an efficient tool to derive error estimates; see, e.g., [28, 29, 30, 31, 44].

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The main novelty of this paper is the derivation of new techniques for the multiscale analysis of nonlinear problems posed in domains with nonperiodic perforations and on the surfaces of nonperiodic microstructures. The l-p unfolding operator allows us to analyze nonlinear differential equations posed on domains with nonperiodic perforations. The l-t-s convergence on oscillating surfaces and the l-p boundary unfolding operator allow us to show strong convergence for sequences defined on oscillating boundaries of nonperiodic microstructures and to derive macroscopic equations for nonlinear equations defined on boundaries of nonperiodic microstructures. Until now, this was not possible using existing methods.

The paper is structured as follows. First, in section 2, we present a mathematical description of l-p microstructures and state the definition of an l-p approximation for a function $\psi \in C(\overline{\Omega}; C_{\rm per}(Y_x))$. In section 3 we introduce all of the main definitions of the paper, i.e., the notion of an l-p unfolding operator, two-scale convergence for sequences defined on oscillating boundaries of l-p microstructures, and the l-p boundary unfolding operator. The main results are summarized in section 4. The central results of this paper are convergence results for sequences bounded in L^p and $W^{1,p}$, with $p \in (1, \infty)$ (see Theorems 4.1–4.4). The proofs of the main results for the l-p unfolding operator are presented in section 5. The properties of the decomposition of a $W^{1,p}$-function, with one part describing the macroscopic behavior and another part of order ε, are shown in section 6. The proofs of the main results for the l-p unfolding operator in perforated domains are given in section 7. The convergence results for l-t-s convergence on oscillating surfaces and the l-p boundary unfolding operator are proved in section 8. In section 9 we apply the l-p unfolding operator to derive macroscopic problems for microscopic models of signaling processes in cell tissues comprising l-p microstructures. As examples of tissues with l-p microstructures we consider plant tissues, epithelial tissues, and nonperiodic fibrous structure of heart tissue. Finally, section 10 contains some concluding remarks.

There are some existing results on the homogenization of problems posed on l-p media. The homogenization of a heat-conductivity problem defined in domains with nonperiodic microstructure consisting of spherical balls was studied in [13] using the Murat–Tartar H-convergence method [40], and in [3] by applying the θ-2 convergence. The nonperiodic distribution of balls is given by a C^2-diffeomorphism θ, transforming the centers of the balls. Estimates for a numerical approximation of this problem were derived in [50]. The notion of a Young measure was used in [37] to extend the concept of periodic two-scale convergence and to define the so-called *scale convergence*. The definition of scale convergence was motivated by the derivation of the Γ-limit for a sequence of nonlinear energy functionals involving nonperiodic oscillations. Formal asymptotic expansions and the technique of two-scale convergence defined for periodic test functions (see, e.g., [4, 42]) were used to derive macroscopic equations for models posed on domains with l-p perforations, i.e., domains consisting of periodic cells with smoothly changing perforations [9, 16, 17, 35, 36, 51]. The H-convergence method [11, 12], the asymptotic expansion method [8], and the method of l-t-s convergence [46] were applied to analyze microscopic models posed on domains consisting of nonperiodic fibrous materials. The optimization of the elastic properties of a material with l-p microstructure was considered in [6, 7].

To illustrate the difference between the formulation of nonperiodic microstructure by using periodic functions and the l-p formulation of the problem, we consider a plywood-like structure, given as the superposition of gradually rotated planes of aligned parallel fibers. We consider layers of cylindrical fibers of radius εa orthogonal to the x_3-axis and rotated around the x_3-axis by an angle γ, constant in each layer

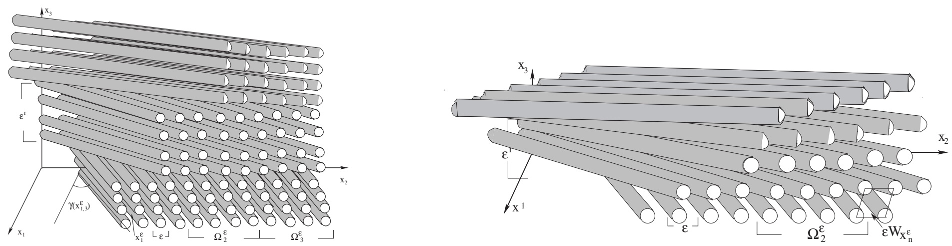

FIG. 1. *Schematic representation of slow rotating and fast rotating plywood-like structures.*

and changing from one layer to another; see Figure 1. To describe the difference in the material properties of fibers and the interfiber space with the help of a periodic function, we define a function

$$A^\varepsilon(x) = A_1 \tilde{\eta}\big(R(\gamma(x_3))x/\varepsilon\big) + A_2\Big[1 - \tilde{\eta}\big(R(\gamma(x_3))x/\varepsilon\big)\Big], \tag{1.1}$$

where A_1, A_2 are constant tensors and $\tilde{\eta}$ denotes the characteristic functions of a fiber of radius a in the direction of the x_1-axis, i.e.,

$$\tilde{\eta}(y) = \begin{cases} 1 & \text{for} \quad |\hat{y} - (1/2, 1/2)| \leq a, \\ 0 & \text{for} \quad |\hat{y} - (1/2, 1/2)| > a, \end{cases} \tag{1.2}$$

and extended $\hat{Y}$-periodic to the whole $\mathbb{R}^3$, with $a < 1/2$, $\hat{y} = (y_2, y_3)$, $Y = [0,1]^3$, and $\hat{Y} = [0,1]^2$. The inverse of the rotation matrix around the x_3-axis with rotation angle α with the x_1-axis is defined as

$$R(\alpha) = \begin{pmatrix} \cos(\alpha) & \sin(\alpha) & 0 \\ -\sin(\alpha) & \cos(\alpha) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{1.3}$$

and $\gamma \in C^1(\mathbb{R})$ is a given function such that $0 \leq \gamma(s) \leq \pi$ for all $s \in \mathbb{R}$. Then, considering, for example, an elliptic problem with a diffusion coefficient or elasticity tensor in the form (1.1) and using a change of variables $\tilde{x} = R(\gamma(x_3))x$, we can apply periodic homogenization techniques to derive corresponding macroscopic equations (see [10, 11] for details). However, in the representation of the microscopic structure by (1.1), every point of a fiber is rotated differently, and the cylindrical structure of the fibers is deformed. Hence, A^ε represent the properties of a material with a microstructure different from the plywood-like structure, and for a correct representation of a plywood-like structure, an l-p formulation of the microscopic problem is essential. Also, applying periodic homogenization techniques, we obtain effective macroscopic coefficients different from the one obtained by using methods of l-p homogenization (see [12, 46] for more details).

To define the characteristic function of the domain occupied by fibers in a domain with an l-p plywood-like structure, we divide $\mathbb{R}^3$ into layers $L_k^\varepsilon = \mathbb{R}^2 \times ((k-1)\varepsilon^r, k\varepsilon^r)$ of height ε^r and perpendicular to the x_3-axis, where $k \in \mathbb{Z}$ and $0 < r < 1$. In each L_k^ε we choose an arbitrary fixed point $x_k^\varepsilon \in L_k^\varepsilon$. Using the l-p approximation of $\eta \in C(\overline{\Omega}, L^\infty_{\text{per}}(Y_x))$, with $\eta(x,y) = \tilde{\eta}(R(x)y)$ for $x \in \Omega$ and $y \in Y_x$, given by

$$(\mathcal{L}^\varepsilon \eta)(x) = \sum_{k \in \mathbb{Z}} \tilde{\eta}\left(R(\gamma(x^\varepsilon_{k,3}))\, x/\varepsilon\right) \chi_{L_k^\varepsilon}(x) \qquad \text{for } x \in \Omega,$$

the characteristic function of the domain occupied by fibers is given by

$$\chi_{\Omega_f^\varepsilon}(x) = \chi_\Omega(x)(\mathcal{L}^\varepsilon \eta)(x). \tag{1.4}$$

Here $\tilde\eta \in L^\infty_{\rm per}(Y)$ is as in (1.2), and $Y_x = R^{-1}(\gamma(x_3))Y$. For a microstructure composed of fast rotating planes of parallel aligned fibers (see Figure 1), we consider an approximation by an l-p plywood-like structure with shifted periodicity $D(x)Y = R^{-1}(x)W(x)Y$; see [12, 46] for more details.

2. Locally periodic microstructures and locally periodic perforated domains. In this section we give a mathematical formulation of l-p microstructures. We also define the approximation of functions, where the periodicity with respect to the fast variable is dependent on the slow variable, by l-p functions, i.e., periodic in subdomains smaller than the domain under consideration but larger than the representative size of the microstructure.

Let $\Omega \subset \mathbb{R}^d$ be a bounded Lipschitz domain. For each $x \in \mathbb{R}^d$ we consider a transformation matrix $D(x) \in \mathbb{R}^{d\times d}$ and its inverse $D^{-1}(x)$ such that $D, D^{-1} \in \mathrm{Lip}(\mathbb{R}^d;\mathbb{R}^{d\times d})$ and $0 < D_1 \le |\det D(x)| \le D_2 < \infty$ for all $x \in \overline{\Omega}$. We consider the continuous family of parallelepipeds $Y_x = D_x Y$ on $\overline{\Omega}$, where $Y = (0,1)^d$ is the "unit cell," and denote $D_x := D(x)$ and $D_x^{-1} := D^{-1}(x)$.

For $\varepsilon > 0$, in a manner similar to that of [13, 46], we consider the partition covering of Ω by a family of open nonintersecting cubes $\{\Omega_n^\varepsilon\}_{1\le n\le N_\varepsilon}$ of side ε^r, with $0 < r < 1$:

$$\Omega \subset \bigcup_{n=1}^{N_\varepsilon} \overline{\Omega_n^\varepsilon} \qquad \text{and} \qquad \Omega_n^\varepsilon \cap \Omega \neq \emptyset.$$

For arbitrary chosen fixed points $x_n^\varepsilon, \tilde x_n^\varepsilon \in \Omega_n^\varepsilon \cap \Omega$ we consider a covering of Ω_n^ε by parallelepipeds $\varepsilon D_{x_n^\varepsilon} Y$:

$$\Omega_n^\varepsilon \subset \tilde x_n^\varepsilon + \bigcup_{\xi\in\Xi_n^\varepsilon} \varepsilon D_{x_n^\varepsilon}(\overline{Y}+\xi), \quad \text{where } \Xi_n^\varepsilon = \{\xi \in \mathbb{Z}^d : \ \tilde x_n^\varepsilon + \varepsilon D_{x_n^\varepsilon}(Y+\xi)\cap\Omega_n^\varepsilon \neq \emptyset\},$$

with $D_{x_n^\varepsilon} = D(x_n^\varepsilon)$ and $1 \le n \le N_\varepsilon$. For each $n = 1,\dots,N_\varepsilon$, $\tilde x_n^\varepsilon$ is a fixed shift in the representation of the microscopic structure of Ω_n^ε. Often we can consider $\tilde x_n^\varepsilon = \varepsilon D_{x_n^\varepsilon}\xi$ for some $\xi \in \mathbb{Z}^d$.

We consider the space $C(\overline{\Omega}; C_{\rm per}(Y_x))$ given in a standard way; i.e., for any $\widetilde\psi \in C(\overline{\Omega}; C_{\rm per}(Y))$ the relation $\psi(x,y) = \widetilde\psi(x, D_x^{-1}y)$ with $x \in \Omega$ and $y \in Y_x$ yields $\psi \in C(\overline{\Omega}; C_{\rm per}(Y_x))$. In the same way the spaces $L^p(\Omega; C_{\rm per}(Y_x))$, $L^p(\Omega; L^q_{\rm per}(Y_x))$, and $C(\overline{\Omega}; L^q_{\rm per}(Y_x))$, for $1 \le p \le \infty$, $1 \le q < \infty$, are defined.

To describe l-p microscopic properties of a composite material and to specify test functions associated with the l-p microstructure of a material, as well as for the definition of the l-t-s convergence, we shall consider an l-p approximation of functions with space-dependent periodicity, i.e., of functions in $C(\overline{\Omega}; C_{\rm per}(Y_x))$, $L^p(\Omega; C_{\rm per}(Y_x))$, or $C(\overline{\Omega}; L^q_{\rm per}(Y_x))$. Locally periodic approximated functions are $Y_{x_n^\varepsilon}$-periodic in each subdomain Ω_n^ε, with $n = 1,\dots,N_\varepsilon$, and are related to test functions associated with the periodic structure of Ω_n^ε. Since the microscopic structure of Ω_n^ε is represented by a union of periodicity cells $\varepsilon Y_{x_n^\varepsilon}$ shifted by a fixed point $\tilde x_n^\varepsilon \in \Omega_n^\varepsilon \cap \Omega$, with $n = 1,\dots,N_\varepsilon$, this shift is also reflected in the definition of the l-p approximation.

Often coefficients in a microscopic model posed in a domain with an l-p microstructure depend only on the microscopic fast variables x/ε and the points $x_n^\varepsilon, \tilde x_n^\varepsilon \in$

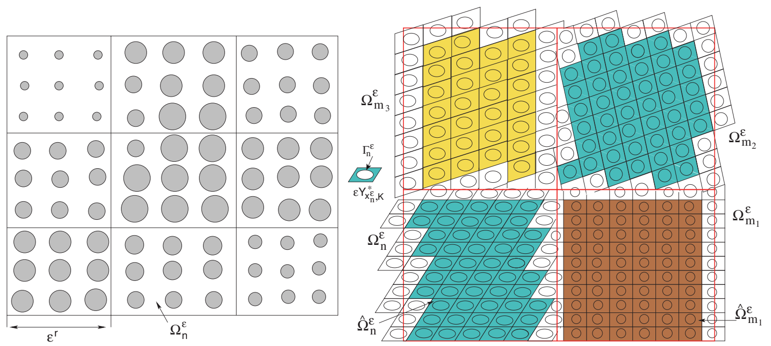

FIG. 2. *Schematic representation of subdomains* Ω_n^ε *and* $\hat\Omega_n^\varepsilon$.

$\Omega_n^\varepsilon \cap \Omega$, describing the periodic microstructure in each Ω_n^ε, with $n = 1, \dots, N_\varepsilon$, and are independent of the macroscopic slow variables x. To define such functions we shall introduce the notion of an l-p approximation $\mathcal{L}_0^\varepsilon$ of a function $\psi \in C(\overline{\Omega}; C_{\rm per}(Y_x))$ (or in $L^p(\Omega; C_{\rm per}(Y_x))$, $C(\overline{\Omega}; L^q_{\rm per}(Y_x))$). In each Ω_n^ε the function $\mathcal{L}_0^\varepsilon(\psi)$ is $Y_{x_n^\varepsilon}$-periodic and depends only on the fast variables x/ε. This specific l-p approximation is important for the derivation of macroscopic equations for a microscopic problem with coefficients discontinuous with respect to the fast variables, since for $\psi \in C(\overline{\Omega}; L^p(Y_x))$ we have that $\mathcal{L}_0^\varepsilon(\psi)$ converges strongly l-t-s; see [46].

As an l-p approximation of ψ we name $\mathcal{L}^\varepsilon : C(\overline{\Omega}; C_{\rm per}(Y_x)) \to L^\infty(\Omega)$ given by

$$(\mathcal{L}^\varepsilon \psi)(x) = \sum_{n=1}^{N_\varepsilon} \widetilde{\psi}\left(x, \frac{D^{-1}_{x_n^\varepsilon}(x - \tilde{x}_n^\varepsilon)}{\varepsilon}\right) \chi_{\Omega_n^\varepsilon}(x) \quad \text{ for } x \in \Omega. \tag{2.1}$$

We consider also the map $\mathcal{L}_0^\varepsilon : C(\overline{\Omega}; C_{\rm per}(Y_x)) \to L^\infty(\Omega)$ defined for $x \in \Omega$ as

$$(\mathcal{L}_0^\varepsilon \psi)(x) = \sum_{n=1}^{N_\varepsilon} \psi\Big(x_n^\varepsilon, \frac{x - \tilde{x}_n^\varepsilon}{\varepsilon}\Big) \chi_{\Omega_n^\varepsilon}(x) = \sum_{n=1}^{N_\varepsilon} \widetilde{\psi}\left(x_n^\varepsilon, \frac{D^{-1}_{x_n^\varepsilon}(x - \tilde{x}_n^\varepsilon)}{\varepsilon}\right) \chi_{\Omega_n^\varepsilon}(x).$$

If we choose $\tilde{x}_n^\varepsilon = D_{x_n^\varepsilon} \varepsilon \xi$ for some $\xi \in \mathbb{Z}^d$, then the periodicity of $\widetilde{\psi}$ implies

$$(\mathcal{L}^\varepsilon \psi)(x) = \sum_{n=1}^{N_\varepsilon} \widetilde{\psi}\left(x, \frac{D^{-1}_{x_n^\varepsilon} x}{\varepsilon}\right) \chi_{\Omega_n^\varepsilon}(x) \;\; \text{ and } \;\; (\mathcal{L}_0^\varepsilon \psi)(x) = \sum_{n=1}^{N_\varepsilon} \widetilde{\psi}\left(x_n^\varepsilon, \frac{D^{-1}_{x_n^\varepsilon} x}{\varepsilon}\right) \chi_{\Omega_n^\varepsilon}(x)$$

for $x \in \Omega$.

In the following, we shall consider the case $\tilde{x}_n^\varepsilon = \varepsilon D_{x_n^\varepsilon} \xi$, with $\xi \in \mathbb{Z}^d$. However, all results hold for arbitrary chosen $\tilde{x}_n^\varepsilon \in \Omega_n^\varepsilon$ with $n = 1, \dots, N_\varepsilon$; see [46]. In a similar way we define $\mathcal{L}^\varepsilon \psi$ and $\mathcal{L}_0^\varepsilon \psi$ for ψ in $C(\overline{\Omega}; L^q_{\rm per}(Y_x))$ or $L^p(\Omega; C_{\rm per}(Y_x))$.

The l-p approximation reflects the microscopic properties of Ω, where in each Ω_n^ε the microstructure is represented by a "unit cell" $Y_{x_n^\varepsilon} = D_{x_n^\varepsilon} Y$ for an arbitrary fixed $x_n^\varepsilon \in \Omega_n^\varepsilon$; see Figures 1 and 2.

In the context of admissible test functions in a weak formulation of partial differential equations, we define a regular approximation of $\mathcal{L}^\varepsilon \psi$ by

$$(\mathcal{L}_\rho^\varepsilon \psi)(x) = \sum_{n=1}^{N_\varepsilon} \widetilde{\psi}\left(x, \frac{D^{-1}_{x_n^\varepsilon} x}{\varepsilon}\right) \phi_{\Omega_n^\varepsilon}(x) \quad \text{ for } x \in \Omega,$$

where $\phi_{\Omega_n^\varepsilon}$ are approximations of $\chi_{\Omega_n^\varepsilon}$ such that $\phi_{\Omega_n^\varepsilon} \in C_0^\infty(\Omega_n^\varepsilon)$ and

$$\sum_{n=1}^{N_\varepsilon} |\phi_{\Omega_n^\varepsilon} - \chi_{\Omega_n^\varepsilon}| \to 0 \text{ in } L^2(\Omega), \ ||\nabla^m \phi_{\Omega_n^\varepsilon}||_{L^\infty(\mathbb{R}^d)} \le C\varepsilon^{-\rho m} \text{ for } 0 < r < \rho < 1; \tag{2.2}$$

see, e.g., [11, 13, 46]. In the definition of the l-p unfolding operator we shall use subdomains of Ω_n^ε given by unit cells $\varepsilon D_{x_n^\varepsilon}(Y+\xi)$ that are completely included in $\Omega_n^\varepsilon \cap \Omega$ (see Figure 2):

$$\hat{\Omega}^\varepsilon = \bigcup_{n=1}^{N_\varepsilon} \hat{\Omega}_n^\varepsilon, \text{ with } \hat{\Omega}_n^\varepsilon = \mathrm{Int}\Bigg(\bigcup_{\xi \in \hat{\Xi}_n^\varepsilon} \varepsilon D_{x_n^\varepsilon}(\overline{Y}+\xi) \Bigg) \text{ and } \Lambda^\varepsilon = \bigcup_{n=1}^{N_\varepsilon} \Lambda_n^\varepsilon \cap \Omega, \tag{2.3}$$

where $\Lambda_n^\varepsilon = \Omega_n^\varepsilon \setminus \hat{\Omega}_n^\varepsilon$ and $\hat{\Xi}_n^\varepsilon = \{\xi \in \Xi_n^\varepsilon \ : \ \varepsilon D_{x_n^\varepsilon}(Y+\xi) \subset (\Omega_n^\varepsilon \cap \Omega)\}$.

As is known from the periodic case, the unfolding operator provides a powerful technique for the multiscale analysis of problems posed in perforated domains and nonlinear equations defined on oscillating surfaces of microstructures. Thus, the main emphasis of this paper will be on the development of the unfolding method for domains with l-p perforations. Therefore, next we introduce perforated domains with l-p changes in the distribution and in the shape of perforations.

We consider $Y_0 \subset Y$ with a Lipschitz boundary $\Gamma = \partial Y_0$ and a matrix K with $K, K^{-1} \in \mathrm{Lip}(\mathbb{R}^d; \mathbb{R}^{d\times d})$, where $0 < K_1 \le |\det K(x)| \le K_2 < \infty$, $K_x Y_0 \subset Y$, and $Y^* = Y \setminus \overline{Y}_0$ and $\widetilde{Y}^*_{K_x} = Y \setminus K_x \overline{Y}_0$ are connected for all $x \in \overline{\Omega}$. Define $Y^*_{x,K} = D_x \widetilde{Y}^*_{K_x}$ with the boundary $\Gamma_x = D_x K_x \Gamma$, where $K_x = K(x)$ and $D_x = D(x)$. Then, a domain with l-p perforations is defined as

$$\Omega^*_{\varepsilon,K} = \mathrm{Int}\left(\bigcup_{n=1}^{N_\varepsilon} \Omega^{*,\varepsilon}_{n,K} \right) \cap \Omega, \quad \text{ where } \quad \Omega^{*,\varepsilon}_{n,K} = \bigcup_{\xi \in \Xi_n^{*,\varepsilon}} \varepsilon D_{x_n^\varepsilon} \left(\overline{\widetilde{Y}^*_{K_{x_n^\varepsilon}}} + \xi \right) \cup \overline{\Lambda_n^{*,\varepsilon}}.$$

Here $\Lambda_n^{*,\varepsilon} = \Omega_n^\varepsilon \setminus \bigcup_{\xi \in \Xi_n^{*,\varepsilon}} \varepsilon D_{x_n^\varepsilon}(\overline{Y}+\xi)$, with $\Xi_n^{*,\varepsilon} = \{\xi \in \Xi_n^\varepsilon \ : \ \varepsilon D_{x_n^\varepsilon}(Y+\xi) \subset \Omega_n^\varepsilon\}$, $\widetilde{Y}^*_{K_{x_n^\varepsilon}} = Y \setminus K_{x_n^\varepsilon} \overline{Y}_0$, and $K_{x_n^\varepsilon} = K(x_n^\varepsilon)$ for $n = 1, \dots, N_\varepsilon$ and $x_n^\varepsilon \in \hat{\Omega}_n^\varepsilon$. The boundaries of the l-p microstructure of $\Omega^*_{\varepsilon,K}$ are denoted by

$$\Gamma^\varepsilon = \bigcup_{n=1}^{N_\varepsilon} \Gamma_n^\varepsilon, \quad \text{where} \quad \Gamma_n^\varepsilon = \bigcup_{\xi \in \Xi_n^{*,\varepsilon}} \varepsilon D_{x_n^\varepsilon} (\widetilde{\Gamma}_{K_{x_n^\varepsilon}} + \xi) \cap \Omega \quad \text{with} \quad \widetilde{\Gamma}_{K_{x_n^\varepsilon}} = K_{x_n^\varepsilon} \Gamma.$$

Notice that changes in the microstructure of $\Omega^*_{\varepsilon,K}$ are defined by changes in the periodicity given by $D(x)$ and additional changes in the shape of perforations described by $K(x)$ for $x \in \Omega$.

Along with plywood-like structures (see Figure 1), examples of l-p microstructures are, e.g., concrete materials with space-dependent perforations, and plant and epithelial tissues; see Figure 3. In the definition of microstructure of concrete materials with space-dependent perforations we have, e.g., $D(x) = \mathbf{I}$ and $K(x) = \rho(x)\mathbf{I}$ for $0 < \rho_1 \le \rho(x) \le \rho_2 < \infty$ such that $K(x)\overline{Y}_0 \subset Y$, where $\mathbf{I}$ denotes the identity matrix; see, e.g., [16, 51] and Figure 2. For plant or epithelial tissues additionally we have space-dependent deformations of cells given by $D(x) \neq \mathbf{I}$; see Figure 3.

Using the mathematical definition of general l-p microstructures, next we introduce the definition of the l-p unfolding operator, mapping functions defined on ε-dependent domains to functions depending on two variables (i.e., a microscopic variable and a macroscopic variable), but defined on fixed domains.

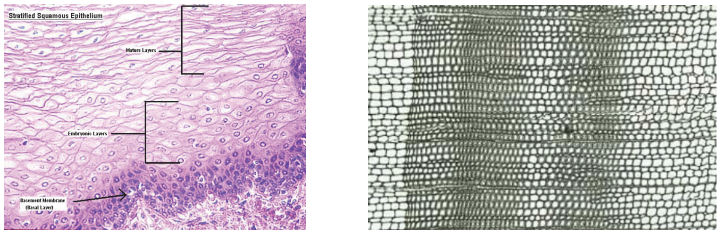

FIG. 3. *Examples of l-p microstructures with local changes in the shape and the periodicity of a microstructure. We observe changes in shape and size of cells in an epithelial tissue due to maturation (left) and changes in the size of plant cells in a wood tissue (right). Reproduced with permission from Anatomy & Physiology (http:// anatomyandphysiologyi.com) (left) and from W. Schoch, I. Heller, F. H. Schweingruber, and F. Kienast,* 2004 [48] *(right).*

3. Definitions of lp unfolding operator and l-t-s convergence on oscillating surfaces. The main idea of the two-scale convergence is to consider test functions which comprise the information about the microstructure and the microscopic properties of a composite material and of model equations. The same idea is used in the definition of l-t-s by considering an l-p approximation of $\psi \in L^q(\Omega; C_{\rm per}(Y_x))$ (reflecting the l-p properties of microscopic problems) as a test function.

DEFINITION 3.1 (see [46]). *Let $u^\varepsilon \in L^p(\Omega)$ for all $\varepsilon > 0$ and $p \in (1, +\infty)$. We say the sequence $\{u^\varepsilon\}$ converges l-t-s to $u \in L^p(\Omega; L^p(Y_x))$ as $\varepsilon \to 0$ if $\|u^\varepsilon\|_{L^p(\Omega)} \le C$ and for any $\psi \in L^q(\Omega; C_{\rm per}(Y_x))$*

$$\lim_{\varepsilon \to 0} \int_\Omega u^\varepsilon(x) \mathcal{L}^\varepsilon \psi(x) dx = \int_\Omega \fint_{Y_x} u(x, y) \psi(x, y) dy dx,$$

where $\mathcal{L}^\varepsilon$ is the l-p approximation of ψ, defined as in (2.1)*, and* $1/p + 1/q = 1$.

Remark. Notice that the definition of l-t-s and convergence results presented in [46] for $p = 2$ are directly generalized to $p \in (1, \infty)$.

Motivated by the notion of the periodic unfolding operator and l-t-s convergence, we define the l-p unfolding operator in the following way.

DEFINITION 3.2. *For any function ψ that is Lebesgue-measurable on Ω, the l-p unfolding operator $\mathcal{T}^\varepsilon_\mathcal{L}$ is defined as*

$$\mathcal{T}^\varepsilon_\mathcal{L}(\psi)(x, y) = \sum_{n=1}^{N_\varepsilon} \psi\big(\varepsilon D_{x_n^\varepsilon} \big[D_{x_n^\varepsilon}^{-1} x/\varepsilon\big]_Y + \varepsilon D_{x_n^\varepsilon} y\big) \chi_{\hat\Omega_n^\varepsilon}(x) \quad \text{for } \ x \in \Omega \text{ and } \ y \in Y.$$

The definition implies that $\mathcal{T}^\varepsilon_\mathcal{L}(\psi)$ is Lebesgue-measurable on $\Omega \times Y$ and is zero for $x \in \Lambda^\varepsilon$.

For perforated domains with local changes in the distribution of perforations, but without additional changes in the shape of perforations, i.e., $K = \mathbf{I}$ and

$$\Omega^*_\varepsilon = \operatorname{Int}\left(\bigcup_{n=1}^{N_\varepsilon} \Omega_n^{*,\varepsilon}\right) \cap \Omega, \qquad \text{where} \qquad \Omega_n^{*,\varepsilon} = \bigcup_{\xi \in \Xi_n^{*,\varepsilon}} \varepsilon D_{x_n^\varepsilon}(\overline{Y^*} + \xi) \cup \overline{\Lambda_n^{*,\varepsilon}}$$

and $Y^* = Y \setminus \overline{Y}_0$, we define the l-p unfolding operator in the following way.

DEFINITION 3.3. *For any function ψ that is Lebesgue-measurable on Ω^*_ε, the l-p unfolding operator $\mathcal{T}^{*,\varepsilon}_\mathcal{L}$ is defined as*

$$\mathcal{T}^{*,\varepsilon}_\mathcal{L}(\psi)(x, y) = \sum_{n=1}^{N_\varepsilon} \psi\big(\varepsilon D_{x_n^\varepsilon} \big[D_{x_n^\varepsilon}^{-1} x/\varepsilon\big]_Y + \varepsilon D_{x_n^\varepsilon} y\big) \chi_{\hat\Omega_n^\varepsilon}(x) \quad \text{for } \ x \in \Omega \text{ and } \ y \in Y^*.$$

The definition implies that $\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}(\psi)$ is Lebesgue-measurable on $\Omega \times Y^*$ and is zero for $x \in \Lambda^\varepsilon$.

In mathematical models posed in perforated domains we often have some processes defined on the surfaces of the microstructure (e.g., nonhomogeneous Neumann conditions or equations defined on the boundaries of the microstructure). Therefore it is important to have a notion of a convergence for sequences defined on oscillating surfaces of l-p microstructures. Applying the same idea as in the definition of l-t-s convergence for sequences in $L^p(\Omega)$ (i.e., considering l-p approximations of functions with space-dependent periodicity as test functions), we define the l-t-s convergence on surfaces of l-p microstructures.

DEFINITION 3.4. *A sequence $\{u^\varepsilon\} \subset L^p(\Gamma^\varepsilon)$, with $p \in (1,+\infty)$, is said to converge l-t-s to $u \in L^p(\Omega; L^p(\Gamma_x))$ if $\varepsilon \|u^\varepsilon\|^p_{L^p(\Gamma^\varepsilon)} \leq C$ and for any $\psi \in C(\overline{\Omega}; C_{\rm per}(Y_x))$*

$$\lim_{\varepsilon \to 0} \varepsilon \int_{\Gamma^\varepsilon} u^\varepsilon(x)\, \mathcal{L}^\varepsilon \psi(x)\, d\sigma_x = \int_\Omega \frac{1}{|Y_x|} \int_{\Gamma_x} u(x,y)\, \psi(x,y)\, d\sigma_y dx,$$

where $\mathcal{L}^\varepsilon$ is the l-p approximation of ψ defined in (2.1).

Often, to show the strong convergence of a sequence defined on oscillating boundaries of a microstructure, we need to map it to a sequence defined on a fixed domain. This can be achieved by using the boundary unfolding operator.

DEFINITION 3.5. *For any function ψ that is Lebesgue-measurable on Γ^ε, the l-p boundary unfolding operator $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}$ is defined as*

$$\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\psi)(x,y) = \sum_{n=1}^{N_\varepsilon} \psi\big(\varepsilon D_{x_n^\varepsilon} \big[D_{x_n^\varepsilon}^{-1} x/\varepsilon\big]_Y + \varepsilon D_{x_n^\varepsilon} K_{x_n^\varepsilon} y\big) \chi_{\hat{\Omega}_n^\varepsilon}(x) \quad \textit{for } x \in \Omega \textit{ and } y \in \Gamma.$$

The definition implies that $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\psi)$ is Lebesgue-measurable on $\Omega \times \Gamma$ and is zero for $x \in \Lambda^\varepsilon$.

The l-p boundary unfolding operator is a generalization of the periodic boundary unfolding operator; see, e.g., [18, 20, 21, 43]. Similar to the periodic unfolding operator, the l-p unfolding operator maps functions defined in domains depending on ε (on Ω_ε^* or Γ^ε) to functions defined on fixed domains ($\Omega \times Y^*$ or $\Omega \times \Gamma$). The l-p microstructures of domains are reflected in the definition of the l-p unfolding operator.

4. Main convergence results for the l-p unfolding operator and l-t-s convergence on oscillating surfaces.

In this section we summarize the main results of the paper. Similar to the periodic case [18, 21], we obtain compactness results for the l-t-s convergence on oscillating boundaries, for the l-p unfolding operator, and for the l-p boundary unfolding operator. We prove convergence results for sequences bounded in $L^p(\Gamma^\varepsilon)$, $H^1(\Omega)$, and $H^1(\Omega_\varepsilon^*)$, respectively. The properties of the transformation matrices D and K, assumed in section 3, are used to prove the convergence results stated in this section.

THEOREM 4.1. *For a sequence $\{w^\varepsilon\} \subset L^p(\Omega)$, with $p \in (1,+\infty)$, satisfying*

$$\|w^\varepsilon\|_{L^p(\Omega)} + \varepsilon \|\nabla w^\varepsilon\|_{L^p(\Omega)} \leq C,$$

there exist a subsequence (denoted again by $\{w^\varepsilon\}$) and $w \in L^p(\Omega; W^{1,p}_{\rm per}(Y_x))$ such that

$$\begin{aligned} \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon) &\rightharpoonup w(\cdot, D_x \cdot) && \textit{weakly in } L^p(\Omega; W^{1,p}(Y)), \\ \varepsilon \mathcal{T}_{\mathcal{L}}^\varepsilon(\nabla w^\varepsilon) &\rightharpoonup D_x^{-T} \nabla_y w(\cdot, D_x \cdot) && \textit{weakly in } L^p(\Omega \times Y). \end{aligned}$$

For a uniformly in ε bounded sequence in $W^{1,p}(\Omega)$, in addition we obtain the weak convergence of the unfolded sequence of derivatives, important for the homogenization of equations comprising elliptic operators of second order.

THEOREM 4.2. *For a sequence $\{w^\varepsilon\} \subset W^{1,p}(\Omega)$, with $p \in (1, +\infty)$, that converges weakly to w in $W^{1,p}(\Omega)$, there exist a subsequence (denoted again by $\{w^\varepsilon\}$) and a function $w_1 \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y_x))$ such that*

$$\begin{aligned} \mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon) &\rightharpoonup w && \textit{weakly in } L^p(\Omega; W^{1,p}(Y)),\\ \mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla w^\varepsilon)(\cdot,\cdot) &\rightharpoonup \nabla_x w(\cdot) + D_x^{-T}\nabla_y w_1(\cdot, D_x\cdot) && \textit{weakly in } L^p(\Omega\times Y). \end{aligned}$$

Two of the main advantages of the unfolding operator are that it helps to overcome one of the difficulties of perforated domains—the use of extension operators—and it allows us to prove strong convergence for sequences defined on boundaries of microstructures. Thus, next we formulate convergence results for the l-p unfolding operator in perforated domains and the l-p boundary unfolding operator.

THEOREM 4.3. *For a sequence $\{w^\varepsilon\} \subset W^{1,p}(\Omega^*_\varepsilon)$, where $p \in (1, +\infty)$, satisfying*

$$\|w^\varepsilon\|_{L^p(\Omega^*_\varepsilon)} + \varepsilon\|\nabla w^\varepsilon\|_{L^p(\Omega^*_\varepsilon)} \le C, \tag{4.1}$$

there exist a subsequence (denoted again by $\{w^\varepsilon\}$) and $w \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y^*_x))$ such that

$$\begin{aligned} \mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(w^\varepsilon) &\rightharpoonup w(\cdot, D_x\cdot) && \textit{weakly in } L^p(\Omega; W^{1,p}(Y^*)),\\ \varepsilon\mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(\nabla w^\varepsilon) &\rightharpoonup D_x^{-T}\nabla_y w(\cdot, D_x\cdot) && \textit{weakly in } L^p(\Omega\times Y^*). \end{aligned} \tag{4.2}$$

In the case when w^ε is bounded in $W^p(\Omega^*_\varepsilon)$ uniformly with respect to ε, we obtain weak convergence of $\mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(\nabla w^\varepsilon)$ in $L^p(\Omega \times Y^*)$ and local strong convergence of $\mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(w^\varepsilon)$.

THEOREM 4.4. *For a sequence $\{w^\varepsilon\} \subset W^{1,p}(\Omega^*_\varepsilon)$, where $p \in (1, +\infty)$, satisfying*

$$\|w^\varepsilon\|_{W^{1,p}(\Omega^*_\varepsilon)} \le C,$$

there exist a subsequence (denoted again by $\{w^\varepsilon\}$) and functions $w \in W^{1,p}(\Omega)$ and $w_1 \in L^p(\Omega; W^{1,p}_{\mathrm{per}}(Y^*_x))$ such that

$$\begin{aligned} \mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(w^\varepsilon) &\rightharpoonup w && \textit{weakly in } L^p(\Omega; W^{1,p}(Y^*)),\\ \mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(\nabla w^\varepsilon) &\rightharpoonup \nabla w + D_x^{-T}\nabla_y w_1(\cdot, D_x\cdot) && \textit{weakly in } L^p(\Omega\times Y^*),\\ \mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(w^\varepsilon) &\to w && \textit{strongly in } L^p_{\mathrm{loc}}(\Omega; W^{1,p}(Y^*)). \end{aligned}$$

Notice that the weak limit of $\varepsilon\mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(\nabla w^\varepsilon)$ and $\mathcal{T}^{*,\varepsilon}_{\mathcal{L}}(\nabla w^\varepsilon)$ reflects the l-p microstructure of Ω^*_ε and depends on the transformation matrix D.

For l-t-s convergence on oscillating surfaces of microstructures we have the following compactness result.

THEOREM 4.5. *For a sequence $\{w^\varepsilon\} \subset L^p(\Gamma^\varepsilon)$, with $p \in (1, +\infty)$, satisfying*

$$\varepsilon\|w^\varepsilon\|^p_{L^p(\Gamma^\varepsilon)} \le C,$$

there exist a subsequence (denoted again by $\{w^\varepsilon\}$) and $w \in L^p(\Omega; L^p(\Gamma_x))$ such that

$$w^\varepsilon \to w \quad \textit{l-t-s.}$$

Similar to the periodic case [18, 21], we show the relation between the l-t-s convergence on oscillating surfaces and the weak convergence of a sequence obtained by applying the l-p boundary unfolding operator.

THEOREM 4.6. *Let* $\{w^\varepsilon\} \subset L^p(\Gamma^\varepsilon)$ *with* $\varepsilon\|w^\varepsilon\|^p_{L^p(\Gamma^\varepsilon)} \le C$, *where* $p \in (1,+\infty)$. *The following assertions are equivalent:*

$$\begin{aligned} &\text{(i)} && w^\varepsilon \to w && \textit{l-t-s}, \qquad w \in L^p(\Omega; L^p(\Gamma_x)),\\ &\text{(ii)} && \mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(w^\varepsilon) \rightharpoonup w(\cdot, D_x K_x \cdot) && \textit{weakly in } L^p(\Omega\times\Gamma). \end{aligned}$$

Theorems 4.5 and 4.6 imply that for $\{w^\varepsilon\} \subset L^p(\Gamma^\varepsilon)$ with $\varepsilon\|w^\varepsilon\|^p_{L^p(\Gamma^\varepsilon)} \le C$ we have the weak convergence of $\{\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(w^\varepsilon)\}$ in $L^p(\Omega\times\Gamma)$, where $p \in (1,+\infty)$.

The definition of the l-p boundary unfolding operator and the relation between the l-t-s convergence of sequences defined on l-p oscillating boundaries and the l-p boundary unfolding operator allow us to obtain homogenization results for equations posed on the boundaries of l-p microstructures.

5. The l-p unfolding operator: Proofs of convergence results. First we prove some properties of the l-p unfolding operator. Similar to the periodic case, we obtain that the l-p unfolding operator is linear and preserves strong convergence.

LEMMA 5.1.

(i) *For* $\phi \in L^p(\Omega)$, *with* $p \in [1,+\infty)$, *it holds that*

$$\begin{aligned} &\frac{1}{|Y|}\int_{\Omega\times Y} \mathcal{T}^\varepsilon_{\mathcal{L}}(\phi)(x,y)\,dydx = \int_\Omega \phi(x)\,dx - \int_{\Lambda^\varepsilon}\phi(x)\,dx,\\ &\int_{\Omega\times Y} |\mathcal{T}^\varepsilon_{\mathcal{L}}(\phi)(x,y)|^p\,dydx \le |Y|\int_\Omega |\phi(x)|^p\,dx. \end{aligned} \tag{5.1}$$

(ii) $\mathcal{T}^\varepsilon_{\mathcal{L}} : L^p(\Omega) \to L^p(\Omega\times Y)$ *is a linear continuous operator, where* $p \in [1,+\infty)$.

(iii) *For* $\phi \in L^p(\Omega)$, *with* $p \in [1,+\infty)$, *we have strong convergence*

$$\mathcal{T}^\varepsilon_{\mathcal{L}}(\phi) \to \phi \quad \textit{in} \quad L^p(\Omega\times Y). \tag{5.2}$$

(iv) *If* $\phi^\varepsilon \to \phi$ *in* $L^p(\Omega)$, *with* $p \in [1,+\infty)$, *then* $\mathcal{T}^\varepsilon_{\mathcal{L}}(\phi^\varepsilon) \to \phi$ *in* $L^p(\Omega\times Y)$.

Proof. Using the definition of the l-p unfolding operator we obtain

$$\begin{aligned} \int_{\Omega\times Y} |\mathcal{T}^\varepsilon_{\mathcal{L}}(\phi)(x,y)|^p dydx &= \sum_{n=1}^{N_\varepsilon}\sum_{\xi\in\hat{\Xi}^\varepsilon_n} \varepsilon^d |D_{x^\varepsilon_n} Y| \int_Y |\phi(D_{x^\varepsilon_n}(\varepsilon\xi+\varepsilon y))|^p\,dy\\ &= \sum_{n=1}^{N_\varepsilon} |Y| \sum_{\xi\in\hat{\Xi}^\varepsilon_n} \int_{\varepsilon D_{x^\varepsilon_n}(\xi+Y)} |\phi(x)|^p\,dx = \sum_{n=1}^{N_\varepsilon}|Y|\int_{\hat{\Omega}^\varepsilon_n}|\phi(x)|^p\,dx. \end{aligned} \tag{5.3}$$

Then the equality and estimate in (5.1) follow from the definition of Λ^ε and the properties of the covering of Ω by $\{\Omega^\varepsilon_n\}_{n=1}^{N_\varepsilon}$.

The result in (ii) is ensured by the definition of the l-p unfolding operator and inequality (5.1).

(iii) Using the fact that $\phi \in L^p(\Omega)$ and $|\Lambda^\varepsilon| \to 0$ as $\varepsilon \to 0$ (ensured by the properties of the covering of Ω by $\{\Omega^\varepsilon_n\}_{n=1}^{N_\varepsilon}$ and the definition of Λ^ε) and applying Lebesgue's dominated convergence theorem (see, e.g., [26]), we obtain that $\int_{\Lambda^\varepsilon}|\phi(x)|^p\,dx \to 0$ as $\varepsilon \to 0$.

Then considering the approximation of L^p-functions by continuous functions and using the definition of $\mathcal{T}^\varepsilon_{\mathcal{L}}$, equality (5.3) and the estimate in (5.1) imply the convergence stated in (iii).

(iv) The linearity of the l-p unfolding operator along with (5.1) and (5.2) yields

$$\|\mathcal{T}^\varepsilon_{\mathcal{L}}(\phi^\varepsilon) - \phi\|_{L^p(\Omega\times Y)} \leq |Y|^{\frac{1}{p}} \|\phi^\varepsilon - \phi\|_{L^p(\Omega)} + \|\mathcal{T}^\varepsilon_{\mathcal{L}}(\phi) - \phi\|_{L^p(\Omega\times Y)} \to 0 \text{ as } \varepsilon \to 0. \qquad \square$$

Similar to l-t-s convergence, the average of the weak limit of the unfolded sequence with respect to microscopic variables is equal to the weak limit of the original sequence.

Lemma 5.2. *For $\{w^\varepsilon\}$ bounded in $L^p(\Omega)$, with $p \in (1, +\infty)$, we have that $\{\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)\}$ is bounded in $L^p(\Omega \times Y)$, and if*

$$\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon) \rightharpoonup \tilde{w} \quad \text{weakly in } L^p(\Omega \times Y),$$

then

$$w^\varepsilon \rightharpoonup ⨍_Y \tilde{w}\, dy \quad \text{weakly in } L^p(\Omega).$$

Proof. The boundedness of $\{\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)\}$ in $L^p(\Omega \times Y)$ follows directly from the boundedness of $\{w^\varepsilon\}$ in $L^p(\Omega)$ and the estimate (5.1). For $\psi \in L^q(\Omega)$, $1/p + 1/q = 1$, using the definition of $\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)$ we have

$$\int_\Omega w^\varepsilon\, \psi\, dx = \frac{1}{|Y|} \int_{\Omega\times Y} \mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)\, \mathcal{T}^\varepsilon_{\mathcal{L}}(\psi)\, dy\, dx + \mathcal{A}_\varepsilon, \qquad \text{where } \mathcal{A}_\varepsilon = \int_{\Lambda^\varepsilon} w^\varepsilon \psi\, dx.$$

For $\{w^\varepsilon\}$ bounded in $L^p(\Omega)$ and $\psi \in L^q(\Omega)$, using the properties of the covering of Ω and the definition of $\hat{\Omega}^\varepsilon$ and Λ^ε, we obtain $\mathcal{A}_\varepsilon \to 0$ as $\varepsilon \to 0$. Then, the weak convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)$ and the strong convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(\psi)$, shown in Lemma 5.1, imply

$$\lim_{\varepsilon\to 0} \int_\Omega w^\varepsilon(x)\, \psi(x)\, dx = \frac{1}{|Y|} \int_\Omega \int_Y \tilde{w}(x,y)\, \psi(x)\, dy\, dx$$

for any $\psi \in L^q(\Omega)$. $\square$

For the periodic unfolding operator we have that $\mathcal{T}^\varepsilon(\psi(\cdot, \cdot/\varepsilon)) \to \psi$ in $L^q(\Omega \times Y)$ for $\psi \in L^q(\Omega, C_{\text{per}}(Y))$. A similar result holds for the l-p unfolding operator and $\psi \in L^q(\Omega, C_{\text{per}}(Y_x))$, but with $\psi(\cdot, \cdot/\varepsilon)$ replaced by the l-p approximation $\mathcal{L}^\varepsilon\psi(\cdot)$.

Lemma 5.3.

(i) *For $\psi \in L^q(\Omega; C_{\text{per}}(Y_x))$, with $q \in [1, +\infty)$, we have*

$$\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon \psi) \to \psi(\cdot, D_x \cdot) \quad \text{strongly in } L^q(\Omega \times Y).$$

(ii) *For $\psi \in C(\overline{\Omega}; L^q_{\text{per}}(Y_x))$, with $q \in [1, +\infty)$, we have*

$$\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon_0 \psi) \to \psi(\cdot, D_x \cdot) \quad \text{strongly in } L^q(\Omega \times Y).$$

Proof. (i) For $\psi \in C(\overline{\Omega}; C_{\text{per}}(Y_x))$ using the definition of $\mathcal{L}^\varepsilon$ and $\mathcal{T}^\varepsilon_{\mathcal{L}}$, we obtain

$$\int_{\Omega\times Y} |\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon \psi)|^q dy\, dx = \sum_{n=1}^{N_\varepsilon} \int_{\hat{\Omega}^\varepsilon_n \times Y} \left| \widetilde{\psi}\left(\varepsilon D_{x^\varepsilon_n} \left[\frac{D^{-1}_{x^\varepsilon_n} x}{\varepsilon} \right]_Y + \varepsilon D_{x^\varepsilon_n} y, y \right) \right|^q dy\, dx,$$

where $q \in [1, +\infty)$ and $\widetilde{\psi} \in C(\overline{\Omega}; C_{\rm per}(Y))$ such that $\psi(x,y) = \widetilde{\psi}(x, D_x^{-1} y)$ for $x \in \Omega$ and $y \in Y_x$. Then, using the properties of the covering of Ω_n^ε by $\varepsilon Y^\xi_{x_n^\varepsilon} = \varepsilon D_{x_n^\varepsilon}(Y+\xi)$, with $\xi \in \Xi_n^\varepsilon$, and considering fixed points $y_\xi \in Y + \xi$ for $\xi \in \hat{\Xi}_n^\varepsilon$, we obtain

$$\int_{\Omega\times Y} |\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon \psi)|^q dy\, dx = \sum_{n=1}^{N_\varepsilon} \sum_{\xi \in \hat{\Xi}_n^\varepsilon} \varepsilon^d |Y_{x_n^\varepsilon}| \int_Y |\widetilde{\psi}(\varepsilon D_{x_n^\varepsilon}(\xi + y_\xi), y)|^q \, dy + \delta(\varepsilon),$$

where, due to the continuity of ψ and the properties of the covering of Ω by $\{\Omega_n^\varepsilon\}_{n=1}^{N_\varepsilon}$,

$$\delta(\varepsilon) = \sum_{n=1}^{N_\varepsilon} \sum_{\xi \in \hat{\Xi}_n^\varepsilon} \varepsilon^d |Y_{x_n^\varepsilon}| \int_Y \Big(|\widetilde{\psi}(\varepsilon D_{x_n^\varepsilon}(\xi + y_\xi), y)|^q - |\widetilde{\psi}(\varepsilon D_{x_n^\varepsilon}(\xi + y), y)|^q \Big) dy \to 0$$

as $\varepsilon \to 0$. Then, using the continuity of ψ and D together with the relation between ψ and $\widetilde{\psi}$, we obtain

$$\lim_{\varepsilon\to 0} \int_{\Omega\times Y} |\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon \psi)|^q dy\, dx = \int_{\Omega\times Y} |\widetilde{\psi}(x,y)|^q \, dy\, dx = \int_{\Omega\times Y} |\psi(x, D_x y)|^q dy\, dx.$$

The continuity of ψ with respect to x yields the pointwise convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon \psi)(x,y)$ to $\psi(x, D_x y)$ a.e. in $\Omega \times Y$.

Considering an approximation of $\psi \in L^q(\Omega; C_{\rm per}(Y_x))$ by $\psi_m \in C(\overline{\Omega}; C_{\rm per}(Y_x))$ and the convergences

$$\lim_{m\to\infty} \lim_{\varepsilon\to 0} \int_\Omega |\mathcal{L}^\varepsilon \psi_m(x) - \mathcal{L}^\varepsilon \psi(x)|^q dx = 0,$$
$$\lim_{m\to\infty} \lim_{\varepsilon\to 0} \int_\Omega \big(|\mathcal{L}^\varepsilon \psi_m(x)|^q - |\mathcal{L}^\varepsilon \psi(x)|^q \big) dx = 0$$

(see [46, Lemma 3.4] for the proof) implies $\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon \psi)(\cdot,\cdot) \to \psi(\cdot, D_x \cdot)$ in $L^q(\Omega \times Y)$ for $\psi \in L^q(\Omega; C_{\rm per}(Y_x))$.

(ii) For $\psi \in C(\overline{\Omega}; L^q_{\rm per}(Y_x))$ we can prove the strong convergence only of $\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon_0 \psi)$. Consider

$$\lim_{\varepsilon\to 0} \int_{\Omega\times Y} |\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon_0 \psi)(x,y)|^q dy dx = |Y| \lim_{\varepsilon\to 0} \bigg[\int_\Omega |\mathcal{L}^\varepsilon_0 \psi(x)|^q dx - \int_{\Lambda^\varepsilon} |\mathcal{L}^\varepsilon_0 \psi(x)|^q dx \bigg].$$

Then, using Lemma 3.4 in [46] along with the regularity of ψ and the properties of Λ^ε, we obtain

$$|Y| \lim_{\varepsilon\to 0} \int_\Omega |\mathcal{L}^\varepsilon_0 \psi(x)|^q dx = \int_{\Omega\times Y} |\psi(x, D_x y)|^q dy dx, \qquad \lim_{\varepsilon\to 0} \int_{\Lambda^\varepsilon} |\mathcal{L}^\varepsilon_0 \psi(x)|^q dx = 0.$$

The continuity of ψ with respect to $x \in \Omega$ implies $\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon_0 \psi)(x,y) \to \psi(x, D_x y)$ pointwise a.e. in $\Omega \times Y$. ☐

Remark. Notice that for $\psi \in C(\overline{\Omega}; L^q_{\rm per}(Y_x))$ we have the strong convergence only of $\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon_0 \psi)$. However, this convergence result is sufficient for the derivation of homogenization results, since the microscopic properties of the considered processes or domains can be represented by coefficients in the form $B\mathcal{L}^\varepsilon_0 A$, with some given functions $B \in L^\infty(\Omega)$ and $A \in C(\overline{\Omega}; L^q_{\rm per}(Y_x))$.

The strong convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon \psi)$ for $\psi \in L^q(\Omega; C_{\rm per}(Y_x))$ is now used to show the equivalence between the weak convergence of the l-p unfolded sequence and the l-t-s convergence of the original sequence. Notice that $L^q(\Omega; C_{\rm per}(Y_x))$ represents the set of test functions admissible in the definition of the l-t-s convergence.

LEMMA 5.4. *Let $\{w^\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$, where $p \in (1, +\infty)$. Then the following assertions are equivalent:*

$$
\begin{array}{llll}
\text{(i)} & w^\varepsilon \to w & \textit{l-t-s}, & w \in L^p(\Omega; L^p(Y_x)),\\
\text{(ii)} & \mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)(\cdot,\cdot) \rightharpoonup w(\cdot, D_x \cdot) & \textit{weakly in } L^p(\Omega \times Y). &
\end{array}
$$

Proof. [(ii) $\Rightarrow$ (i)] Since $\{w^\varepsilon\}$ is bounded in $L^p(\Omega)$, there exists (up to a subsequence) an l-t-s limit of w^ε as $\varepsilon \to 0$. For an arbitrary $\psi \in L^q(\Omega; C_{\rm per}(Y_x))$, the weak convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)$ and the strong convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon(\psi))$ ensure

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \int_\Omega w^\varepsilon \mathcal{L}^\varepsilon(\psi) dx &= \lim_{\varepsilon\to 0} \Big[\int_\Omega \fint_Y \mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)\, \mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon(\psi)) dy dx + \int_{\Lambda^\varepsilon} w^\varepsilon \mathcal{L}^\varepsilon(\psi) dx \Big]\\
&= \int_\Omega \fint_Y w(x, D(x)y)\, \psi(x, D_x y)\, dy dx = \int_\Omega \fint_{Y_x} w\, \psi\, dy dx.
\end{aligned}
$$

Thus the whole sequence w^ε converges l-t-s to w.

[(i) $\Rightarrow$ (ii)] On the other hand, the boundedness of $\{w^\varepsilon\}$ in $L^p(\Omega)$ implies the boundedness of $\{\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)\}$ and (up to a subsequence) the weak convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)$ in $L^p(\Omega \times Y)$. If $w^\varepsilon \to w$ l-t-s, then

$$
\begin{aligned}
\lim_{\varepsilon\to 0} \int_\Omega \fint_Y \mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)\, \mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon(\psi))\, dy dx &= \lim_{\varepsilon\to 0} \Big[\int_\Omega w^\varepsilon \mathcal{L}^\varepsilon(\psi)\, dx - \int_{\Lambda^\varepsilon} w^\varepsilon \mathcal{L}^\varepsilon(\psi)\, dx \Big]\\
&= \int_\Omega \fint_{Y_x} w\, \psi\, dy dx
\end{aligned}
$$

for $\psi \in L^q(\Omega; C_{\rm per}(Y_x))$. Since $\mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon(\psi))(\cdot,\cdot) \to \psi(\cdot, D_x \cdot)$ in $L^q(\Omega \times Y)$, we obtain the weak convergence of the whole sequence $\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)$ to $w(\cdot, D_x \cdot)$ in $L^p(\Omega \times Y)$. Notice that the boundedness of $\{w^\varepsilon\}$ in $L^p(\Omega)$ and the fact that $|\Lambda^\varepsilon| \to 0$ as $\varepsilon \to 0$ imply

$$
\int_{\Lambda^\varepsilon} |w^\varepsilon\, \mathcal{L}^\varepsilon(\psi)|\, dx \le C \Big(\int_{\Lambda^\varepsilon} \sup_{y \in Y} |\psi(x, D_x y)|^q dx \Big)^{1/q} \to 0 \quad \text{as} \quad \varepsilon \to 0
$$

for $\psi \in L^q(\Omega; C_{\rm per}(Y_x))$ and $1/p + 1/q = 1$. ∎

Next, we prove the main convergence results for the l-p unfolding operator, i.e., convergence results for $\{\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)\}$, $\{\varepsilon \mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla w^\varepsilon)\}$, and $\{\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla w^\varepsilon)\}$.

The definition of the l-p unfolding operator yields that for $w \in W^{1,p}(\Omega)$

$$
\nabla_y \mathcal{T}^\varepsilon_{\mathcal{L}}(w) = \varepsilon \sum_{n=1}^{N_\varepsilon} D^T_{x^\varepsilon_n}\, \mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla w)\, \chi_{\Omega^\varepsilon_n}. \tag{5.4}
$$

Due to the regularity of D, the boundedness of $\varepsilon \nabla w^\varepsilon$ implies the boundedness of $\nabla_y \mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)$. Thus, assuming the boundedness of $\{\varepsilon \nabla w^\varepsilon\}$, we obtain convergence of the derivatives with respect to the microscopic variables, but have no information about the macroscopic derivatives.

Proof of Theorem 4.1. The assumptions on $\{w^\varepsilon\}$, together with inequality (5.1), equality (5.4), and regularity of D, ensure that $\{\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)\}$ is bounded in $L^p(\Omega; W^{1,p}(Y))$. This implies that there exist a subsequence, denoted again by $\{w^\varepsilon\}$, and a function $\widetilde{w} \in L^p(\Omega; W^{1,p}(Y))$ such that $\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon) \rightharpoonup \widetilde{w}$ in $L^p(\Omega; W^{1,p}(Y))$. We define

$w(x,y) = \widetilde{w}(x, D_x^{-1}y)$ for a.a. $x \in \Omega$, $y \in Y_x$. Due to the regularity of D, we have $w \in L^p(\Omega; W^{1,p}(Y_x))$. For $\phi \in C_0^\infty(\Omega \times Y)$, using the convergence of $\mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)$, we have

$$\lim_{\varepsilon\to 0}\int_{\Omega\times Y} \varepsilon \mathcal{T}_\mathcal{L}^\varepsilon(\nabla w^\varepsilon)\,\phi\, dydx = -\lim_{\varepsilon\to 0}\int_{\Omega\times Y} \mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)\sum_{n=1}^{N_\varepsilon} \operatorname{div}_y(D_{x_n^\varepsilon}^{-1}\phi(x,y))\chi_{\Omega_n^\varepsilon}\, dydx$$
$$= -\int_{\Omega\times Y} w(x, D_x y)\operatorname{div}_y(D_x^{-1}\phi(x,y))dydx = \int_{\Omega\times Y} D_x^{-T}\nabla_y w(x, D_x y)\,\phi(x,y)\,dydx.$$

Hence, $\varepsilon\mathcal{T}_\mathcal{L}^\varepsilon(\nabla w^\varepsilon)(\cdot,\cdot) \rightharpoonup D_x^{-T}\nabla_y w(\cdot, D_x\cdot)$ in $L^p(\Omega\times Y)$ as $\varepsilon \to 0$. To show the Y_x-periodicity of w, i.e., Y-periodicity of $\widetilde{w}$, we show first the periodicity in the e_d-direction. Then, considering similar calculations in each e_j-direction, with $j = 1,\dots,d-1$ and $\{e_j\}_{j=1,\dots,d}$ being the canonical basis of $\mathbb{R}^d$, we obtain the Y_x-periodicity of w. For $\psi \in C_0^\infty(\Omega\times Y')$ we consider

$$I = \int_{\Omega\times Y'} \left[\mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)(x,(y',1)) - \mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)(x,(y',0))\right]\psi(x,y')dy'dx,$$

where $Y' = (0,1)^{d-1}$. For $j = 1,\dots,d$ we define

$$\widetilde{\Omega}_n^{\varepsilon,j} = \operatorname{Int}\Bigg(\bigcup_{\xi\in\overline{\Xi}_{n,1}^{\varepsilon,j}} \varepsilon D_{x_n^\varepsilon}(\overline{Y}+\xi)\Bigg), \quad \widetilde{\Lambda}_{n,l}^{\varepsilon,j} = \operatorname{Int}\Bigg(\bigcup_{\xi\in\widetilde{\Xi}_{n,l}^{\varepsilon,j}} \varepsilon D_{x_n^\varepsilon}(\overline{Y}+\xi)\Bigg) \quad \text{for } l = 1,2,$$

where $\overline{\Xi}_{n,1}^{\varepsilon,j} = \{\xi \in \hat{\Xi}_n^\varepsilon : \varepsilon D_{x_n^\varepsilon}(Y+\xi-e_j) \subset \hat{\Omega}_n^\varepsilon\}$, $\overline{\Xi}_n^{\varepsilon,j} = \{\xi \in \hat{\Xi}_n^\varepsilon : \varepsilon D_{x_n^\varepsilon}(Y+\xi+e_j) \subset \hat{\Omega}_n^\varepsilon$ and $\varepsilon D_{x_n^\varepsilon}(Y+\xi-e_j) \subset \hat{\Omega}_n^\varepsilon\}$, and $\widetilde{\Xi}_n^{\varepsilon,j} = \hat{\Xi}_n^\varepsilon \setminus \overline{\Xi}_n^{\varepsilon,j}$. We write $\widetilde{\Xi}_n^{\varepsilon,j} = \widetilde{\Xi}_{n,1}^{\varepsilon,j}\cup\widetilde{\Xi}_{n,2}^{\varepsilon,j}$, where $\widetilde{\Xi}_{n,1}^{\varepsilon,j}$ corresponds to upper and $\widetilde{\Xi}_{n,2}^{\varepsilon,j}$ corresponds to lower cells in $\hat{\Omega}_n^\varepsilon$ in the $D_{x_n^\varepsilon}e_j$-direction. Using the definition of $\mathcal{T}_\mathcal{L}^\varepsilon$ we can write

$$I = \sum_{n=1}^{N_\varepsilon}\int_{\widetilde{\Omega}_n^{\varepsilon,d}\times Y'} \mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)(x,y^0)\left[\psi(x-\varepsilon D_{x_n^\varepsilon}e_d, y') - \psi(x,y')\right]dy'dx$$
$$+\sum_{n=1}^{N_\varepsilon}\Bigg[\int_{\widetilde{\Lambda}_{n,1}^{\varepsilon,d}\times Y'} \mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)(x,y^1)\psi(x,y')dy'dx - \int_{\widetilde{\Lambda}_{n,2}^{\varepsilon,d}\times Y'} \mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)(x,y^0)\psi(x,y')dy'dx\Bigg],$$

where $y^1 = (y',1)$ and $y^0 = (y',0)$. Using the continuity of ψ, the boundedness of the trace of $\mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)$ in $L^p(\Omega\times Y')$, ensured by the assumptions on w^ε, and the fact that $\sum_{n=1}^{N_\varepsilon}|\widetilde{\Lambda}_{n,l}^{\varepsilon,d}| \le C\varepsilon^{1-r} \to 0$ as $\varepsilon\to 0$, with $r\in[0,1)$ and $l = 1,2$, we obtain that $I \to 0$ as $\varepsilon \to 0$. Similar calculations for e_j, with $j = 1,\dots,d-1$, and the convergence of the trace of $\mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)$ in $L^p(\Omega\times Y')$, ensured by the weak convergence of $\mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)$ in $L^p(\Omega; W^{1,p}(Y))$, imply the Y_x-periodicity of w. ∎

If $\|w^\varepsilon\|_{W^{1,p}(\Omega)}$ is bounded uniformly in ε, we have the weak convergence of w^ε in $W^{1,p}(\Omega)$ and of $\mathcal{T}_\mathcal{L}^\varepsilon(\nabla w^\varepsilon)$ in $L^p(\Omega\times Y)$. Hence we have information about the macroscopic and microscopic gradients of limit functions. The proof of the convergence results for $\mathcal{T}_\mathcal{L}^\varepsilon(\nabla w^\varepsilon)$ makes use of the Poincaré inequality for an auxiliary sequence. For this purpose we define a local average operator $\mathcal{M}_\mathcal{L}^\varepsilon$, i.e., an average of the unfolded function with respect to the microscopic variables.

DEFINITION 5.5. *The local average operator* $\mathcal{M}_\mathcal{L}^\varepsilon : L^p(\Omega) \to L^p(\Omega)$, $p \in [1,+\infty]$, *is defined as*

$$\mathcal{M}_\mathcal{L}^\varepsilon(\psi)(x) = \fint_Y \mathcal{T}_\mathcal{L}^\varepsilon(\psi)(x,y)dy = \sum_{n=1}^{N_\varepsilon}\fint_Y \psi\big(\varepsilon D_{x_n^\varepsilon}\big([D_{x_n^\varepsilon}^{-1}x/\varepsilon]+y\big)\big)dy\,\chi_{\hat{\Omega}_n^\varepsilon}(x). \tag{5.5}$$

Proof of Theorem 4.2. The proof of the convergence of $\mathcal{T}_\mathcal{L}^\varepsilon(\nabla w^\varepsilon)$ follows ideas similar to those in the case of the periodic unfolding operator. However, the proof of the periodicity of the corrector w_1 involves new ideas and technical details.

The convergence of $\mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)$ follows from Lemma 5.2 and the fact that due to the assumption on $\{w^\varepsilon\}$ and regularity of D, we have

$$\|\nabla_y \mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon)\|_{L^p(\Omega\times Y)} \le C\varepsilon \to 0 \quad \text{ as } \varepsilon \to 0.$$

To show the convergence of $\mathcal{T}_\mathcal{L}^\varepsilon(\nabla w^\varepsilon)$ we consider a function $V^\varepsilon : \Omega \times Y \to \mathbb{R}$ defined as

$$V^\varepsilon = \varepsilon^{-1}\left(\mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon) - \mathcal{M}_\mathcal{L}^\varepsilon(w^\varepsilon)\right). \tag{5.6}$$

Then, the definition of $\mathcal{T}_\mathcal{L}^\varepsilon$ and $\mathcal{M}_\mathcal{L}^\varepsilon$ implies

$$\nabla_y V^\varepsilon = \frac{1}{\varepsilon}\nabla_y \mathcal{T}_\mathcal{L}^\varepsilon(w^\varepsilon) = \sum_{n=1}^{N_\varepsilon} D^T_{x_n^\varepsilon} \mathcal{T}_\mathcal{L}^\varepsilon(\nabla w^\varepsilon)\, \chi_{\Omega_n^\varepsilon}.$$

The boundedness of $\{w^\varepsilon\}$ in $W^{1,p}(\Omega)$, together with (5.1) and regularity assumptions on D, implies that the sequence $\{\nabla_y V^\varepsilon\}$ is bounded in $L^p(\Omega\times Y)$. Considering

$$\fint_Y V^\varepsilon \, dy = 0 \quad \text{ and } \quad \fint_Y y_c^\varepsilon \cdot \nabla w \, dy = 0 \quad \text{with} \quad y_c^\varepsilon = \sum_{n=1}^{N_\varepsilon} D_{x_n^\varepsilon} y_c \, \chi_{\Omega_n^\varepsilon},$$

where $y_c = (y_1 - \frac{1}{2}, \dots, y_d - \frac{1}{2})$ for $y \in Y$, and applying the Poincaré inequality to $V^\varepsilon - y_c^\varepsilon \cdot \nabla w$ yields

$$\|V^\varepsilon - y_c^\varepsilon \cdot \nabla w\|_{L^p(\Omega\times Y)} \le C_1 \left\| \nabla_y V^\varepsilon - \sum_{n=1}^{N_\varepsilon} D^T_{x_n^\varepsilon} \nabla w \, \chi_{\Omega_n^\varepsilon} \right\|_{L^p(\Omega\times Y)} \le C_2.$$

Thus, there exist a subsequence (denoted again by $\{V^\varepsilon - y_c^\varepsilon\cdot\nabla w\}$) and a function $\widetilde{w}_1 \in L^p(\Omega; W^{1,p}(Y))$ such that

$$V^\varepsilon - y_c^\varepsilon \cdot \nabla w \rightharpoonup \widetilde{w}_1 \quad \text{ weakly in } L^p(\Omega; W^{1,p}(Y)). \tag{5.7}$$

For $\phi \in W^{1,p}(\Omega)$ we have the following relation:

$$\mathcal{T}_\mathcal{L}^\varepsilon(\nabla\phi)(x,y) = \varepsilon^{-1} \sum_{n=1}^{N_\varepsilon} D^{-T}_{x_n^\varepsilon} \nabla_y \mathcal{T}_\mathcal{L}^\varepsilon(\phi)(x,y)\, \chi_{\Omega_n^\varepsilon}(x).$$

Then the convergence in (5.7) and the continuity of D yield

$$\mathcal{T}_\mathcal{L}^\varepsilon(\nabla w^\varepsilon) = \sum_{n=1}^{N_\varepsilon} D^{-T}_{x_n^\varepsilon} \nabla_y V^\varepsilon \chi_{\Omega_n^\varepsilon} \rightharpoonup \nabla w + D_x^{-T}\nabla_y \widetilde{w}_1 \quad \text{ weakly in } L^p(\Omega\times Y). \tag{5.8}$$

We show now that $\widetilde{w}_1(x,y)$ is Y-periodic. Then the function $w_1(x,y) = \widetilde{w}_1(x, D_x^{-1}y)$ for a.a. $x\in\Omega$, $y \in Y_x$ will be Y_x-periodic. For $\psi \in C_0^\infty(\Omega\times Y')$ we consider

$$\int_\Omega \int_{Y'} \left[V^\varepsilon(x,y^1) - V^\varepsilon(x,y^0)\right] \psi(x,y') dy' dx = \sum_{n=1}^{N_\varepsilon} \left(\mathcal{I}_{1,n} + \mathcal{I}_{2,n}\right)$$

with

$$\begin{aligned}\mathcal{I}_{1,n} &= \int_{\widetilde{\Omega}_n^{\varepsilon,d}} \int_{Y'} \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)(x,y^0) \frac{1}{\varepsilon}\left[\psi(x-\varepsilon D_{x_n^\varepsilon} e_d, y') - \psi(x,y')\right] dy' dx,\\ \mathcal{I}_{2,n} &= \frac{1}{\varepsilon}\bigg[\int_{\widetilde{\Lambda}_{n,1}^{\varepsilon,d}\times Y'} \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)(x,y^1)\psi(x,y')dy'dx - \int_{\widetilde{\Lambda}_{n,2}^{\varepsilon,d}\times Y'} \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)(x,y^0)\psi(x,y')dy'dx\bigg]\\ &= \mathcal{I}_{2,n}^U - \mathcal{I}_{2,n}^L,\end{aligned}$$

where y^1, y^0, $\widetilde{\Omega}_n^{\varepsilon,d}$, and $\widetilde{\Lambda}_{n,l}^{\varepsilon,d}$, with $l=1,2$, are defined in the proof of Theorem 4.1. Then Lemma 5.1 and the strong convergence of $\{w^\varepsilon\}$ in $L^p(\Omega)$, ensured by the boundedness of $\{w^\varepsilon\}$ in $W^{1,p}(\Omega)$, imply the strong convergence of $\{\mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)\}$ to w in $L^p(\Omega\times Y)$. The boundedness of $\{\nabla_y \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)\}$ (ensured by the boundedness of $\{\nabla w^\varepsilon\}$) yields the weak convergence of $\{\mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)\}$ in $L^p(\Omega; W^{1,p}(Y))$ to the same w. Applying the trace theorem in $W^{1,p}(Y)$, we obtain that the trace of $\mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)$ on $\Omega\times Y'$ converges weakly to w in $L^p(\Omega\times Y')$ as $\varepsilon\to 0$. This together with the regularity of ψ and D gives

$$\lim_{\varepsilon\to 0}\sum_{n=1}^{N_\varepsilon} \mathcal{I}_{1,n} = -\int_\Omega\int_{Y'} w(x)\, D_d(x)\cdot\nabla_x\psi(x,y')\, dy'dx,$$

where $D_j(x) = (D_{1j}(x),\dots,D_{dj}(x))^T$, with $j=1,\dots,d$. Next we consider the integrals over the upper cells $\mathcal{I}_{2,n_1}^U$ and over the lower cells $\mathcal{I}_{2,n_2}^L$ in neighboring $\Omega_{n_1}^\varepsilon$ and $\Omega_{n_2}^\varepsilon$ (in the e_j-direction, with $e_j\cdot D_{x_{n_1}^\varepsilon} e_d \neq 0$, $j=1,\dots,d$), i.e., for $1\le n_1,n_2\le N_\varepsilon$ such that $\Theta_{n_{1,2}} = (\partial\Omega_{n_1}^\varepsilon \cap \partial\Omega_{n_2}^\varepsilon)\cap\{x_j = \text{const}\}\neq\emptyset$, $\dim(\Theta_{n_{1,2}}) = d-1$, and $x_{n_1,j}^\varepsilon < x_{n_2,j}^\varepsilon$, and write

$$\begin{aligned}\mathcal{I}_{2,n_1}^U - \mathcal{I}_{2,n_2}^L &= \frac{1}{\varepsilon}\bigg[\int_{\widetilde{\Lambda}_{n_1,1}^{\varepsilon,d}\times Y'} \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)(x,y^0)\psi dy'dx - \int_{\widetilde{\Lambda}_{n_2,2}^{\varepsilon,d}\times Y'} \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)(x,y^0)\psi dy'dx\bigg]\\ &\quad + \int_{\widetilde{\Lambda}_{n_1,1}^{\varepsilon,d}} \frac{1}{\varepsilon}\bigg[\int_{Y'} \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)(x,y^1)\psi\, dy' - \int_{Y'} \mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)(x,y^0)\psi\, dy'\bigg]dx = \mathcal{I}_{2,n}^{1,2} + \mathcal{I}_{2,n}^1.\end{aligned}$$

The second integral $\mathcal{I}_{2,n}^1$ can be rewritten as

$$\mathcal{I}_{2,n}^1 = \frac{1}{\varepsilon}\int_{\widetilde{\Lambda}_{n_1,1}^{\varepsilon,d}\times Y} \partial_{y_d}\mathcal{T}_{\mathcal{L}}^\varepsilon(w^\varepsilon)(x,y)\,\psi(x,y')dydx = \int_{\widetilde{\Lambda}_{n_1,1}^{\varepsilon,d}\times Y} D_d(x_{n_1}^\varepsilon)\cdot\mathcal{T}_{\mathcal{L}}^\varepsilon(\nabla w^\varepsilon)\psi\, dydx.$$

Using the boundedness of $\{\nabla w^\varepsilon\}$ in $L^p(\Omega)$ and $\sum_{n_1=1}^{N_\varepsilon}|\widetilde{\Lambda}_{n_1,1}^{\varepsilon,d}|\le C\varepsilon^{1-r}$, we conclude that $\sum_{n=1}^{N_\varepsilon}\mathcal{I}_{2,n}^1\to 0$ as $\varepsilon\to 0$ and $r<1$.

In $\mathcal{I}_{2,n}^{1,2}$ we distinguish between variations in the $D_{x_n^\varepsilon} e_j$-direction, for $1\le j\le d-1$, and in the $D_{x_n^\varepsilon} e_d$-direction. For an arbitrary fixed $x_{n_{1,2}}^\varepsilon\in\Theta_{n_{1,2}}$ we define $\hat{D}^l_{x_{n_{1,2}}^\varepsilon} = (D_1(x_{n_{1,2}}^\varepsilon),\dots,D_{d-1}(x_{n_{1,2}}^\varepsilon), D_d(x_{n_l}^\varepsilon))$, with $l=1,2$, and introduce

$$\hat{\Lambda}_{n_l}^\varepsilon = \operatorname{Int}\Bigg(\bigcup_{\xi\in\widetilde{\Xi}_{n_{1,2}}^{\varepsilon,l}} \varepsilon\hat{D}^l_{x_{n_{1,2}}^\varepsilon}(\overline{Y}+\xi)\Bigg) \qquad \text{for } l=1,2,$$

where

$$\begin{aligned}\widetilde{\Xi}_{n_{1,2}}^{\varepsilon,1} &= \left\{\xi\in\mathbb{Z}^d : \varepsilon\hat{D}^1_{x_{n_{1,2}}^\varepsilon}(\overline{Y}+\xi+e_d)\cap\Theta_{n_{1,2}}\neq\emptyset \text{ and } \varepsilon\hat{D}^1_{x_{n_{1,2}}^\varepsilon}(Y+\xi)\subset\Omega_{n_1}^\varepsilon\right\},\\ \widetilde{\Xi}_{n_{1,2}}^{\varepsilon,2} &= \left\{\xi\in\mathbb{Z}^d : \varepsilon\hat{D}^2_{x_{n_{1,2}}^\varepsilon}(\overline{Y}+\xi-e_d)\cap\Theta_{n_{1,2}}\neq\emptyset \text{ and } \varepsilon\hat{D}^2_{x_{n_{1,2}}^\varepsilon}(Y+\xi)\subset\Omega_{n_2}^\varepsilon\right\}.\end{aligned}$$

Then each of the integrals in $\mathcal{I}^{1,2}_{2,n}$ is rewritten as

$$\frac{1}{\varepsilon}\int_{\widetilde{\Lambda}^{\varepsilon,d}_{n_l,l}}\int_{Y'}\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)(x,y^0)\psi dy'dx=\frac{1}{\varepsilon}\int_{\hat{\Lambda}^\varepsilon_{n_l}}\int_{Y'}w^\varepsilon(\varepsilon\hat{D}^l_{x^\varepsilon_{n_{1,2}}}([x^l_{D,n}/\varepsilon]+y^0))\psi dy'dx$$
$$+\frac{1}{\varepsilon}\Big[\int_{\widetilde{\Lambda}^{\varepsilon,d}_{n_l,l}}\int_{Y'}\mathcal{T}^\varepsilon_{\mathcal{L}}(w^\varepsilon)(x,y^0)\psi dy'dx-\int_{\hat{\Lambda}^\varepsilon_{n_l}}\int_{Y'}w^\varepsilon(\varepsilon\hat{D}^l_{x^\varepsilon_{n_{1,2}}}([x^l_{D,n}/\varepsilon]+y^0))\psi dy'dx\Big]$$
$$=J^1_{l,n}+J^2_{l,n},$$

where $x^l_{D,n}=(\hat{D}^l_{x^\varepsilon_{n_{1,2}}})^{-1}x$ and $l=1,2$. Using the definition of $\hat{\Lambda}^\varepsilon_{n_l}$ for $l=1,2$ and the fact that $|\widetilde{\Xi}^{\varepsilon,l}_{n_{1,2}}||\hat{D}^l_{x^\varepsilon_{n_{1,2}}}|=I_j|D_d(x^\varepsilon_{n_l})\cdot e_j|$, with $D_d(x^\varepsilon_{n_l})\cdot e_j\neq 0$ and some $I_j>0$, $j=1,\dots,d$, and denoting $|\widetilde{\Xi}^{\varepsilon,1}_{n_{1,2}}|=I^\varepsilon_{n_{1,2}}$ yields

$$J^1_{1,n}-J^1_{2,n}=\varepsilon^d\sum_{i=1}^{I^\varepsilon_{n_{1,2}}}\int_Y\int_{Y'}\frac{1}{\varepsilon}\Big[w^\varepsilon\big(\varepsilon\hat{D}^1_{x^\varepsilon_{n_{1,2}}}(\xi^1_i+y^0)\big)\psi(\varepsilon\tilde{y}^i_{n_1,\xi},y')$$
$$-w^\varepsilon\big(\varepsilon\hat{D}^2_{x^\varepsilon_{n_{1,2}}}(\xi^2_i+y^0)\big)\psi(\varepsilon\tilde{y}^i_{n_2,\xi},y')\Big]\big|\hat{D}^1_{x^\varepsilon_{n_{1,2}}}\big|\,dy'd\tilde{y}$$
$$-\varepsilon^{r-1}\sum_{\xi\in\widetilde{\Xi}^{\varepsilon,2}_{n_{1,2}}}\int_{\varepsilon(Y+\xi)}\int_{Y'}w^\varepsilon\big(\varepsilon\hat{D}^2_{x^\varepsilon_{n_{1,2}}}\big([\tilde{x}/\varepsilon]+y^0\big)\big)\psi\,dy'\,\frac{1}{\varepsilon^r}\Big[d(\hat{D}^2_{x^\varepsilon_{n_{1,2}}}\tilde{x})-d(\hat{D}^1_{x^\varepsilon_{n_{1,2}}}\tilde{x})\Big],$$

where $\tilde{y}^i_{n_l,\xi}=\hat{D}^l_{x^\varepsilon_{n_{1,2}}}(\tilde{y}+\xi^l_i)$ for $l=1,2$. The first integral in the last equality can be estimated by

$$C\varepsilon^{rd+(1-r)}\|w^\varepsilon\|_{W^{1,p}(\Omega)}\|\psi\|_{C^1_0(\Omega\times Y')}.$$

In the second integral we have a discrete derivative in the e_j-direction, $e_j\cdot D_d(x^\varepsilon_{n_1})\neq 0$, $j=1,\dots,d$, of an integral over an evolving domain with the velocity vector D_d. Then, using the fact that $|N_\varepsilon|\le C\varepsilon^{-dr}$ and $x^\varepsilon_{n_1,j}<x^\varepsilon_{n_2,j}$, together with the regularity of D and the definition of $\hat{D}^l_{x^\varepsilon_{n_{1,2}}}$, where $l=1,2$, yields

$$\sum_{n=1}^{N_\varepsilon}\big(J^1_{1,n}-J^1_{2,n}\big)\to-\int_\Omega\int_{Y'}w(x)\,\psi(x,y')\,\mathrm{div}D_d(x)\,dy'dx\qquad\text{as }\varepsilon\to 0.$$

For $J^2_{1,n}-J^2_{2,n}$, using the definition of $\widetilde{\Lambda}^{\varepsilon,d}_{n_l,l}$ and $\hat{\Lambda}^\varepsilon_{n_l}$, with $l=1,2$, the regularity of D and ψ, the boundedness of $\{w^\varepsilon\}$ in $W^{1,p}(\Omega)$, along with the properties of the covering of Ω by $\{\Omega^\varepsilon_n\}_{n=1}^{N_\varepsilon}$, we obtain

$$\sum_{n=1}^{N_\varepsilon}|J^2_{1,n}-J^2_{2,n}|\le C\varepsilon^{1-r}\sum_{j=1}^{d-1}\|\mathrm{div}D_j\|_{L^\infty(\Omega)}\|w^\varepsilon\|_{W^{1,p}(\Omega)}\|\psi\|_{C^1_0(\Omega\times Y')}\to 0$$

as $\varepsilon\to 0$ for $r\in[0,1)$. Combining the obtained results, we conclude that

$$\sum_{n=1}^{N_\varepsilon}(\mathcal{I}_{1,n}+\mathcal{I}_{2,n})\to-\int_{\Omega\times Y'}\big[w(x)D_d(x)\cdot\nabla_x\psi(x,y')+w(x)\psi(x,y')\mathrm{div}D_d(x)\big]dy'dx$$

as $\varepsilon \to 0$. The definition of $y^c_\varepsilon \cdot \nabla w$ implies

$$(y^\varepsilon_c \cdot \nabla w(x))(y',1) - (y^\varepsilon_c \cdot \nabla w(x))(y',0) = \sum_{n=1}^{N_\varepsilon} D_d(x^\varepsilon_n) \cdot \nabla w(x) \chi_{\Omega^\varepsilon_n}(x)$$

for $y' \in Y'$ and $x \in \Omega$. Taking the limit as $\varepsilon \to 0$ yields

$$\begin{aligned}\lim_{\varepsilon\to 0} \int_{\Omega\times Y'} \big[(y^\varepsilon_c \cdot \nabla w)(y^1) - (y^\varepsilon_c \cdot \nabla w)(y^0)\big]\, \psi \, dy'dx &= \int_{\Omega\times Y'} D_d(x) \cdot \nabla w \, \psi \, dy'dx \\ &= -\int_{\Omega\times Y'} w(x)\big[D_d(x)\cdot \nabla_x \psi(x,y') + \operatorname{div} D_d(x)\psi(x,y')\big] dy'dx.\end{aligned}$$

Then, using the convergence of $V^\varepsilon - y^c_\varepsilon \cdot \nabla w$ to $\widetilde{w}_1$ in $L^p(\Omega; W^{1,p}(Y))$, we obtain

$$\begin{aligned}\int_\Omega \int_{Y'} [\widetilde{w}_1(x,(y',1)) - \widetilde{w}_1(x,(y',0))]\psi(x,y')\, dy'dx = \lim_{\varepsilon\to 0}\int_\Omega\int_{Y'} \big[V^\varepsilon(x,(y',1)) \\ -(y^\varepsilon_c\cdot\nabla w)(x,(y',1)) - V^\varepsilon(x,(y',0)) + (y^\varepsilon_c \cdot \nabla w)(x,(y',0))\big]\, \psi(x,y')\, dy'dx = 0.\end{aligned}$$

Carrying out similar calculations for y_j with $j = 1, \dots, d-1$ yields the Y-periodicity of $\widetilde{w}_1$ and, hence, the Y_x-periodicity of w_1, defined by $w_1(x,y) = \widetilde{w}_1(x, D_x^{-1}y)$ for $x \in \Omega$ and $y \in D_x Y$. $\square$

6. Micro-macro decomposition: The interpolation operator $\mathcal{Q}^\varepsilon_{\mathcal{L}}$. Similar to the periodic case [19, 18], in the context of convergence results for the unfolding method in perforated domains as well as for the derivation of error estimates [28, 29, 30, 31, 44], it is important to consider micro-macro decomposition of a function in $W^{1,p}$ and to introduce an interpolation operator $\mathcal{Q}^\varepsilon_{\mathcal{L}}$. For any $\varphi \in W^{1,p}(\Omega)$ we consider the splitting $\varphi = \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi) + \mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi)$ and show that $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)$ has a behavior similar to that of φ, whereas $\mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi)$ is of order ε.

We consider a continuous extension operator $\mathcal{P} : W^{1,p}(\Omega) \to W^{1,p}(\mathbb{R}^d)$ satisfying

$$\|\mathcal{P}(\varphi)\|_{W^{1,p}(\mathbb{R}^d)} \le C\|\varphi\|_{W^{1,p}(\Omega)} \qquad \text{for all} \quad \phi \in W^{1,p}(\Omega),$$

where the constant C depends only on p and Ω; see, e.g., [26]. In the following we use the same notation for a function in $W^{1,p}(\Omega)$ and its continuous extension into $\mathbb{R}^d$.

We consider a bounded Lipschitz domain $\Omega_1 \subset \mathbb{R}^d$, such that $\Omega \subset \Omega_1$, $\operatorname{dist}(\partial\Omega, \partial\Omega_1) \ge 2\varepsilon^r$, and $\Omega_1 \subset \bigcup_{n=1}^{N_{\varepsilon,1}} \overline{\Omega^\varepsilon_n}$, where Ω^ε_n as in section 2, and identify $N_{\varepsilon,1}$ with N_ε.

We consider $\mathcal{Y} = \operatorname{Int}(\bigcup_{k\in\{0,1\}^d}(\overline{Y}+k))$ and define

$$\Omega^\varepsilon_{\mathcal{Y}} = \operatorname{Int}\Bigg(\bigcup_{n=1}^{N_\varepsilon} \overline{\Omega^\varepsilon_{n,\mathcal{Y}}}\Bigg), \qquad \text{with} \quad \Omega^\varepsilon_{n,\mathcal{Y}} = \operatorname{Int}\Bigg(\bigcup_{\xi\in\Xi^\varepsilon_{n,\mathcal{Y}}} \varepsilon D_{x^\varepsilon_n}(\overline{Y}+\xi)\Bigg),$$
$$\Lambda^\varepsilon_{\mathcal{Y}} = \Omega \setminus \Omega^\varepsilon_{\mathcal{Y}},$$

where $\Xi^\varepsilon_{n,\mathcal{Y}} = \{\xi \in \Xi^\varepsilon_n : \varepsilon D_{x^\varepsilon_n}(\mathcal{Y}+\xi) \subset (\Omega^\varepsilon_n \cap \Omega_1)\}$.

In order to define an interpolation between two neighboring Ω^ε_n and Ω^ε_m we introduce $\mathcal{Y}^- = \operatorname{Int}(\bigcup_{k\in\{0,1\}^d}(\overline{Y}-k))$.

For $1 \le n \le N_\varepsilon$ and $m \in Z_n = \{1 \le m \le N_\varepsilon : \partial\Omega^\varepsilon_n \cap \partial\Omega^\varepsilon_m \ne \emptyset\}$ we shall consider unit cells near the corresponding neighboring parts of the boundaries $\partial\hat{\Omega}^\varepsilon_n$ and $\partial\hat{\Omega}^\varepsilon_m$, respectively. For $\xi_n \in \bar{\Xi}^\varepsilon_n$, where $\bar{\Xi}^\varepsilon_n = \{\xi \in \hat{\Xi}^\varepsilon_n : \varepsilon D_{x^\varepsilon_n}(\overline{Y}+\xi) \cap \partial\hat{\Omega}^\varepsilon_n \ne \emptyset\}$, we consider

$$\widetilde{\Xi}^\varepsilon_{n,m} = \big\{\xi_m \in \bar{\Xi}^\varepsilon_m : \ \varepsilon D_{x^\varepsilon_n}(\mathcal{Y}+\xi_n) \cap \varepsilon D_{x^\varepsilon_m}(\mathcal{Y}^- + \xi_m) \ne \emptyset\big\}$$

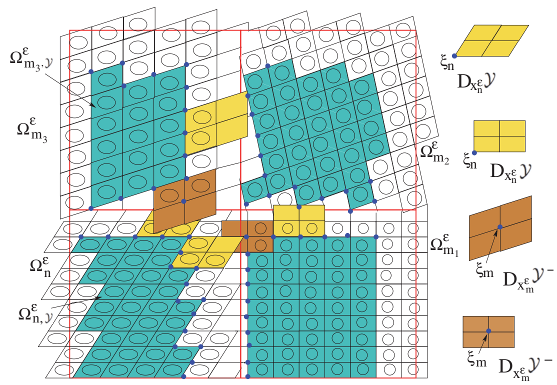

FIG. 4. *Schematic diagram of the covering of Ω by Ω^ε_n, of $D_{x^\varepsilon_n}\mathcal{Y}$ and $D_{x^\varepsilon_m}\mathcal{Y}^-$, and of the interpolation points ξ_n and ξ_m for $\mathcal{Q}^\varepsilon_{\mathcal{L}}$ and $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}$.*

and

$$\hat{K}_n = \{k \in \{0,1\}^d : \ \xi_n + k \in \bar{\Xi}^\varepsilon_n\}, \qquad \hat{K}^-_m = \{k \in \{0,1\}^d : \ \xi_m - k \in \bar{\Xi}^\varepsilon_m\}.$$

One of the important parts in the definition of $\mathcal{Q}^\varepsilon_{\mathcal{L}}$ is to define an interpolation between neighboring Ω^ε_n and Ω^ε_m. For two neighboring Ω^ε_n and Ω^ε_m we consider triangular interpolations between such vertices of $\varepsilon D_{x^\varepsilon_n}(Y+\xi_n)$ and $\varepsilon D_{x^\varepsilon_m}(Y+\xi_m)$ that are lying on $\partial\Omega^\varepsilon_{n,\mathcal{Y}}$ and $\partial\Omega^\varepsilon_{m,\mathcal{Y}}$, respectively.

DEFINITION 6.1. *The operator $\mathcal{Q}^\varepsilon_{\mathcal{L}} : L^p(\Omega) \to W^{1,\infty}(\Omega)$, for $p \in [1,+\infty]$, is defined by*

$$\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi) = \fint_Y \varphi(D_{x^\varepsilon_n}(\varepsilon\xi + \varepsilon y))dy \qquad \text{for } \xi \in \Xi^\varepsilon_n \text{ and } 1 \le n \le N_\varepsilon, \tag{6.1}$$

and for $x \in \Omega^\varepsilon_{n,\mathcal{Y}} \cap \Omega$ we define $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(x)$ as the Q_1-interpolant of $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi)$ at the vertices of $\varepsilon[D^{-1}_{x^\varepsilon_n} x/\varepsilon]_Y + \varepsilon Y$, where $1 \le n \le N_\varepsilon$.

For $x \in \Lambda^\varepsilon_{\mathcal{Y}}$ we define $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(x)$ as a triangular Q_1-interpolant of the values of $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi)$ at $\xi_n + k_n$ and ξ_m such that $\xi_n \in \bar{\Xi}^\varepsilon_n$, $\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}$ for $m \in Z_n$, and $k_n \in \hat{K}_n$, where $1 \le n \le N_\varepsilon$ and $\Omega^\varepsilon_n \cap \Omega \ne \emptyset$ or $\Omega^\varepsilon_m \cap \Omega \ne \emptyset$.

The vertices of $\varepsilon D_{x^\varepsilon_n}(Y + \xi_n + k_n)$ and $\varepsilon D_{x^\varepsilon_m}(Y+\xi_m)$ for $\xi_n \in \bar{\Xi}^\varepsilon_n$, $\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}$, and $k_n \in \hat{K}_n$, in the definition of $\mathcal{Q}^\varepsilon_{\mathcal{L}}$, belong to $\partial\Omega^\varepsilon_{n,\mathcal{Y}}$ and $\partial\Omega^\varepsilon_{m,\mathcal{Y}}$; see Figure 4.

For $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)$ and $\mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi) = \varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)$ we have the following estimates.

LEMMA 6.2. *For every $\varphi \in W^{1,p}(\Omega)$, where $p \in [1,+\infty)$, we have*

$$\begin{aligned} &\|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\Omega)} \le C\|\varphi\|_{L^p(\Omega)}, \qquad \|\mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\Omega)} \le C\varepsilon\|\nabla\varphi\|_{L^p(\Omega)}, \\ &\|\nabla\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\Omega)} + \|\nabla\mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\Omega)} \le C\|\nabla\varphi\|_{L^p(\Omega)}, \end{aligned} \tag{6.2}$$

where the constant C is independent of ε and depends only on Y, D, and $d = \dim(\Omega)$.

Proof. Similar to the periodic case [19], we use the fact that the space of Q_1-interpolants is a finite-dimensional space of dimension 2^d and all norms are equivalent. Then, for $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$, where $n = 1, \dots, N_\varepsilon$, we obtain

$$\|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|^p_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))} \le C_1 \varepsilon^d \sum_{k\in\{0,1\}^d} \left|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi + \varepsilon k)\right|^p. \tag{6.3}$$

For $\xi_n \in \bar{\Xi}^\varepsilon_n$ and triangular elements $\omega^\varepsilon_{\xi_{n,m}}$ between $\Omega^\varepsilon_{n,\mathcal{Y}}$ and $\Omega^\varepsilon_{m,\mathcal{Y}}$, $m \in Z_n$, it holds that

$$\|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|^p_{L^p(\omega^\varepsilon_{\xi_{n,m}})} \leq C_2 \varepsilon^d \sum_{k\in \hat{K}_n, m\in Z_n} \sum_{\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}} \Big[\big|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_n + \varepsilon k)\big|^p + \big|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon \xi_m)\big|^p\Big],$$

where $|Z_n| \leq 2^d$ and $|\widetilde{\Xi}^\varepsilon_{n,m}| \leq 2^{2d}$ for every $n = 1, \dots, N_\varepsilon$. Thus, for $\Lambda^\varepsilon_{\mathcal{Y}}$ it holds that

$$\begin{aligned} &\|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|^p_{L^p(\Lambda^\varepsilon_{\mathcal{Y}})} \\ &\leq C_3 \varepsilon^d \sum_{n=1}^{N_\varepsilon} \sum_{\xi_n \in \bar{\Xi}^\varepsilon_n, k \in \hat{K}_n} \sum_{m\in Z_n, \xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}} \Big[\big|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_n + \varepsilon k)\big|^p + \big|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon \xi_m)\big|^p\Big]. \end{aligned} \tag{6.4}$$

From the definition of $\mathcal{Q}^\varepsilon_{\mathcal{L}}$ it follows that

$$|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi)|^p \leq \fint_Y |\varphi(\varepsilon D_{x^\varepsilon_n}(\xi + y))|^p dy = \frac{1}{\varepsilon^d |D_{x^\varepsilon_n} Y|} \int_{\varepsilon D_{x^\varepsilon_n}(\xi+Y)} |\varphi(x)|^p dx$$

for $\xi \in \Xi^\varepsilon_n$ and $n = 1, \dots, N_\varepsilon$. Then, using (6.3) and (6.4) implies

$$\|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|^p_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))} \leq C_4 \sum_{k\in\{0,1\}^d} \int_{\varepsilon D_{x^\varepsilon_n}(\xi + k + Y)} |\varphi(x)|^p dx \tag{6.5}$$

for $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$ and $n = 1, \dots, N_\varepsilon$, and in $\Lambda^\varepsilon_{\mathcal{Y}}$ we have

$$\|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|^p_{L^p(\Lambda^\varepsilon_{\mathcal{Y}})} \leq C_5 \sum_{n=1}^{N_\varepsilon} \sum_{m\in Z_n} \sum_{j=n,m} \sum_{\xi \in \bar{\Xi}^\varepsilon_j} \int_{\varepsilon D_{x^\varepsilon_j}(\xi+Y)} |\varphi(x)|^p dx. \tag{6.6}$$

Summing up in (6.5) over $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$ and $n = 1, \dots, N_\varepsilon$, and adding (6.6), we obtain the estimate for the L^p-norm of $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)$, as stated in the lemma.

From the definition of Q_1-interpolants we obtain that for $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$

$$\|\nabla \mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)\|^p_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))} \leq C\varepsilon^{d-p} \sum_{k\in\{0,1\}^d} |\mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)(\varepsilon\xi + \varepsilon k) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)(\varepsilon\xi)|^p. \tag{6.7}$$

For the triangular regions $\omega^\varepsilon_{\xi_{n,m}}$ between neighboring $\Omega^\varepsilon_{n,\mathcal{Y}}$ and $\Omega^\varepsilon_{m,\mathcal{Y}}$ we have

$$\begin{aligned} \|\nabla \mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)\|^p_{L^p(\omega^\varepsilon_{\xi_{n,m}})} \leq C\varepsilon^{d-p} \sum_{\substack{m\in Z_n \\ \xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}}} \sum_{k_n \in \hat{K}_n, k_m \in \hat{K}^-_m} \Big[&|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)(\varepsilon(\xi_n + k_n)) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)(\varepsilon\xi_n)|^p \\ + |\mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)(\varepsilon(\xi_n + k_n)) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)(\varepsilon(\xi_m - k_m))|^p &+ |\mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)(\varepsilon(\xi_m - k_m)) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)(\varepsilon\xi_m)|^p \Big]. \end{aligned}$$

For $\phi \in W^{1,p}(D_{x^\varepsilon_n} Y)$ (and $W^{1,p}(D_{x^\varepsilon_n}\mathcal{Y})$, $W^{1,p}(D_{x^\varepsilon_n}\mathcal{Y}^-)$), using the regularity of D

and the Poincaré inequality, we obtain

$$
\begin{aligned}
\Big\| \phi - \fint_{D_{x_n^\varepsilon} Y} \phi \, dy \Big\|_{L^p(D_{x_n^\varepsilon} Y)} &\le C \|\nabla_y \phi\|_{L^p(D_{x_n^\varepsilon} Y)}, \\
\Big\| \phi - \fint_{D_{x_n^\varepsilon} Y} \phi \, dy \Big\|_{L^p(D_{x_n^\varepsilon} \mathcal{Y})} &\le C \|\nabla_y \phi\|_{L^p(D_{x_n^\varepsilon} \mathcal{Y})}, \\
\Big| \fint_{D_{x_n^\varepsilon} Y} \phi \, dy - \fint_{D_{x_n^\varepsilon} \mathcal{Y}} \phi \, dy \Big|^p + \Big| \fint_{D_{x_n^\varepsilon} (Y+k)} \phi \, dy - \fint_{D_{x_n^\varepsilon} \mathcal{Y}} \phi \, dy \Big|^p &\le C \|\nabla_y \phi\|^p_{L^p(D_{x_n^\varepsilon} \mathcal{Y})}, \\
\Big| \fint_{D_{x_n^\varepsilon} Y} \phi \, dy - \fint_{D_{x_n^\varepsilon} \mathcal{Y}^-} \phi \, dy \Big|^p + \Big| \fint_{D_{x_n^\varepsilon} (Y-k)} \phi \, dy - \fint_{D_{x_n^\varepsilon} \mathcal{Y}^-} \phi \, dy \Big|^p &\le C \|\nabla_y \phi\|^p_{L^p(D_{x_n^\varepsilon} \mathcal{Y}^-)},
\end{aligned} \tag{6.8}
$$

where $1 \le n \le N_\varepsilon$, $k \in \{0,1\}^d$, and the constant C depends on D and is independent of ε and n. Using a scaling argument, we obtain for every $\xi \in \Xi_n^\varepsilon$

$$
\Big\| \phi - \fint_{\varepsilon D_{x_n^\varepsilon}(\xi+Y)} \phi \, dx \Big\|_{L^p(\varepsilon D_{x_n^\varepsilon}(\xi+Y))} \le C\varepsilon \|\nabla \phi\|_{L^p(\varepsilon D_{x_n^\varepsilon}(\xi+Y))}. \tag{6.9}
$$

Hence, for $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$ and $k \in \{0,1\}^d$ as well as for $\xi_j \in \bar{\Xi}^\varepsilon_j$, with $j = n, m$ and $k_n \in \hat{K}_n$, $k_m \in \hat{K}_m^-$, using a scaling argument in (6.8), we have

$$
\begin{aligned}
|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi + \varepsilon k) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi)|^p &= \Big| \fint_{Y+k} \varphi(\varepsilon D_{x_n^\varepsilon}(\xi+y)) dy - \fint_Y \varphi(\varepsilon D_{x_n^\varepsilon}(\xi+y)) dy \Big|^p \\
&\le C\varepsilon^{p-d} \|\nabla \varphi\|^p_{L^p(\varepsilon D_{x_n^\varepsilon}(\xi+\mathcal{Y}))}, \\
|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_n + \varepsilon k_n) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_n)|^p &\le C\varepsilon^{p-d} \|\nabla \varphi\|^p_{L^p(\varepsilon D_{x_n^\varepsilon}(\xi_n+\mathcal{Y}))}, \\
|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_m - \varepsilon k_m) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_m)|^p &\le C\varepsilon^{p-d} \|\nabla \varphi\|^p_{L^p(\varepsilon D_{x_m^\varepsilon}(\xi_m+\mathcal{Y}^-))},
\end{aligned} \tag{6.10}
$$

where C depends on D and is independent of ε, n, and m.

For $\xi_n \in \bar{\Xi}^\varepsilon_n$, $\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}$ and $k_n \in \hat{K}_n$, $k_m \in \hat{K}_m^-$, using that $\varepsilon D_{x_m^\varepsilon}(\xi_m + \mathcal{Y}^-) \cap \varepsilon D_{x_n^\varepsilon}(\xi_n + \mathcal{Y}) \neq \emptyset$ and applying the inequalities (6.8) with a connected domain

$$
\widetilde{\mathcal{Y}}_{\xi_n} = \bigcup_{m \in Z_n, \xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}} \bigcup_{k \in \{0,1\}^d} D_{x_m^\varepsilon}(\xi_m + \mathcal{Y}^- + k) \cup D_{x_n^\varepsilon}(\xi_n + \mathcal{Y} - k),
$$

instead of $\mathcal{Y}$ and $\mathcal{Y}^-$, together with a scaling argument, yields

$$
\begin{aligned}
|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_n + \varepsilon k_n) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_m - \varepsilon k_m)|^p &\le \Big| \fint_{D_{x_n^\varepsilon}(\xi_n+Y+k_n)} \varphi(\varepsilon y) dy - \fint_{\widetilde{\mathcal{Y}}_{\xi_n}} \varphi(\varepsilon y) dy \Big|^p \\
&+ \Big| \fint_{D_{x_m^\varepsilon}(\xi_m+Y-k_m)} \varphi(\varepsilon y) dy - \fint_{\widetilde{\mathcal{Y}}_{\xi_n}} \varphi(\varepsilon y) dy \Big|^p \le C\varepsilon^{p-d} \|\nabla\varphi\|^p_{L^p(\varepsilon\widetilde{\mathcal{Y}}_{\xi_n})},
\end{aligned} \tag{6.11}
$$

where C depends on D and is independent of ε, n, and m. Thus, using (6.11) and the last two estimates in (6.10), we obtain

$$
\|\nabla \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|^p_{L^p(\Lambda^\varepsilon_{\mathcal{Y}})} \le C_1 \sum_{n=1}^{N_\varepsilon} \sum_{\substack{\xi_n \in \bar{\Xi}^\varepsilon_n \\ m \in Z_n}} \sum_{\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}} \|\nabla\varphi\|^p_{L^p(\varepsilon\widetilde{\mathcal{Y}}_{\xi_n})} \le C_2 \|\nabla\varphi\|^p_{L^p(\Omega)}. \tag{6.12}
$$

Applying (6.10) in (6.7), summing up over $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$ and $n = 1, \dots, N_\varepsilon$, and combining with the estimate for $\|\nabla \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\Lambda^\varepsilon_{\mathcal{Y}})}$ in (6.12), we obtain the estimate for $\|\nabla \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\Omega)}$ in terms of $\|\nabla \varphi\|_{L^p(\Omega)}$, as stated in the lemma.

To show the estimates for $\mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi)$ first we consider

$$\begin{aligned}\|\varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))} \leq \|\varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))}\\ + \|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))}\end{aligned}$$

for $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$. Using the definition of $\mathcal{Q}^\varepsilon_{\mathcal{L}}$ and (6.9), we obtain

$$\|\varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))} \leq C\varepsilon \|\nabla\varphi\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))} \quad \text{for } \ \xi \in \Xi^\varepsilon_{n,\mathcal{Y}}.$$

The definition of $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)$ and the properties of Q_1-interpolants along with (6.10) imply

$$\|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))} \leq C\varepsilon \|\nabla\varphi\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+\mathcal{Y}))} \quad \text{for } \xi \in \Xi^\varepsilon_{n,\mathcal{Y}}.$$

For triangular elements $\omega^\varepsilon_{\xi_{n,m}} \subset \Lambda^\varepsilon_{\mathcal{Y}}$ with $\xi_n \in \bar{\Xi}^\varepsilon_n$ and $\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}$ we have $\omega^\varepsilon_{\xi_{n,m}} \subset \varepsilon\widetilde{\mathcal{Y}}_{\xi_n}$. Then, the second inequality in (6.8) with $\widetilde{\mathcal{Y}}_{\xi_n}$ and a scaling argument yield

$$\|\varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_n)\|_{L^p(\omega^\varepsilon_{\xi_{n,m}})} \leq \|\varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_n)\|_{L^p(\varepsilon\widetilde{\mathcal{Y}}_{\xi_n})} \leq C\varepsilon\|\nabla\varphi\|_{L^p(\varepsilon\widetilde{\mathcal{Y}}_{\xi_n})},$$

whereas (6.10) and (6.11) together with the properties of Q_1-interpolants ensure

$$\|\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi) - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)(\varepsilon\xi_n)\|_{L^p(\omega^\varepsilon_{\xi_{n,m}})} \leq C\varepsilon\|\nabla\varphi\|_{L^p(\varepsilon\widetilde{\mathcal{Y}}_{\xi_n})}.$$

Thus, combining the estimates from above, we obtain

$$\begin{aligned}\|\mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\Omega)} &\leq \sum_{n=1}^{N_\varepsilon} \|\varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\Omega^\varepsilon_n)} \leq \sum_{n=1}^{N_\varepsilon} \sum_{\xi\in\Xi^\varepsilon_{n,\mathcal{Y}}} \|\varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\xi+Y))}\\ &+ \sum_{n=1}^{N_\varepsilon} \sum_{\xi_n\in\bar{\Xi}^\varepsilon_n, m\in Z_n} \sum_{\xi_m\in\tilde{\Xi}^\varepsilon_{n,m}} \|\varphi - \mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)\|_{L^p(\omega^\varepsilon_{\xi_{n,m}})} \leq C\varepsilon\|\nabla\varphi\|_{L^p(\Omega)}.\end{aligned}$$

Then the estimate for $\nabla\mathcal{Q}^\varepsilon_{\mathcal{L}}(\varphi)$ and the definition of $\mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi)$ yield the estimate for $\nabla\mathcal{R}^\varepsilon_{\mathcal{L}}(\varphi)$. ∎

To show convergence results for sequences obtained by applying the l-p unfolding operator to sequences of functions defined on l-p perforated domains, we have to introduce the interpolation operator $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}$ for functions in $L^p(\Omega^*_\varepsilon)$. We define

$$\hat{\Omega}^*_\varepsilon = \operatorname{Int}\left(\bigcup_{n=1}^{N_\varepsilon} \hat{\Omega}^{*,\varepsilon}_n\right), \qquad \Lambda^*_\varepsilon = \Omega^*_\varepsilon \setminus \hat{\Omega}^*_\varepsilon, \quad \text{ where } \ \hat{\Omega}^{*,\varepsilon}_n = \bigcup_{\xi\in\hat{\Xi}^\varepsilon_n} \varepsilon D_{x^\varepsilon_n}(\overline{Y^*} + \xi),$$

and

$$\Omega^*_{\varepsilon,\mathcal{Y}} = \operatorname{Int}\left(\bigcup_{n=1}^{N_\varepsilon} \overline{\Omega^{*,\varepsilon}_{n,\mathcal{Y}}}\right),\ \Lambda^*_{\varepsilon,\mathcal{Y}} = \Omega^*_\varepsilon \setminus \Omega^*_{\varepsilon,\mathcal{Y}},\ \text{where } \Omega^{*,\varepsilon}_{n,\mathcal{Y}} = \operatorname{Int}\left(\bigcup_{\xi\in\Xi^\varepsilon_{n,\mathcal{Y}}} \varepsilon D_{x^\varepsilon_n}(\overline{Y^*}+\xi)\right),$$

with Ω instead of Ω_1 in the definition of $\Xi^\varepsilon_{n,\mathcal{Y}}$, as well as $\widetilde{\Omega}^*_\varepsilon = \Omega^*_\varepsilon \cap \widetilde{\Omega}_\varepsilon$, where $\widetilde{\Omega}_\varepsilon$ is defined as

$$\widetilde{\Omega}_\varepsilon = \Big\{x \in \Omega : \ \operatorname{dist}(x, \partial\Omega) > 4\varepsilon \max_{x\in\partial\Omega} \operatorname{diam}(D(x)Y)\Big\}. \tag{6.13}$$

We also consider $\mathcal{Y}^* = \mathrm{Int}(\bigcup_{k\in\{0,1\}^d}(\overline{Y}^* + k))$ and $\mathcal{Y}^{*,-} = \mathrm{Int}(\bigcup_{k\in\{0,1\}^d}(\overline{Y}^* - k))$.

Similar to $\mathcal{Q}^\varepsilon_{\mathcal{L}}$, in the definition of the interpolation operator $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}$ we shall distinguish between $\Omega^\varepsilon_{\mathcal{Y}}$ and $\Lambda^\varepsilon_{\mathcal{Y}} \cap \widetilde{\Omega}_\varepsilon$. For $x \in \Omega^\varepsilon_{\mathcal{Y}}$ we can consider Q_1-interpolation between vertices of the corresponding unit cells, whereas for $x \in \Lambda^\varepsilon_{\mathcal{Y}} \cap \widetilde{\Omega}_\varepsilon$ we consider triangular Q_1-interpolation between vertices of unit cells in two neighboring Ω^ε_n and Ω^ε_m. This approach ensures that $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)$ is continuous in $\widetilde{\Omega}_\varepsilon$.

Definition 6.3. *The operator $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}} : L^p(\Omega^*_\varepsilon) \to W^{1,\infty}(\widetilde{\Omega}_\varepsilon)$, for $p \in [1,+\infty]$, is defined by*

$$\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi) = \fint_{Y^*} \phi(D_{x^\varepsilon_n}(\varepsilon\xi + \varepsilon y))dy \qquad \text{for } \xi \in \hat{\Xi}^\varepsilon_n \text{ and } n = 1,\dots,N_\varepsilon, \tag{6.14}$$

and for $x \in \Omega^\varepsilon_{n,\mathcal{Y}} \cap \widetilde{\Omega}_\varepsilon$ we define $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(x)$ as the Q_1-interpolant of the values of $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi)$ at vertices of $\varepsilon[D^{-1}_{x^\varepsilon_n} x/\varepsilon]_Y + \varepsilon Y$, where $1 \le n \le N_\varepsilon$.

For $x \in \Lambda^\varepsilon_{\mathcal{Y}} \cap \widetilde{\Omega}_\varepsilon$ we define $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(x)$ as a triangular Q_1-interpolant of the values of $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi)$ at $\xi_n + k_n$ and ξ_m such that $\xi_n \in \bar{\Xi}^{*,\varepsilon}_n = \{\xi \in \bar{\Xi}^\varepsilon_n : \varepsilon D_{x^\varepsilon_n}(Y+\xi) \cap \widetilde{\Omega}_{\varepsilon/2} \neq \emptyset\}$, $\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}$ for $m \in Z_n$, and $k_n \in \hat{K}_n$, where $1 \le n \le N_\varepsilon$; see Figure 4.

In a similar way as for $\mathcal{Q}^\varepsilon_{\mathcal{L}}(\phi)$ and $\mathcal{R}^\varepsilon_{\mathcal{L}}(\phi)$ we obtain estimates for $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)$ and $\mathcal{R}^{*,\varepsilon}_{\mathcal{L}}(\phi) = \phi - \mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)$.

Lemma 6.4. *For every $\phi \in W^{1,p}(\Omega^*_\varepsilon)$, where $p \in [1,+\infty)$, we have*

$$\begin{aligned}
&\|\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C\|\phi\|_{L^p(\Omega^*_\varepsilon)}, &&\|\nabla\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)\|_{L^p(\widetilde{\Omega}_\varepsilon)} \le C\|\nabla\phi\|_{L^p(\Omega^*_\varepsilon)},\\
&\|\mathcal{R}^{*,\varepsilon}_{\mathcal{L}}(\phi)\|_{L^p(\widetilde{\Omega}^*_\varepsilon)} \le C\varepsilon\|\nabla\phi\|_{L^p(\Omega^*_\varepsilon)}, &&\|\nabla\mathcal{R}^{*,\varepsilon}_{\mathcal{L}}(\phi)\|_{L^p(\widetilde{\Omega}^*_\varepsilon)} \le C\|\nabla\phi\|_{L^p(\Omega^*_\varepsilon)},
\end{aligned}$$

where the constant C is independent of ε.

Proof. The proof for the first estimate follows along the same lines as the proof of the corresponding estimate in Lemma 6.2. To show the estimates for $\nabla\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)$ and $\mathcal{R}^{*,\varepsilon}_{\mathcal{L}}(\phi)$ we have to estimate the differences $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi) - \mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi + k)$ for $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$ and $k \in \{0,1\}^d$, and $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi_n + \varepsilon k_n) - \mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi_m - \varepsilon k_m)$ for $\xi_n \in \bar{\Xi}^{*,\varepsilon}_n$ and $\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}$, with $m \in Z_n$, and $k_n \in \hat{K}_n$, $k_m \in \hat{K}^-_m$, where $1 \le n \le N_\varepsilon$.

As in the proof of Lemma 6.2, by considering the estimate (6.7), applying the Poincaré inequality, and using the estimates similar to (6.8) and (6.10), with Y^* and $\mathcal{Y}^*$ instead of Y and $\mathcal{Y}$, we obtain

$$\begin{aligned}
&\left|\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi) - \mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi + k)\right|^p \le C\varepsilon^{p-d}\|\nabla\phi\|^p_{L^p(\varepsilon D_{x^\varepsilon_n}(\mathcal{Y}^*+\xi))},\\
&\quad\|\nabla\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(Y+\xi))} \le C\|\nabla\phi\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\mathcal{Y}^*+\xi))},\\
&\quad\|\phi - \mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(Y^*+\xi))} \le \|\phi - \mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(Y^*+\xi))}\\
&\quad+ \|\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi) - \mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi)\|_{L^p(\varepsilon D_{x^\varepsilon_n}(Y+\xi))} \le C\varepsilon\|\nabla\phi\|_{L^p(\varepsilon D_{x^\varepsilon_n}(\mathcal{Y}^*+\xi))}
\end{aligned} \tag{6.15}$$

for $\xi \in \Xi^\varepsilon_{n,\mathcal{Y}}$ and $n = 1,\dots,N_\varepsilon$. For $\xi_n \in \bar{\Xi}^{*,\varepsilon}_n$ and $\xi_m \in \widetilde{\Xi}^\varepsilon_{n,m}$, with $m \in Z_n$, we consider $\varepsilon D_{x^\varepsilon_j}(Y_0 + \xi)$ for such $\varepsilon D_{x^\varepsilon_j}(Y + \xi)$, with $\xi \in \hat{\Xi}^\varepsilon_j$, that have possible nonempty intersections with a triangular element $\omega^\varepsilon_{\xi_{n,m}}$ between neighboring $\Omega^{*,\varepsilon}_{n,\mathcal{Y}}$ and $\Omega^{*,\varepsilon}_{m,\mathcal{Y}}$, i.e.,

$$\hat{\mathcal{Y}}^0_{\xi_n} = \bigcup_{\substack{k_n^+\in \widetilde{K}_n\\ k\in \widetilde{K}_n^-}} \bigcup_{\substack{l\in Z_n, \xi_l \in \widetilde{\Xi}^{\varepsilon,+}_{l,n}\\ k_l\in\widetilde{K}_l}} D_{x^\varepsilon_n}(\overline{Y}_0+\xi_n+k_n^+-k)\cup D_{x^\varepsilon_l}(\overline{Y}_0+\xi_l+k_l),$$
$$\hat{\mathcal{Y}}^{0,-}_{\xi_n} = \bigcup_{\substack{m\in Z_n, \xi_m\in\widetilde{\Xi}^\varepsilon_{n,m}\\ k_m^-\in\widetilde{K}^-_m, k\in\widetilde{K}_m}} \bigcup_{\substack{l\in Z_m, \xi_l\in\widetilde{\Xi}^{\varepsilon,+}_{l,m}\\ k_l\in\widetilde{K}_l}} D_{x^\varepsilon_m}(\overline{Y}_0+\xi_m-k_m^-+k)\cup D_{x^\varepsilon_l}(\overline{Y}_0+\xi_l+k_l),$$
$$\hat{\mathcal{Y}}^{0,+}_{\xi_n} = \bigcup_{m\in Z_n, \xi_m\in\widetilde{\Xi}^\varepsilon_{n,m}} \bigcup_{s\in Z_m, \xi_s\in\widetilde{\Xi}^\varepsilon_{m,s}} \bigcup_{k\in\widetilde{K}_s^-} D_{x^\varepsilon_s}(\overline{Y}_0+\xi_s-k),$$

where $\widetilde{K}_n^- = \{k\in\{0,1\}^d: \ \xi_n - k\in\hat{\Xi}^\varepsilon_n\}$, $\widetilde{K}_m=\{k\in\{0,1\}^d: \ \xi_m+k\in\hat{\Xi}^\varepsilon_m\}$, and $\widetilde{\Xi}^{\varepsilon,+}_{l,n} = \big\{\xi_l\in\bar{\Xi}^\varepsilon_l: \ \varepsilon D_{x^\varepsilon_l}(\mathcal{Y}+\xi_l)\cap\varepsilon D_{x^\varepsilon_n}(\mathcal{Y}^-+\xi_n)\neq\emptyset\big\}$, assemble a set of such cells $\varepsilon D_{x^\varepsilon_n}(Y+\xi)$ and $\varepsilon D_{x^\varepsilon_m}(Y+\xi)$ that have possible nonempty intersections with $\omega^\varepsilon_{\xi_{n,m}}$, i.e.,

$$\hat{\mathcal{Y}}_{\xi_n} = \bigcup_{m\in Z_n, \xi_m\in\widetilde{\Xi}^\varepsilon_{n,m}} \bigcup_{k\in\{0,1\}^d} D_{x^\varepsilon_m}(\overline{\mathcal{Y}^-}+\xi_m+k)\cup D_{x^\varepsilon_n}(\overline{\mathcal{Y}}+\xi_n-k),$$

and define $\widetilde{\mathcal{Y}}^*_{\xi_n} = \mathrm{Int}(\hat{\mathcal{Y}}_{\xi_n}\setminus(\hat{\mathcal{Y}}^0_{\xi_n}\cup\hat{\mathcal{Y}}^{0,-}_{\xi_n}\cup\hat{\mathcal{Y}}^{0,+}_{\xi_n}))$. We have that $\widetilde{\mathcal{Y}}^*_{\xi_n}$ is connected and $\varepsilon\widetilde{\mathcal{Y}}^*_{\xi_n}\subset\Omega^*_\varepsilon$ for all $\xi_n\in\bar{\Xi}^{*,\varepsilon}_n$ and $n=1,\dots,N_\varepsilon$. Applying the Poincaré inequality in $\widetilde{\mathcal{Y}}^*_{\xi_n}$ and using the regularity of D yields

$$\begin{aligned}
&\Big|\fint_{D_{x^\varepsilon_n}(Y^*+\xi_n+k_n)}\phi(y)dy - \fint_{\widetilde{\mathcal{Y}}^*_{\xi_n}}\phi(y)dy\Big|^p \le C\int_{\widetilde{\mathcal{Y}}^*_{\xi_n}}|\nabla_y\phi(y)|^p dy,\\
&\Big|\fint_{D_{x^\varepsilon_m}(Y^*+\xi_m-k_m)}\phi(y)dy - \fint_{\widetilde{\mathcal{Y}}^*_{\xi_n}}\phi(y)dy\Big|^p \le C\int_{\widetilde{\mathcal{Y}}^*_{\xi_n}}|\nabla_y\phi(y)|^p dy,\\
&\Big\|\phi-\fint_{D_{x^\varepsilon_n}(Y^*+\xi_n)}\phi(y)dy\Big\|_{L^p(\widetilde{\mathcal{Y}}^*_{\xi_n})}\le C\|\nabla_y\phi\|_{L^p(\widetilde{\mathcal{Y}}^*_{\xi_n})}
\end{aligned}\tag{6.16}$$

for $\xi_n\in\bar{\Xi}^{*,\varepsilon}_n$, $\xi_m\in\widetilde{\Xi}^\varepsilon_{n,m}$, with $m\in Z_n$, and $k_n\in\hat{K}_n$, $k_m\in\hat{K}^-_m$, where the constant C depends on D and is independent of ε, n, and m. Then, using a scaling argument in (6.16) implies

$$\begin{aligned}
&\big|\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi_n+\varepsilon k_n)-\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi_n)\big|^p + \big|\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi_m-\varepsilon k_m)-\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi_m)\big|^p\\
&+\big|\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi_n+\varepsilon k_n)-\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)(\varepsilon\xi_m-\varepsilon k_m)\big|^p\le C\varepsilon^{p-d}\|\nabla\phi\|^p_{L^p(\varepsilon\widetilde{\mathcal{Y}}^*_{\xi_n})}
\end{aligned}\tag{6.17}$$

for $\xi_n\in\bar{\Xi}^{*,\varepsilon}_n$, $\xi_m\in\widetilde{\Xi}^\varepsilon_{n,m}$, with $m\in Z_n$, and $k_n\in\hat{K}_n$, $k_m\in\hat{K}^-_m$. Hence, taking into account that $|Z_n|\le 2^d$ and $|\widetilde{\Xi}^\varepsilon_{n,m}|\le 2^{2d}$, we obtain

$$\|\nabla\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(\phi)\|^p_{L^p(\Lambda^\varepsilon_{\mathcal{Y}}\cap\widetilde{\Omega}_\varepsilon)}\le C_1\sum_{n=1}^{N_\varepsilon}\sum_{\xi_n\in\bar{\Xi}^{*,\varepsilon}_n}\|\nabla\phi\|^p_{L^p(\varepsilon\widetilde{\mathcal{Y}}^*_{\xi_n})}\le C_2\|\nabla\phi\|^p_{L^p(\Omega^*_\varepsilon)}.\tag{6.18}$$

Applying a scaling argument in (6.16) and using the properties of Q_1-interpolants and the estimate (6.17) yields

$$\|\phi - \mathcal{Q}_{\mathcal{L}}^{*,\varepsilon}(\phi)\|_{L^p(\Lambda^*_{\varepsilon,\mathcal{Y}}\cap\widetilde{\Omega}_\varepsilon)} \le \sum_{n=1}^{N_\varepsilon}\sum_{\xi_n\in\bar{\Xi}_n^{*,\varepsilon}}\Big[\big\|\phi - \mathcal{Q}_{\mathcal{L}}^{*,\varepsilon}(\phi)(\varepsilon\xi_n)\big\|_{L^p(\varepsilon\widetilde{\mathcal{Y}}^*_{\xi_n})}$$
$$+ \sum_{m\in Z_n,\xi_m\in\widetilde{\Xi}^\varepsilon_{n,m}}\big\|\mathcal{Q}_{\mathcal{L}}^{*,\varepsilon}(\phi)(\varepsilon\xi_n) - \mathcal{Q}_{\mathcal{L}}^{*,\varepsilon}(\phi)\big\|_{L^p(\omega^\varepsilon_{\xi_{n,m}})}\Big] \le C\varepsilon\|\nabla\phi\|_{L^p(\Omega^*_\varepsilon)}. \tag{6.19}$$

Summing in (6.15) over $\Xi^\varepsilon_{n,\mathcal{Y}}$ and $1\le n\le N_\varepsilon$, adding (6.18) or (6.19), respectively, and using the definition of $\mathcal{R}_{\mathcal{L}}^{*,\varepsilon}(\phi)$, we obtain the estimates stated in the lemma. ∎

7. The l-p unfolding operator in perforated domains: Proofs of convergence results. In this section we prove convergence results for the l-p unfolding operator in domains with l-p perforations. First, we show some properties of the l-p unfolding operator in perforated domains.

Lemma 7.1.

(i) $\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}$ *is linear and continuous from* $L^p(\Omega^*_\varepsilon)$ *to* $L^p(\Omega\times Y^*)$, *where* $p\in[1,+\infty)$, *and*

$$\|\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}(w)\|_{L^p(\Omega\times Y^*)} \le |Y|^{1/p}\|w\|_{L^p(\Omega^*_\varepsilon)}\ .$$

(ii) *For* $w\in L^p(\Omega)$, *with* $p\in[1,+\infty)$, $\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}(w)\to w$ *strongly in* $L^p(\Omega\times Y^*)$.

(iii) *Let* $w^\varepsilon\in L^p(\Omega^*_\varepsilon)$, *with* $p\in(1,+\infty)$, *such that* $\|w^\varepsilon\|_{L^p(\Omega^*_\varepsilon)}\le C$. *If*

$$\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}(w^\varepsilon)\rightharpoonup\hat{w}\quad\textit{weakly in } L^p(\Omega\times Y^*)\ ,$$

then

$$\widetilde{w}^\varepsilon\rightharpoonup\frac{1}{|Y|}\int_{Y^*}\hat{w}\,dy\quad\textit{weakly in } L^p(\Omega).$$

(iv) *For* $w\in L^p(\Omega;C_{\rm per}(Y^*_x))$ *we have* $\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}(\mathcal{L}^\varepsilon w)\to w(\cdot,D_x\cdot)$ *in* $L^p(\Omega\times Y^*)$, *where* $p\in[1,+\infty)$.

(v) *For* $w\in C(\overline{\Omega};L^p_{\rm per}(Y^*_x))$ *we have* $\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}(\mathcal{L}^\varepsilon_0 w)\to w(\cdot,D_x\cdot)$ *in* $L^p(\Omega\times Y^*)$, *where* $p\in[1,+\infty)$.

By $\widetilde{w}$ we denote the extension of w by zero from Ω^*_ε into Ω.

Sketch of the proof. The proof of (i) follows directly from the definition of $\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}$ and by using calculations similar to those in the proof of Lemma 5.1.

For $w_k\in C_0^\infty(\Omega)$ the convergence in (ii) results from the definition of $\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}$, the properties of the covering of Ω^*_ε by $\Omega^{*,\varepsilon}_n$, and the following simple calculations:

$$\lim_{\varepsilon\to 0}\int_{\Omega\times Y^*}|\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}(w_k)|^p dydx = \lim_{\varepsilon\to0}\left[\sum_{n=1}^{N_\varepsilon}|\hat{\Omega}^\varepsilon_n||Y^*||w_k(x^\varepsilon_n)|^p+\delta_\varepsilon\right] = \int_{\Omega\times Y^*}|w_k(x)|^p dydx.$$

We used the fact that $|\Lambda^\varepsilon|\to0$ as $\varepsilon\to0$ and, due to the continuity of w_k, we have

$$\delta^\varepsilon = \sum_{n=1}^{N_\varepsilon}\sum_{\xi\in\hat{\Xi}^\varepsilon_n}|Y|\int_{\varepsilon D_{x^\varepsilon_n}(\xi+Y^*)}|w_k(x)-w_k(x^\varepsilon_n)|^p\,dx\ \to 0 \quad\text{as}\quad \varepsilon\to0.$$

The approximation of $w\in L^p(\Omega)$ by $\{w_k\}\subset C_0^\infty(\Omega)$ and the estimate for the norm of $\mathcal{T}_{\mathcal{L}}^{*,\varepsilon}(w-w_k)$ in (i) yield the convergence for $w\in L^p(\Omega)$.

The proof of the convergence in (iii) is similar to the proof of Lemma 5.2 and the corresponding result for the periodic unfolding operator.

The proof of (iv) follows along the same lines as the proof of the corresponding result for $\mathcal{T}^\varepsilon_\mathcal{L}$ in Lemma 5.3. In a way similar to that in [46, Lemma 3.4], we obtain that

$$\lim_{\varepsilon\to 0}\int_{\Omega^*_\varepsilon} |\mathcal{L}^\varepsilon_0(w)(x)|^p dx = \int_\Omega \frac{1}{|Y_x|}\int_{Y^*_x} |w(x,y)|^p dy dx = \int_\Omega \frac{1}{|Y|}\int_{Y^*} |w(x, D_x\tilde{y})|^p d\tilde{y} dx,$$
$$\lim_{\varepsilon\to 0}\int_{\Lambda^*_\varepsilon} |\mathcal{L}^\varepsilon_0(w)(x)|^p dx = 0.$$

Then, the last two convergence results together with the equality

$$\lim_{\varepsilon\to 0}\int_{\Omega\times Y^*} |\mathcal{T}^{*,\varepsilon}_\mathcal{L}(\mathcal{L}^\varepsilon_0 w)|^p dy dx = |Y| \lim_{\varepsilon\to 0}\left[\int_{\Omega^*_\varepsilon} |\mathcal{L}^\varepsilon_0 w|^p dx - \int_{\Lambda^*_\varepsilon} |\mathcal{L}^\varepsilon_0 w|^p dx\right]$$

and the continuity of w with respect to x imply the convergence result stated in (v). ◻

Similar to $\mathcal{T}^\varepsilon_\mathcal{L}$ we have $\nabla_y \mathcal{T}^{*,\varepsilon}_\mathcal{L}(w) = \varepsilon \sum_{n=1}^{N_\varepsilon} D^T_{x^\varepsilon_n}\, \mathcal{T}^{*,\varepsilon}_\mathcal{L}(\nabla w)\chi_{\Omega^\varepsilon_n}$ for $w \in W^{1,p}(\Omega^*_\varepsilon)$. Using the definition and properties of $\mathcal{T}^{*,\varepsilon}_\mathcal{L}$, we prove convergence results for $\mathcal{T}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)$, $\varepsilon\mathcal{T}^{*,\varepsilon}_\mathcal{L}(\nabla w^\varepsilon)$, and $\mathcal{T}^{*,\varepsilon}_\mathcal{L}(\nabla w^\varepsilon)$. We start with the proof of Theorem 4.3. Here the proof of the weak convergence follows the same steps as for $\mathcal{T}^\varepsilon_\mathcal{L}$ in Theorem 4.1, whereas the periodicity of the limit-function is shown in a different way.

Proof of Theorem 4.3. The boundedness of $\{\mathcal{T}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)\}$ and $\{\nabla_y \mathcal{T}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)\}$, ensured by (4.1) and the regularity of D, imply the weak convergences in (4.2). To show the periodicity of w we consider for $\phi \in C^\infty_0(\Omega\times Y^*)$ and $j = 1,\dots,d$

$$\int_{\Omega\times Y^*} \mathcal{T}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)(x, \tilde{y}+e_j)\phi d\tilde{y} dx = \int_{\Omega\times Y^*} \sum_{n=1}^{N_\varepsilon} \mathcal{T}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)\phi(x - \varepsilon D_{x^\varepsilon_n} e_j, \tilde{y})\chi_{\widetilde{\Omega}^{\varepsilon,j}_n} d\tilde{y} dx$$
$$+ \sum_{n=1}^{N_\varepsilon} \int_{\widetilde{\Lambda}^{\varepsilon,j}_{n,1}\times Y^*} \mathcal{T}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)(x, \tilde{y}+e_j)\phi\, d\tilde{y} dx,$$

where $\widetilde{\Omega}^{\varepsilon,j}_n$ and $\widetilde{\Lambda}^{\varepsilon,j}_{n,l}$, with $l = 1, 2$, are defined in the proof of Theorem 4.1 in section 5. Considering the weak convergence of $\mathcal{T}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)$ along with $\sum_{n=1}^{N_\varepsilon} |\widetilde{\Lambda}^{\varepsilon,j}_{n,l}| \le C\varepsilon^{1-r}$, for $l = 1, 2$, and taking the limit as $\varepsilon \to 0$ implies

$$\int_{\Omega\times Y^*} w(x, D_x(\tilde{y}+e_j))\phi(x,\tilde{y}) d\tilde{y} dx = \int_{\Omega\times Y^*} w(x, D_x\tilde{y})\phi(x,\tilde{y}) d\tilde{y} dx$$

for all $\phi \in C^\infty_0(\Omega\times Y^*)$ and $j = 1,\dots,d$. Thus, we obtain that w is Y_x-periodic. ◻

Similar to the periodic case, we use the micro-macro decomposition of a function $\phi \in W^{1,p}(\Omega^*_\varepsilon)$, i.e., $\phi = \mathcal{Q}^{*,\varepsilon}_\mathcal{L}(\phi) + \mathcal{R}^{*,\varepsilon}_\mathcal{L}(\phi)$, to show the weak convergence of $\mathcal{T}^{*,\varepsilon}_\mathcal{L}(\nabla w^\varepsilon)$. Here we use the fact that for $\{w^\varepsilon\}$ bounded in $W^{1,p}(\Omega^*_\varepsilon)$, the sequence $\{\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)\}$ is bounded in $W^{1,p}(G)$ for every relatively compact open subset $G \subset \Omega$.

Notice that for $w^\varepsilon \in W^{1,p}(\Omega^*_\varepsilon)$ the function $\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)$ is defined on $\widetilde{\Omega}_\varepsilon$. Thus, we can apply $\mathcal{T}^\varepsilon_\mathcal{L}$ to $\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)$ and use the convergence results for the l-p unfolding operator $\mathcal{T}^\varepsilon_\mathcal{L}$ (shown in Theorems 4.1 and 4.2) to prove the weak convergence of $\mathcal{T}^\varepsilon_\mathcal{L}(\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)^\sim)$ and $\mathcal{T}^\varepsilon_\mathcal{L}([\nabla\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)]^\sim)$, where $^\sim$ denotes an extension by zero from $\widetilde{\Omega}_\varepsilon$ to Ω.

LEMMA 7.2. *Let* $\|w^\varepsilon\|_{W^{1,p}(\Omega^*_\varepsilon)} \le C$, *where* $p \in (1, +\infty)$. *Then there exist a* *subsequence (denoted again by $\{w^\varepsilon\}$) and a function $w \in W^{1,p}(\Omega)$ such that*

$$\begin{aligned} &\mathcal{T}^\varepsilon_\mathcal{L}(\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)^\sim) \to w && \textit{strongly in } L^p_{\rm loc}(\Omega; W^{1,p}(Y)),\\ &\mathcal{T}^\varepsilon_\mathcal{L}(\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)^\sim) \rightharpoonup w && \textit{weakly in } L^p(\Omega; W^{1,p}(Y)),\\ &\mathcal{T}^\varepsilon_\mathcal{L}([\nabla\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)]^\sim) \rightharpoonup \nabla w && \textit{weakly in } L^p(\Omega\times Y). \end{aligned}$$

Proof. Similar to the periodic case [18], the estimates for $\mathcal{Q}^{*,\varepsilon}_\mathcal{L}$ in Lemma 6.4 ensure that there exists a function $w \in W^{1,p}(\Omega)$ such that, up to a subsequence,

$$\begin{aligned} &\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)^\sim \to w && \text{strongly in } L^p_{\rm loc}(\Omega) \text{ and weakly in } L^p(\Omega),\\ &[\nabla\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)]^\sim \rightharpoonup \nabla w && \text{weakly in } L^p(\Omega). \end{aligned}$$

Then, the first two convergences stated in the lemma follow directly from the estimate $\|\nabla_y\mathcal{T}^\varepsilon_\mathcal{L}(\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)^\sim)\|_{L^p(\Omega\times Y)} \le C_1\varepsilon\|[\nabla\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)]^\sim\|_{L^p(\Omega)} \le C\varepsilon$, and convergence results for $\mathcal{T}^\varepsilon_\mathcal{L}$ in Lemmas 5.1 and 5.2 and Theorem 4.1. To prove the final convergence stated in the lemma, we observe that $\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)|_G$ is uniformly bounded in $W^{1,p}(G)$, where $G \subset \Omega$ is a relatively compact open set; see Lemma 6.4. Then, by Theorem 4.2 there exists $\hat w_{1,G} \in L^p(G; W^{1,p}_{\rm per}(Y_x))$ such that

$$\mathcal{T}^\varepsilon_\mathcal{L}(\nabla\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)|_G) \rightharpoonup \nabla w + D_x^{-T}\nabla_y\hat w_{1,G}(\cdot, D_x\cdot) \quad \text{weakly in } L^p(G\times Y).$$

The definition of $\mathcal{Q}^\varepsilon_\mathcal{L}$ implies that $\hat w_{1,G}$ is a polynomial in y of degree less than or equal to one with respect to each variable $y_1, \dots, y_d$. Thus, the Y_x-periodicity of $\hat w_{1,G}$ yields that it is constant with respect to y and

$$\mathcal{T}^\varepsilon_\mathcal{L}([\nabla\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)]^\sim) \rightharpoonup \nabla w \quad \text{weakly in } L^p_{\rm loc}(\Omega; L^p(Y)).$$

The boundedness of $[\nabla\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)]^\sim$ in $L^p(\Omega)$ implies the boundedness of $\mathcal{T}^\varepsilon_\mathcal{L}([\nabla\mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)]^\sim)$ in $L^p(\Omega\times Y)$, and we obtain the last convergence stated in the lemma. ∎

For $\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon) = w^\varepsilon - \mathcal{Q}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)$ we have the following convergence results.

LEMMA 7.3. *Consider a sequence $\{w^\varepsilon\} \subset W^{1,p}(\Omega^*_\varepsilon)$, with $p \in (1,+\infty)$, satisfying $\|\nabla w^\varepsilon\|_{L^p(\Omega^*_\varepsilon)} \le C$. Then, there exist a subsequence (denoted again by $\{w^\varepsilon\}$) and a function $w_1 \in L^p(\Omega; W^{1,p}_{\rm per}(Y^*_x))$ such that*

$$\begin{aligned} &\varepsilon^{-1}\,\mathcal{T}^{*,\varepsilon}_\mathcal{L}(\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)^\sim) \rightharpoonup w_1(\cdot, D_x\cdot) && \textit{weakly in } L^p(\Omega; W^{1,p}(Y^*)),\\ &\mathcal{T}^{*,\varepsilon}_\mathcal{L}(\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)^\sim) \to 0 && \textit{strongly in } L^p(\Omega; W^{1,p}(Y^*)),\\ &\mathcal{T}^{*,\varepsilon}_\mathcal{L}([\nabla\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)]^\sim) \rightharpoonup D_x^{-T}\nabla_y w_1(\cdot, D_x\cdot) && \textit{weakly in } L^p(\Omega\times Y^*), \end{aligned} \tag{7.1}$$

where $^\sim$ denotes the extension by zero from $\widetilde\Omega^*_\varepsilon$ to Ω^*_ε.

Proof. The estimates in Lemma 6.4 imply that $\varepsilon^{-1}\,\mathcal{T}^{*,\varepsilon}_\mathcal{L}(\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)^\sim)$ is bounded in $L^p(\Omega; W^{1,p}(Y^*))$ and there exist $\widetilde w_1 \in L^p(\Omega; W^{1,p}(Y^*))$ and $w_1(x,y) = \widetilde w_1(x, D_x^{-1}y)$ for $x \in \Omega$, $y \in Y^*_x$, where $Y^*_x = D(x)Y^*$, such that the convergences in (7.1) are satisfied. To show that w_1 is Y_x-periodic, we consider the restriction of $\varepsilon^{-1}\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)$ on G^*_ε, which belongs to $W^{1,p}(G^*_\varepsilon)$. Here $G^*_\varepsilon = G \cap \Omega^*_\varepsilon$, and $G \subset \Omega$ is a relatively compact open subset of Ω. Using Lemma 6.4 we obtain

$$\|\varepsilon^{-1}\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)\|_{L^p(G^*_\varepsilon)} + \varepsilon\|\varepsilon^{-1}\nabla\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)\|_{L^p(G^*_\varepsilon)} \le C.$$

Applying Theorem 4.3 to $\varepsilon^{-1}\mathcal{R}^{*,\varepsilon}_\mathcal{L}(w^\varepsilon)|_{G^*_\varepsilon}$ yields $w_1|_{G\times Y^*_x} \in L^p(G; W^{1,p}_{\rm per}(Y^*_x))$. Since G can be chosen arbitrarily we obtain that $w_1 \in L^p(\Omega; W^{1,p}_{\rm per}(Y^*_x))$. ∎

Combining the convergence results from above, we obtain directly the main convergence theorem for the l-p unfolding operator in l-p perforated domains.

Proof of Theorem 4.4. Similar to the periodic case, the convergence results stated in Theorem 4.4 follow directly from the fact that $w^\varepsilon = \mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(w^\varepsilon) + \mathcal{R}^{*,\varepsilon}_{\mathcal{L}}(w^\varepsilon)$ and from the convergence results in Lemmas 7.2 and 7.3. ▯

Remark. In the definition of Ω^*_ε we assumed that there are no perforations in layers $(\Omega^{*,\varepsilon}_n \setminus \overline{\hat{\Omega}^{*,\varepsilon}_n}) \cap \widetilde{\Omega}_{\varepsilon/2}$, with $\widetilde{\Omega}_{\varepsilon/2} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 2\varepsilon \max_{x\in\partial\Omega} \operatorname{diam}(D(x)Y)\}$ and $1 \leq n \leq N_\varepsilon$. In the proofs of convergence results only local estimates for $\mathcal{Q}^{*,\varepsilon}_{\mathcal{L}}(w^\varepsilon)$ and $\mathcal{R}^{*,\varepsilon}_{\mathcal{L}}(w^\varepsilon)$ are used; thus no restrictions on the distribution of perforations near $\partial\Omega$ are needed. For the macroscopic description of microscopic processes this assumption is not restrictive since $|\bigcup_{n=1}^{N_\varepsilon}(\Omega^{*,\varepsilon}_n \setminus \overline{\hat{\Omega}^{*,\varepsilon}_n}) \cap \Omega| \leq C\varepsilon^{1-r} \to 0$ as $\varepsilon \to 0$, $r < 1$. If we allow perforations in layers between two neighboring $\hat{\Omega}^{*,\varepsilon}_n$ and $\hat{\Omega}^{*,\varepsilon}_m$ in $\widetilde{\Omega}_{\varepsilon/2}$, then using that $Y^* = Y \setminus \overline{Y}_0$ is connected, the transformation matrix D is Lipschitz continuous, and $\operatorname{dist}(\widetilde{\Omega}_{\varepsilon/2}, \partial\Omega) > 0$, it is possible to construct an extension of $w^\varepsilon \in W^{1,p}(\Omega^*_\varepsilon)$ from $(\Omega^{*,\varepsilon}_n \setminus \overline{\hat{\Omega}^{*,\varepsilon}_n}) \cap \widetilde{\Omega}_{\varepsilon/2}$ to $(\Omega^{\varepsilon}_n \setminus \overline{\hat{\Omega}^{\varepsilon}_n}) \cap \widetilde{\Omega}_{\varepsilon/2}$ such that the $W^{1,p}$-norm of the extension is controlled by the $W^{1,p}$-norm of the original function, uniform in ε, and apply Lemmas 7.2 and 7.3 and Theorem 4.4 to the sequence of extended functions.

8. Two-scale convergence on oscillating surfaces and the l-p boundary unfolding operator. To derive macroscopic equations for the microscopic problems posed on boundaries of l-p microstructures or with nonhomogeneous Neumann conditions on boundaries of l-p microstructures, we have to show convergence properties for sequences defined on oscillating surfaces. To show the compactness result for l-t-s convergence on oscillating surfaces (see Theorem 4.5), we first prove the convergence of the $L^p(\Gamma^\varepsilon)$-norm of the l-p approximation of $\psi \in C(\overline{\Omega}; C_{\text{per}}(Y_x))$.

LEMMA 8.1. *For $\psi \in C(\overline{\Omega}; C_{\text{per}}(Y_x))$ and $p \in [1, +\infty)$, we have that*

$$\lim_{\varepsilon\to 0} \varepsilon \int_{\Gamma^\varepsilon} |\mathcal{L}^\varepsilon \psi(x)|^p \, d\sigma_x = \int_\Omega \frac{1}{|Y_x|} \int_{\Gamma_x} |\psi(x, y)|^p d\sigma_y dx.$$

Proof. The definition of the l-p approximation $\mathcal{L}^\varepsilon$ implies

$$\begin{aligned}
\varepsilon \int_{\Gamma^\varepsilon} |\mathcal{L}^\varepsilon \psi|^p d\sigma_x &= \varepsilon \sum_{n=1}^{N_\varepsilon} \sum_{\xi \in \hat{\Xi}^\varepsilon_n} \int_{\varepsilon\Gamma^\xi_{x_n}} \left|\widetilde{\psi}\Big(x, \frac{D^{-1}_{x^\varepsilon_n} x}{\varepsilon}\Big)\right|^p - \left|\widetilde{\psi}\Big(x^\varepsilon_n, \frac{D^{-1}_{x^\varepsilon_n} x}{\varepsilon}\Big)\right|^p d\sigma_x \\
&+ \varepsilon \sum_{n=1}^{N_\varepsilon} \left[\sum_{\xi \in \hat{\Xi}^\varepsilon_n} \int_{\varepsilon\Gamma^\xi_{x_n}} \left|\widetilde{\psi}\Big(x^\varepsilon_n, \frac{D^{-1}_{x^\varepsilon_n} x}{\varepsilon}\Big)\right|^p d\sigma_x + \sum_{\xi \in \widetilde{\Xi}^\varepsilon_n} \int_{\varepsilon\Gamma^\xi_{x_n}} \left|\widetilde{\psi}\Big(x, \frac{D^{-1}_{x^\varepsilon_n} x}{\varepsilon}\Big)\right|^p \chi_{\Omega^\varepsilon_n} \chi_\Omega d\sigma_x \right] \\
&= I_1 + I_2 + I_3,
\end{aligned}$$

where $\widetilde{\Xi}^\varepsilon_n = \Xi^\varepsilon_n \setminus \hat{\Xi}^\varepsilon_n$ and $\Gamma^\xi_{x_n} = D_{x^\varepsilon_n}(\xi + \widetilde{\Gamma}_{K_{x^\varepsilon_n}})$. Then, the continuity of ψ, the properties of Ω^ε_n, and the inequality $||a|^p - |b|^p| \leq p|a - b|(|a|^{p-1} + |b|^{p-1})$ imply $I_1 \to 0$ as $\varepsilon \to 0$. Using the properties of the covering of Ω by $\{\Omega^\varepsilon_n\}_{n=1}^{N_\varepsilon}$, we obtain

$$|I_3| \leq C\varepsilon^{-rd} \sup_{1\leq n\leq N_\varepsilon} \varepsilon^d |\widetilde{\Xi}^\varepsilon_n| |D_{x^\varepsilon_n} \widetilde{\Gamma}_{K_{x^\varepsilon_n}}| \leq C\varepsilon^{1-r} \to 0 \quad \text{as } \varepsilon \to 0 \text{ for } 0 \leq r < 1.$$

Considering the properties of the covering of $\hat{\Omega}_n^\varepsilon$ by $D_{x_n^\varepsilon}(Y+\xi)$, where $\xi \in \hat{\Xi}_n^\varepsilon$ and $1 \le n \le N_\varepsilon$, and the Y-periodicity of $\widetilde{\psi}$, the second integral can be rewritten as

$$I_2 = \sum_{n=1}^{N_\varepsilon} \varepsilon^d |\hat{\Xi}_n^\varepsilon| \int_{D_{x_n^\varepsilon} \widetilde{\Gamma}_{K_{x_n^\varepsilon}}} |\widetilde{\psi}(x_n^\varepsilon, D_{x_n^\varepsilon}^{-1} y)|^p d\sigma_y = \sum_{n=1}^{N_\varepsilon} \frac{|\hat{\Omega}_n^\varepsilon|}{|Y_{x_n^\varepsilon}|} \int_{D_{x_n^\varepsilon} \widetilde{\Gamma}_{K_{x_n^\varepsilon}}} |\psi(x_n^\varepsilon, y)|^p d\sigma_y.$$

Then, the regularity assumptions on ψ, D, and K, the definition of $\hat{\Omega}_n^\varepsilon$, and the properties of the covering of Ω by $\{\Omega_n^\varepsilon\}_{n=1}^{N_\varepsilon}$ imply the convergence result stated in the lemma. $\square$

Similar to l-t-s convergence and two-scale convergence for sequences defined on surfaces of periodic microstructures, the convergence of l-p approximations (shown in Lemma 8.1) and the Riesz representation theorem ensure the compactness result for sequences $\{w^\varepsilon\} \subset L^p(\Gamma^\varepsilon)$ with $\varepsilon \|w^\varepsilon\|_{L^p(\Gamma^\varepsilon)}^p \le C$.

Proof of Theorem 4.5. The Banach space $C(\overline{\Omega}; C_{\mathrm{per}}(Y_x))$ is separable and dense in $L^p(\Omega; L^p(\Gamma_x))$. Thus, using the convergence result in Lemma 8.1, the Riesz representation theorem, and arguments similar to those in [46, Theorem 3.2], we obtain l-t-s convergence of $\{w^\varepsilon\} \subset L^p(\Gamma^\varepsilon)$ to $w \in L^p(\Omega; L^p_{\mathrm{per}}(\Gamma_x))$, as stated in the theorem. $\square$

Using the structure of $\Omega_{n,K}^{*,\varepsilon}$ and the covering properties of $\Omega_{\varepsilon,K}^*$ by $\{\Omega_{n,K}^{*,\varepsilon}\}_{n=1}^{N_\varepsilon}$, we can derive the trace inequalities for functions defined on Γ^ε. Applying first the trace inequality in $Y_{x_n^\varepsilon,K}^{*,\xi} = D_{x_n^\varepsilon}(\widetilde{Y}_{K_{x_n^\varepsilon}}^* + \xi)$, with $\xi \in \hat{\Xi}_n^\varepsilon$, yields

$$\|u\|^p_{L^p(D_{x_n^\varepsilon}(\widetilde{\Gamma}_{K_{x_n^\varepsilon}}+\xi))} \le \mu_\Gamma \left[\|u\|^p_{L^p(Y^{*,\xi}_{x_n^\varepsilon,K})} + \|\nabla u\|^p_{L^p(Y^{*,\xi}_{x_n^\varepsilon,K})} \right],$$
$$\|u\|^p_{L^p(D_{x_n^\varepsilon}(\widetilde{\Gamma}_{K_{x_n^\varepsilon}}+\xi))} \le \mu_\Gamma \left[\|u\|^p_{L^p(Y^{*,\xi}_{x_n^\varepsilon,K})} + \int_{Y^{*,\xi}_{x_n^\varepsilon,K}} \int_{Y^{*,\xi}_{x_n^\varepsilon,K}} \frac{|u(y_1)-u(y_2)|^p}{|y_1-y_2|^{d+\beta p}} dy_1 dy_2 \right]$$

for $u \in W^{1,p}(Y^{*,\xi}_{x_n^\varepsilon,K})$ or $u \in W^{\beta,p}(Y^{*,\xi}_{x_n^\varepsilon,K})$, for $1/2 < \beta < 1$, respectively, where the constant μ_Γ depends only on D, K, and Y^*; see, e.g., [26, 52]. Then, scaling by ε and summing up over $\xi \in \hat{\Xi}_n^\varepsilon$ and $1 \le n \le N_\varepsilon$ implies the estimates

$$\varepsilon \|u\|^p_{L^p(\hat{\Gamma}^\varepsilon)} \le \mu_\Gamma \left[\|u\|^p_{L^p(\Omega^*_{\varepsilon,K})} + \varepsilon^p \|\nabla u\|^p_{L^p(\Omega^*_{\varepsilon,K})} \right] \quad \text{for } u \in W^{1,p}(\Omega^*_{\varepsilon,K}),\ p \in [1,+\infty), \tag{8.1}$$

$$\varepsilon \|u\|^p_{L^p(\hat{\Gamma}^\varepsilon)} \le \mu_\Gamma \left[\|u\|^p_{L^p(\Omega^*_{\varepsilon,K})} + \varepsilon^{\beta p} \int_{\Omega^*_{\varepsilon,K}} \int_{\Omega^*_{\varepsilon,K}} \frac{|u(x_1)-u(x_2)|^p}{|x_1-x_2|^{d+\beta p}} dx_1 dx_2 \right] \quad \text{for } u \in W^{\beta,p}(\Omega^*_{\varepsilon,K}) \text{ with } 1/2<\beta<1,\ p \in [1,+\infty), \tag{8.2}$$

where the constant μ_Γ depends on D, K, and Y^* and is independent of ε, and

$$\hat{\Gamma}^\varepsilon = \bigcup_{n=1}^{N_\varepsilon} \hat{\Gamma}_n^\varepsilon \quad \text{with} \quad \hat{\Gamma}_n^\varepsilon = \bigcup_{\xi \in \hat{\Xi}_n^\varepsilon} \varepsilon D_{x_n^\varepsilon}(\widetilde{\Gamma}_{K_{x_n^\varepsilon}} + \xi).$$

Since $\Gamma_{x_n^\varepsilon}$ is given by a linear transformation of Γ, for a parametrization $y = y(w)$ of Γ, where $w \in \mathbb{R}^{d-1}$, we obtain by $x(w) = \varepsilon D_{x_n^\varepsilon} K_{x_n^\varepsilon} y(w)$ the parametrization of $\varepsilon\Gamma_{x_n^\varepsilon}$. We consider for Γ that $d\sigma_y = \sqrt{g}dw$ with $w \in \mathbb{R}^{d-1}$, and for $\Gamma^\varepsilon_{x_n^\varepsilon}$ we have $d\sigma_x^n = \varepsilon^{d-1}\sqrt{g_{x_n^\varepsilon}}dw$, where $g = \det(g_{ij})$, $g_{x_n^\varepsilon} = \det(g_{x_n^\varepsilon,ij})$, and (g_{ij}), $(g_{x_n^\varepsilon,ij})$

are the corresponding first fundamental forms (metrics). We have also $\int_{\Gamma^\varepsilon} d\sigma^\varepsilon_x = \sum_{n=1}^{N_\varepsilon} \int_{\Gamma^\varepsilon_n} d\sigma^n_x$ and $\Gamma_x = D(x)K(x)\Gamma$ with $d\sigma_x = \sqrt{g(x)}dw$.

Using the definition of the l-p boundary unfolding operator, the trace inequality (8.1), and the assumptions on D and K, we show the following properties of $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}$.

LEMMA 8.2. *For $\psi \in W^{1,p}(\Omega^*_{\varepsilon,K})$, with $p \in [1,+\infty)$, we have*

$$\text{(i)} \int_{\Omega\times\Gamma} \sum_{n=1}^{N_\varepsilon} \frac{\sqrt{g_{x^\varepsilon_n}}}{\sqrt{g}|Y_{x^\varepsilon_n}|} |\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\psi)(x,y)|^p \chi_{\Omega^\varepsilon_n} d\sigma_y dx = \varepsilon \int_{\hat\Gamma^\varepsilon} |\psi(x)|^p d\sigma^\varepsilon_x,$$

$$\text{(ii)} \int_{\Omega\times\Gamma} |\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\psi)(x,y)|^p\, d\sigma_y dx = \varepsilon \sum_{n=1}^{N_\varepsilon} \int_{\hat\Gamma^\varepsilon_n} \frac{\sqrt{g}|Y_{x^\varepsilon_n}|}{\sqrt{g_{x^\varepsilon_n}}} |\psi(x)|^p d\sigma^n_x \le C\varepsilon \int_{\hat\Gamma^\varepsilon} |\psi(x)|^p d\sigma^\varepsilon_x,$$

$$\text{(iii)} \|\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\psi)\|_{L^p(\Omega\times\Gamma)} \le C\left(\|\psi\|_{L^p(\Omega^*_{\varepsilon,K})} + \varepsilon\|\nabla\psi\|_{L^p(\Omega^*_{\varepsilon,K})}\right),$$

where the constant C depends on D and K and is independent of ε.

Proof. Equality (i) follows directly from the definition of $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}$, i.e.,

$$\begin{aligned}
&\int_{\Omega\times\Gamma} \sum_{n=1}^{N_\varepsilon} \frac{\sqrt{g_{x^\varepsilon_n}}}{\sqrt{g}|Y_{x^\varepsilon_n}|} |\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\psi)|^p \chi_{\Omega^\varepsilon_n} d\sigma_y dx \\
&= \sum_{n=1}^{N_\varepsilon} \sum_{\xi\in\hat\Xi^\varepsilon_n} \varepsilon^d \int_\Gamma \frac{\sqrt{g_{x^\varepsilon_n}}}{\sqrt{g}} |\psi(\varepsilon D_{x^\varepsilon_n}(\xi + K_{x^\varepsilon_n} y))|^p d\sigma_y = \varepsilon \int_{\hat\Gamma^\varepsilon} |\psi(x)|^p d\sigma^\varepsilon_x.
\end{aligned}$$

Similar calculations and the regularity assumptions on D and K imply the equality and the estimate in (ii). The estimate in (iii) is ensured by (ii) and (8.1). ∎

Remark. Due to the second estimate in Lemma 8.2 and the assumptions on D and K, the boundedness of $\varepsilon\|w^\varepsilon\|^p_{L^p(\hat\Gamma^\varepsilon)}$ implies the boundedness of $\|\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(w^\varepsilon)\|^p_{L^p(\Omega\times\Gamma)}$ and, hence, the weak convergence of $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(w^\varepsilon)$ in $L^p(\Omega\times\Gamma)$.

Applying the properties of the l-p boundary unfolding operator shown in Lemma 8.2, we prove the relation between the l-t-s convergence on oscillating boundaries and the l-p boundary unfolding operator.

Proof of Theorem 4.6. Using the definition of $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}$ and considering $\psi \in C(\overline{\Omega}; C_{\text{per}}(Y_x))$ together with the corresponding $\widetilde{\psi} \in C(\overline{\Omega}; C_{\text{per}}(Y))$ yields

$$\begin{aligned}
&\int_\Omega\int_\Gamma \sum_{n=1}^{N_\varepsilon} \frac{\sqrt{g_{x^\varepsilon_n}}}{\sqrt{g}|Y_{x^\varepsilon_n}|} \mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(w^\varepsilon)\, \widetilde{\psi}(x, K_{x^\varepsilon_n} y)\, \chi_{\Omega^\varepsilon_n} d\sigma_y dx \\
&= \sum_{n=1}^{N_\varepsilon} \sum_{\xi\in\hat\Xi^\varepsilon_n} \varepsilon \int_{\varepsilon\Gamma^\xi_{x^\varepsilon_n}} w^\varepsilon(z) \widetilde{\psi}\left(z, D^{-1}_{x^\varepsilon_n}\frac{z}{\varepsilon}\right) d\sigma^n_z \\
&+ \sum_{n=1}^{N_\varepsilon} \sum_{\xi\in\hat\Xi^\varepsilon_n} \varepsilon^{1-d} \frac{1}{|Y_{x^\varepsilon_n}|} \int_{\varepsilon\Gamma^\xi_{x^\varepsilon_n}} w^\varepsilon(z) \int_{\varepsilon Y^\xi_{x^\varepsilon_n}} \left[\widetilde{\psi}\left(x, D^{-1}_{x^\varepsilon_n}\frac{z}{\varepsilon}\right) - \widetilde{\psi}\left(z, D^{-1}_{x^\varepsilon_n}\frac{z}{\varepsilon}\right)\right] dx\, d\sigma^n_z,
\end{aligned}$$

where $\Gamma^\xi_{x^\varepsilon_n} = D_{x^\varepsilon_n}(\widetilde{\Gamma}_{K_{x^\varepsilon_n}} + \xi)$ and $Y^\xi_{x^\varepsilon_n} = D_{x^\varepsilon_n}(Y + \xi)$. The continuity of ψ and the boundedness of $\varepsilon\|w^\varepsilon\|^p_{L^p(\Gamma^\varepsilon)}$ ensure the convergence of the last integral to zero as $\varepsilon \to 0$. Consider first that $w^\varepsilon \to w$ l-t-s. The assumption on w^ε, i.e., $\varepsilon\|w^\varepsilon\|^p_{L^p(\Gamma^\varepsilon)} \le C$, with $p \in (1,+\infty)$, ensures that, up to a subsequence, $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(w^\varepsilon) \rightharpoonup \hat w$ weakly in

$L^p(\Omega \times \Gamma)$. Using the continuity of ψ, D, and K, along with $|\Gamma^\varepsilon \setminus \hat{\Gamma}^\varepsilon| \to 0$ as $\varepsilon \to 0$, yields

$$\begin{aligned}
&\int_\Omega \int_\Gamma \frac{\sqrt{g_x}}{|Y_x|\sqrt{g}} \hat{w}(x,y)\, \widetilde{\psi}(x, K_x y)\, d\sigma_y dx \\
&= \lim_{\varepsilon\to 0} \int_\Omega \int_\Gamma \sum_{n=1}^{N_\varepsilon} \frac{\sqrt{g_{x_n^\varepsilon}}}{|Y_{x_n^\varepsilon}|\sqrt{g}} \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(w^\varepsilon) \widetilde{\psi}(x, K_{x_n^\varepsilon} y) \chi_{\Omega_n^\varepsilon}\, d\sigma_y dx \qquad (8.3)\\
&= \lim_{\varepsilon\to 0} \varepsilon \int_{\Gamma^\varepsilon} w^\varepsilon(x) \mathcal{L}^\varepsilon(\psi)\, d\sigma_x^\varepsilon = \int_\Omega \frac{1}{|Y_x|} \int_{\Gamma_x} w(x,y)\psi(x,y)\, d\sigma_y dx
\end{aligned}$$

for all $\psi \in C(\overline{\Omega}; C_{\rm per}(Y_x))$. Applying the coordinate transformation in the integral on the right-hand side yields $\hat{w}(x,y) = w(x, D_x K_x y)$ for a.a. $x \in \Omega$, $y \in \Gamma$, and the whole sequence $\{\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(w^\varepsilon)\}$ converges to $w(\cdot, D_x K_x \cdot)$.

Consider $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(w^\varepsilon) \rightharpoonup w(\cdot, D_x K_x \cdot)$ in $L^p(\Omega\times\Gamma)$. The boundedness of $\varepsilon\|w^\varepsilon\|^p_{L^p(\Gamma^\varepsilon)}$ implies that, up to a subsequence, $w^\varepsilon \to \hat{w}$ l-t-s and $\hat{w} \in L^p(\Omega; L^p(\Gamma_x))$. Interchanging $\hat{w}$ and w in (8.3), we obtain that the whole sequence w^ε l-t-s converges to w. ∎

For functions in $W^{\beta,p}(\Omega)$, with $p \in (1,+\infty)$ and $1/2 < \beta \le 1$, or for sequences defined on oscillating boundaries and converging in the $L^p(\Gamma^\varepsilon)$-norm, scaled by $\varepsilon^{1/p}$, we obtain the strong convergence of the corresponding unfolded sequences.

LEMMA 8.3. *For $u \in W^{\beta,p}(\Omega)$, with $p \in (1,+\infty)$ and $1/2 < \beta \le 1$, we have*

$$\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(u) \to u \quad \textit{strongly in} \quad L^p(\Omega\times\Gamma). \qquad (8.4)$$

If for $\{v^\varepsilon\} \subset L^p(\Gamma^\varepsilon)$ and some $v \in C(\overline{\Omega}; C_{\rm per}(Y_x))$ we have $\varepsilon\|v^\varepsilon - \mathcal{L}^\varepsilon v\|^p_{L^p(\Gamma^\varepsilon)} \to 0$ as $\varepsilon \to 0$, then

$$\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(v^\varepsilon) \to v(\cdot, D_x K_x \cdot) \quad \textit{strongly in} \quad L^p(\Omega\times\Gamma). \qquad (8.5)$$

Proof. For an approximation of u by $u_k \in C^1(\overline{\Omega})$ we can write

$$\int_{\Omega\times\Gamma} |\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(u_k)|^p d\sigma_y dx = \sum_{n=1}^{N_\varepsilon} \int_{\Omega\times\Gamma} \left| u_k\left(\varepsilon D_{x_n^\varepsilon}\left[D_{x_n^\varepsilon}^{-1} x/\varepsilon\right]_Y + \varepsilon D_{x_n^\varepsilon} K_{x_n^\varepsilon} y\right)\right|^p \chi_{\hat{\Omega}_n^\varepsilon} d\sigma_y dx$$

$$= \sum_{n=1}^{N_\varepsilon} \sum_{\xi\in\hat{\Xi}_n^\varepsilon} \varepsilon^d |Y_{x_n^\varepsilon}| \int_\Gamma |u_k(\varepsilon D_{x_n^\varepsilon}(\xi + K_{x_n^\varepsilon} y))|^p d\sigma_y = \sum_{n=1}^{N_\varepsilon} \sum_{\xi\in\hat{\Xi}_n^\varepsilon} |\varepsilon Y_{x_n^\varepsilon}||\Gamma||u_k(\tilde{x}_{n,\xi}^\varepsilon)|^p + \delta_\varepsilon$$

for some fixed $\tilde{x}_{n,\xi}^\varepsilon \in \varepsilon D_{x_n^\varepsilon}(K_{x_n^\varepsilon}\Gamma + \xi)$, where, due to the continuity of u_k, we have

$$\delta_\varepsilon = \sum_{n=1}^{N_\varepsilon} \sum_{\xi\in\hat{\Xi}_n^\varepsilon} \varepsilon^d |D_{x_n^\varepsilon} Y| \int_\Gamma |u_k(\varepsilon D_{x_n^\varepsilon}(\xi + K_{x_n^\varepsilon} y)) - u_k(\tilde{x}_{n,\xi}^\varepsilon)|^p d\sigma_y \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

The properties of the covering of Ω by $\{\Omega_n^\varepsilon\}_{n=1}^{N_\varepsilon}$ and $|\Omega \setminus \hat{\Omega}^\varepsilon| \to 0$ as $\varepsilon \to 0$ imply

$$\lim_{\varepsilon\to 0} \sum_{n=1}^{N_\varepsilon} \sum_{\xi\in\hat{\Xi}_n^\varepsilon} \varepsilon^d |D_{x_n^\varepsilon} Y||\Gamma||u_k(\tilde{x}_{n,\xi}^\varepsilon)|^p = \int_\Omega \int_\Gamma |u_k(x)|^p d\sigma_y dx.$$

Then, the density of $C^1(\overline{\Omega})$ in $W^{\beta,p}(\Omega)$, relation (ii) in Lemma 8.2, and the trace estimates (8.1) and (8.2) ensure the convergence result for $u \in W^{\beta,p}(\Omega)$.

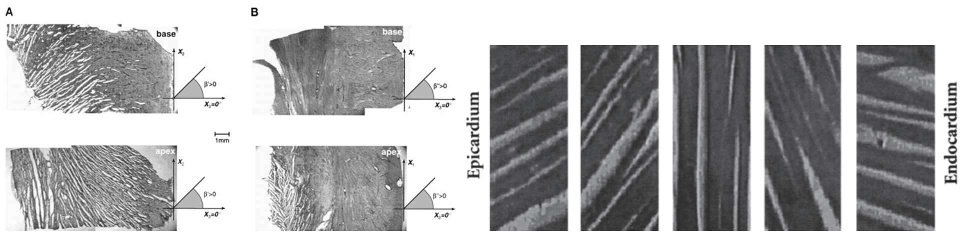

FIG. 5. *Left: Images of laminar cleavage planes in longitudinal-radial* (A) *and circumferential-radial* (B) *tissue sections from basal and apical measurement sites in anterior left ventricular free wall. Republished with permission of The American Physiological Society, from K. D. Costa, Y. Takayana, A. D. McCulloch, and J. W. Covell,* Laminar fiber architecture and three-dimensional systolic mechanics in canine ventricular myocardium, *Am. J. Physiol. Heart Circ. Physiol.,* 276 (1999)*, pp.* H595–H607 [23]*. Right: Cardiac muscle fiber orientations vary continuously through the left ventricular wall from a negative angle at the epicardium to positive values toward the endocardium. Republished with permission of Taylor and Francis Group, from A. D. McCulloch,* Cardiac biomechanics*, in The Biomedical Engineering Handbook,* 2nd *ed., J.D. Bronzino, ed., CRC Press, Boca Raton, FL,* 2000 [39]; *permission conveyed through Copyright Clearance Center, Inc.*

To show the convergence in (8.5) we consider

$$\begin{aligned}\|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(v^\varepsilon) - v(\cdot, D_x K_x \cdot)\|_{L^p(\Omega\times\Gamma)} \leq \|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(v^\varepsilon) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\mathcal{L}^\varepsilon v)\|_{L^p(\Omega\times\Gamma)} \\ + \|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\mathcal{L}^\varepsilon v) - v(\cdot, D_x K_x \cdot)\|_{L^p(\Omega\times\Gamma)}.\end{aligned}$$

Then, estimate (ii) in Lemma 8.2, the regularity of v, D, and K, and the convergence

$$\begin{aligned}\lim_{\varepsilon\to 0}\int_{\Omega\times\Gamma} |\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\mathcal{L}^\varepsilon v)|^p d\sigma_y dx = \lim_{\varepsilon\to 0}\sum_{n=1}^{N_\varepsilon} |\varepsilon Y_{x_n^\varepsilon}| \sum_{\xi\in\hat{\Xi}_n^\varepsilon}\int_\Gamma |\widetilde{v}(\varepsilon D_{x_n^\varepsilon}(\xi + K_{x_n^\varepsilon} y), K_{x_n^\varepsilon} y)|^p d\sigma_y \\ = \int_\Omega\int_\Gamma |v(x, D_x K_x y)|^p d\sigma_y dx,\end{aligned}$$

where $\widetilde{v}(x,y) = v(x, D_x y)$ for $x \in \Omega$ and $y \in Y$, yield (8.5). ∎

The results in Lemma 8.3 will ensure the strong convergence of coefficients in equations defined on oscillating boundaries and are used in the derivation of macroscopic problems for microscopic equations defined on surfaces of l-p microstructures.

9. Homogenization of a model for a signaling process in a tissue with l-p distribution of cells. In this section we apply the methods of the l-p unfolding operator and l-t-s convergence on oscillating surfaces to derive macroscopic equations for microscopic models posed in domains with l-p perforations. We consider a generalization of the model for an intercellular signaling process presented in [34] to tissues with l-p microstructures. As examples for tissues with space-dependent changes in the size and shape of cells, we consider epithelial and plant cell tissues; see Figure 3. As an example of a tissue which has a plywood-like structure we consider the cardiac muscle tissue of the left ventricular wall; see Figure 5.

The microstructure of cardiac muscle is described in the same way as a plywood-like structure considered in the introduction, where $D(x) = R^{-1}(\gamma(x_3))$ and the rotation matrix R is as defined in the introduction. Additionally we assume that the radius of fibers may change locally; i.e., $K(x)Y_0 \subset Y$, with

$$K(x) = \begin{pmatrix} 1 & \mathbf{0}^T \\ \mathbf{0} & \rho(x)\mathbf{I}_2 \end{pmatrix},$$

$Y_0 = \{(y_1, y_2, y_3) \in Y : |y_2|^2 + |y_3|^2 < a^2\}$, $0 < a < 1/2$, $Y = (-1/2, 1/2)^3$, and $\rho \in C^1(\overline{\Omega})$ with $0 < \rho_1 \le \rho(x)a < 1/2$ for all $x \in \overline{\Omega}$. Then, for the plywood-like structure we have $D_{x_n^\varepsilon} = R^{-1}(\gamma(x^\varepsilon_{n,3}))$, $\widetilde{Y}^*_{K_x} = Y \setminus K(x)\overline{Y}_0$, $Y^*_{x,K} = R^{-1}(\gamma(x_3))\widetilde{Y}^*_{K_x}$, and the characteristic function of fibers is given by

$$\chi_{\Omega_f^\varepsilon}(x) = \chi_\Omega(x) \sum_{n=1}^{N_\varepsilon} \tilde{\eta}(x_n^\varepsilon, R(\gamma(x^\varepsilon_{n,3}))x/\varepsilon)\chi_{\Omega_n^\varepsilon},$$

where

$$\tilde{\eta}(x, y) = \begin{cases} 1 & \text{for } |\hat{K}(x)^{-1}\hat{y}| \le a, \\ 0 & \text{elsewhere,} \end{cases}$$

and is extended $\hat{Y}$-periodically to the whole of $\mathbb{R}^3$. Here $\hat{y} = (y_2, y_3)$, $\hat{Y} = (-1/2, 1/2)^2$, and $\hat{K}(x) = \rho(x)\,\mathbf{I}_2$, where $\mathbf{I}_2$ denotes the identity matrix in $\mathbb{R}^{2\times 2}$.

In the case of an epithelial tissue consider $Y_x = D(x)Y$, with, e.g.,

$$D(x) = \begin{pmatrix} \mathbf{I}_2 & \mathbf{0} \\ \mathbf{0}^T & \kappa(x) \end{pmatrix},$$

where $\kappa \in C^1(\overline{\Omega})$ and $0 < \kappa_1 \le \kappa(x) < 1$ for all $x \in \Omega$ defines a compression of cells in the x_3-direction. The changes in the size and shape of cells can be defined by the boundaries of the microstructure $\Gamma_x = S(x)\Gamma \subset Y_x = D_x Y$ for all $x \in \overline{\Omega}$ and $S \in \mathrm{Lip}(\overline{\Omega}; \mathbb{R}^{3\times 3})$. Then, in the definition of the intercellular space $\Omega^*_{\varepsilon,K}$ we have $Y^*_{x,K} = D(x)\widetilde{Y}^*_{K_x} = D(x)(Y \setminus K(x)\overline{Y}_0)$, where $K(x) = D(x)^{-1}S(x)$.

We define the intercellular space in a tissue as

$$\Omega^*_{\varepsilon,K} = \mathrm{Int}\Bigg(\bigcup_{n=1}^{N_\varepsilon} \overline{\Omega^{*,\varepsilon}_{n,K}}\Bigg) \cap \Omega, \quad \text{where} \quad \Omega^{*,\varepsilon}_{n,K} = \Omega^\varepsilon_n \setminus \bigcup_{\xi \in \Xi_n^{*,\varepsilon}} D_{x_n^\varepsilon}(K_{x_n^\varepsilon}\overline{Y}_0 + \xi),$$

$$\hat{\Omega}^*_{\varepsilon,K} = \mathrm{Int}\Bigg(\bigcup_{n=1}^{N_\varepsilon} \overline{\hat{\Omega}^{*,\varepsilon}_{n,K}}\Bigg),\ \hat{\Omega}^{*,\varepsilon}_{n,K} = \mathrm{Int}\Bigg(\bigcup_{\xi \in \hat{\Xi}^\varepsilon_n} \varepsilon D_{x_n^\varepsilon}(\overline{\widetilde{Y}^*_{K_{x_n^\varepsilon}}} + \xi)\Bigg),\ \Lambda^*_{\varepsilon,K} = \Omega^*_{\varepsilon,K} \setminus \hat{\Omega}^*_{\varepsilon,K}.$$

In the model for a signaling process in a cell tissue we consider diffusion of signaling molecules l^ε in the intercellular space and their interactions with free and bound receptors r_f^ε and r_b^ε located on cell surfaces. The microscopic model reads as

$$\begin{aligned} \partial_t l^\varepsilon - \mathrm{div}(A^\varepsilon(x)\nabla l^\varepsilon) &= F^\varepsilon(x, l^\varepsilon) - d_l^\varepsilon(x) l^\varepsilon && \text{in } (0,T)\times\Omega^*_{\varepsilon,K}, \\ A^\varepsilon(x)\nabla l^\varepsilon \cdot \mathbf{n} &= \varepsilon\big[\beta^\varepsilon(x) r_b^\varepsilon - \alpha^\varepsilon(x) l^\varepsilon r_f^\varepsilon\big] && \text{on } (0,T)\times\Gamma^\varepsilon, \\ A^\varepsilon(x)\nabla l^\varepsilon \cdot \mathbf{n} &= 0 && \text{on } (0,T)\times(\partial\Omega\cap\partial\Omega^*_{\varepsilon,K}), \\ l^\varepsilon(0,x) &= l_0(x) && \text{in } \Omega^*_{\varepsilon,K}, \end{aligned} \tag{9.1}$$

where the dynamics in the concentrations of free and bound receptors on cell surfaces are determined by two ordinary differential equations:

$$\begin{aligned} &\partial_t r_f^\varepsilon = p^\varepsilon(x, r_b^\varepsilon) - \alpha^\varepsilon(x) l^\varepsilon r_f^\varepsilon + \beta^\varepsilon(x) r_b^\varepsilon - d_f^\varepsilon(x) r_f^\varepsilon && \text{on } (0,T)\times\Gamma^\varepsilon, \\ &\partial_t r_b^\varepsilon = \qquad\qquad\quad \alpha^\varepsilon(x) l^\varepsilon r_f^\varepsilon - \beta^\varepsilon(x) r_b^\varepsilon - d_b^\varepsilon(x) r_b^\varepsilon && \text{on } (0,T)\times\Gamma^\varepsilon, \\ &r_f^\varepsilon(0,x) = r_{f0}^\varepsilon(x), \qquad r_b^\varepsilon(0,x) = r_{b0}^\varepsilon(x) && \text{on } \Gamma^\varepsilon. \end{aligned} \tag{9.2}$$

The coefficients A^ε, α^ε, β^ε, d_j^ε and the functions $F^\varepsilon(\cdot,\xi)$, $p^\varepsilon(\cdot,\xi)$, r_{i0}^ε are defined as

$$\begin{aligned}
&A^\varepsilon(x) = \mathcal{L}_0^\varepsilon(A(x,y)), && F^\varepsilon(x,\xi) = \mathcal{L}_0^\varepsilon(F(x,y,\xi)), && p^\varepsilon(x,\xi) = \mathcal{L}_0^\varepsilon(p(x,y,\xi)),\\
&\alpha^\varepsilon(x) = \mathcal{L}_0^\varepsilon(\alpha(x,y)), && \beta^\varepsilon(x) = \mathcal{L}_0^\varepsilon(\beta(x,y)), && d_j^\varepsilon(x) = \mathcal{L}_0^\varepsilon(d_j(x,y)),\\
&r_{i0}^\varepsilon(x) = \mathcal{L}^\varepsilon(r_{i0}(x,y)), && && j = l,f,b, \quad i = f,b,
\end{aligned}$$

for $x \in \Omega$, $y \in Y_x$, and $\xi \in \mathbb{R}$, where $A(x,\cdot)$, $\alpha(x,\cdot)$, $\beta(x,\cdot)$, $d_j(x,\cdot)$, $p(x,\cdot,\xi)$, $F(x,\cdot,\xi)$, and $r_{i0}(x,\cdot)$ are Y_x-periodic functions. We assume also that $\alpha^\varepsilon(x) = 0$ and $\beta^\varepsilon(x) = 0$ for $x \in \Lambda^\varepsilon$. The last assumption is not restrictive, since the domain Λ^ε is very small compared to the size of the whole domain Ω and $|\Lambda^\varepsilon| \le C\varepsilon^{1-r} \to 0$ as $\varepsilon \to 0$ for $0 \le r < 1$.

Here, $A^\varepsilon : \Omega \to \mathbb{R}$ denotes the diffusion coefficient for signaling molecules (ligands), $F^\varepsilon : \Omega \times \mathbb{R} \to \mathbb{R}$ models the production of new ligands, $p^\varepsilon : \Omega \to \mathbb{R}$ describes the production of new free receptors, $d_j^\varepsilon : \Omega \to \mathbb{R}$, $j = l,f,b$, denotes the rates of decay of ligands, free receptors, and bound receptors, respectively, $\beta^\varepsilon : \Omega \to \mathbb{R}$ is the rate of dissociation of bound receptors, and $\alpha^\varepsilon : \Omega \to \mathbb{R}$ is the rate of binding of ligands to free receptors.

Assumption 9.1.
- $A \in C(\overline{\Omega}; L^\infty_{\rm per}(Y_x))$ is symmetric with $(A(x,y)\xi,\xi) \ge d_0|\xi|^2$ for $d_0 > 0$, $\xi \in \mathbb{R}^d$, $x \in \Omega$, and a.a. $y \in Y_x$.
- $F(\cdot,\cdot,\xi) \in C(\overline{\Omega}; L^\infty_{\rm per}(Y_x))$ is Lipschitz continuous in ξ with $\xi \ge -\kappa$, for some $\kappa > 0$, uniformly in (x,y) and $F(x,y,\xi) \ge 0$ for $\xi \ge 0$, $x \in \Omega$, and $y \in Y_x$.
- $p(\cdot,\cdot,\xi) \in C(\overline{\Omega}; C_{\rm per}(Y_x))$ is Lipschitz continuous in ξ with $\xi \ge -\kappa$, for some $\kappa > 0$, uniformly in (x,y) and nonnegative for nonnegative ξ.
- Coefficients $\alpha, \beta, d_j \in C(\overline{\Omega}; C_{\rm per}(Y_x))$ are nonnegative, $j = l,f,b$.
- Initial conditions $l_0 \in H^1(\Omega) \cap L^\infty(\Omega)$, $r_{j0} \in C(\overline{\Omega}; C_{\rm per}(Y_x))$ are nonnegative, $j = f,b$.

Notice that these assumptions are satisfied by the physical processes described by our model, since for most signaling processes in biological tissues we have that $A = \text{const}$, $F(x,y,\xi) = \mu_1\xi/(\mu_2 + \mu_3\xi)$, and $p(x,y,\xi) = \kappa_1\xi/(\kappa_2 + \kappa_3\xi)$ with some nonnegative constants μ_i and κ_i for $i = 1,2,3$, and the coefficients α, β, and d_j, with $j = l,f,b$, can be chosen as constant or as some smooth functions.

We shall use the following notation: $\hat\Gamma_T^\varepsilon = (0,T) \times \hat\Gamma^\varepsilon$, $\Gamma_T^\varepsilon = (0,T) \times \Gamma^\varepsilon$, $\Omega_T = (0,T) \times \Omega$, $\Gamma_T = (0,T) \times \Gamma$, and $\Gamma_{x,T} = (0,T) \times \Gamma_x$. For $v \in L^p(0,\sigma; L^q(A))$, $w \in L^{p'}(0,\sigma; L^{q'}(A))$ we denote $\langle v, w\rangle_{A,\sigma} = \int_0^\sigma \int_A v\, w\, dxdt$.

Definition 9.1. *A weak solution of the problem* (9.1)–(9.2) *are functions* $(l^\varepsilon, r_f^\varepsilon, r_b^\varepsilon)$ *such that* $l^\varepsilon \in L^2(0,T; H^1(\Omega^*_{\varepsilon,K})) \cap H^1(0,T; L^2(\Omega^*_{\varepsilon,K}))$, $r_j^\varepsilon \in H^1(0,T; L^2(\Gamma^\varepsilon)) \cap L^\infty(\Gamma_T^\varepsilon)$ *for* $j = f,b$, *satisfying* (9.1) *in the weak form*

$$\begin{aligned}
\langle \partial_t l^\varepsilon, \phi\rangle_{\Omega^*_{\varepsilon,K},T} + \langle A^\varepsilon(x)\nabla l^\varepsilon, \nabla\phi\rangle_{\Omega^*_{\varepsilon,K},T} = \langle F^\varepsilon(x,l^\varepsilon) - d_l^\varepsilon(x)\, l^\varepsilon, \phi\rangle_{\Omega^*_{\varepsilon,K},T}\\
+ \varepsilon\langle \beta^\varepsilon(x) r_b^\varepsilon - \alpha^\varepsilon(x) l^\varepsilon r_f^\varepsilon, \phi\rangle_{\Gamma^\varepsilon,T}
\end{aligned} \tag{9.3}$$

for all $\phi \in L^2(0,T; H^1(\Omega^*_{\varepsilon,K}))$; *equations* (9.2) *are satisfied a.e. on* Γ_T^ε; *and* $l^\varepsilon(t,\cdot) \to l_0(\cdot)$ *in* $L^2(\Omega^*_{\varepsilon,K})$, $r_j^\varepsilon(t,\cdot) \to r_{j0}^\varepsilon(\cdot)$ *in* $L^2(\Gamma^\varepsilon)$ *as* $t \to 0$.

In a similar way as in [15, 34] we obtain the existence and uniqueness results and a priori estimates for a weak solution of the microscopic problem (9.1)–(9.2).

Lemma 9.2. *Under Assumption* 9.1 *there exists a unique nonnegative weak so-*

lution of the microscopic problem (9.1)–(9.2) *satisfying a priori estimates*

$$
\begin{aligned}
(9.4)\qquad &\|l^\varepsilon\|_{L^\infty(0,T;L^2(\Omega^*_{\varepsilon,K}))} + \|\nabla l^\varepsilon\|_{L^\infty(0,T;L^2(\Omega^*_{\varepsilon,K}))} + \|\partial_t l^\varepsilon\|_{L^2((0,T)\times\Omega^*_{\varepsilon,K})} \leq C,\\
&\varepsilon^{1/2}\|l^\varepsilon\|_{L^2(\hat\Gamma^\varepsilon_T)} + \|r^\varepsilon_j\|_{L^\infty(0,T;L^\infty(\Gamma^\varepsilon))} + \varepsilon^{1/2}\|\partial_t r^\varepsilon_j\|_{L^2(\Gamma^\varepsilon_T)} \leq C,
\end{aligned}
$$

with $j = f, b$, *where the constant* C *is independent of* ε. *Additionally, we have that*

$$
(9.5)\qquad \|(l^\varepsilon - Me^{Bt})^+\|_{L^\infty(0,T;L^2(\Omega^*_{\varepsilon,K}))} + \|\nabla(l^\varepsilon - Me^{Bt})^+\|_{L^2((0,T)\times\Omega^*_{\varepsilon,K})} \leq C\varepsilon^{1/2},
$$

where $v^+ = \max\{0, v\}$, $M \geq \sup_\Omega l_0(x)$, $B = B(F, \beta, p)$, *and* C *is independent of* ε.

Sketch of the proof. To prove the existence of a solution of the microscopic model we show the existence of a fixed point of an operator $\mathcal{B}$ defined on $L^2(0,T; H^\varsigma(\Omega^*_{\varepsilon,K}))$, with $1/2 < \varsigma < 1$, by $l^\varepsilon_n = \mathcal{B}(l^\varepsilon_{n-1})$ given as a solution of (9.1)–(9.2) with l^ε_{n-1} in (9.2) and in the nonlinear function $F^\varepsilon(x, l^\varepsilon)$ instead of l^ε_n. For a given nonnegative $l^\varepsilon_{n-1} \in L^2(0,T; H^\varsigma(\Omega^*_{\varepsilon,K}))$ there exists a nonnegative solution $(r^\varepsilon_{f,n}, r^\varepsilon_{b,n})$ of (9.2). Then, the nonnegativity of solutions, the equality

$$
\partial_t(r^\varepsilon_{f,n} + r^\varepsilon_{b,n}) = p^\varepsilon(x, r^\varepsilon_{b,n}) - d^\varepsilon_b(x) r^\varepsilon_{b,n} - d^\varepsilon_f(x) r^\varepsilon_{f,n},
$$

and the Lipschitz continuity of p ensure the boundedness of $r^\varepsilon_{f,n}$ and $r^\varepsilon_{b,n}$. Considering $l^{\varepsilon,-}_n = \min\{0, l^\varepsilon_n\}$ as a test function in (9.3) and using the nonnegativity of $r^\varepsilon_{f,n}$, $r^\varepsilon_{b,n}$ and the initial data, we obtain the nonnegativity of l^ε_n. Applying Galerkin's method and using a priori estimates similar to those in (9.4), we obtain the existence of a weak nonnegative solution $l^\varepsilon_n \in H^1(0,T; L^2(\Omega^*_{\varepsilon,K})) \cap L^2(0,T; H^1(\Omega^*_{\varepsilon,K}))$. The compactness of the embedding $H^1(0,T; L^2(\Omega^*_{\varepsilon,K})) \cap L^2(0,T; H^1(\Omega^*_{\varepsilon,K})) \subset L^2(0,T; H^\varsigma(\Omega^*_{\varepsilon,K}))$ and Schauder's theorem imply the existence of a fixed point l^ε of $\mathcal{B}$. Notice that the strong convergence of l^ε_n in $L^2(\Gamma^\varepsilon_T)$, as $n \to \infty$, implies the strong convergence of $r^\varepsilon_{j,n}$, $j = f, b$. Taking l^ε_n and $\partial_t l^\varepsilon_n$ as test functions in (9.3) and using the trace estimate (8.1), we obtain a priori estimates for l^ε_n and $\partial_t l^\varepsilon_n$. Testing (9.2) by $\partial_t r^\varepsilon_{f,n}$ and $\partial_t r^\varepsilon_{b,n}$, respectively, yields the estimates for the time derivatives. Then, using the lower semicontinuity of a norm, we obtain the a priori estimates (9.4) for $l^\varepsilon, r^\varepsilon_f$, and r^ε_b.

Especially for the derivation of a priori estimates for $\partial_t l^\varepsilon$ we consider

$$
\begin{aligned}
\varepsilon \int_{\Gamma^\varepsilon} (\beta^\varepsilon\, r^\varepsilon_b - \alpha^\varepsilon r^\varepsilon_f\, l^\varepsilon)\partial_t l^\varepsilon d\sigma_x &= \varepsilon \frac{d}{dt}\int_{\Gamma^\varepsilon} \beta^\varepsilon\, r^\varepsilon_b\, l^\varepsilon\, d\sigma_x - \varepsilon \int_{\Gamma^\varepsilon} \beta^\varepsilon\, \partial_t r^\varepsilon_b\, l^\varepsilon\, d\sigma_x\\
&\quad - \frac{\varepsilon}{2}\frac{d}{dt}\int_{\Gamma^\varepsilon} \alpha^\varepsilon r^\varepsilon_f\, |l^\varepsilon|^2 d\sigma_x + \frac{\varepsilon}{2}\int_{\Gamma^\varepsilon} \alpha^\varepsilon \partial_t r^\varepsilon_f\, |l^\varepsilon|^2 d\sigma_x.
\end{aligned}
$$

Using the equation for $\partial_t r^\varepsilon_f$, the last integral can be rewritten as

$$
\frac{\varepsilon}{2}\int_{\Gamma^\varepsilon} \alpha^\varepsilon \big(p^\varepsilon(x, r^\varepsilon_b) - \alpha^\varepsilon\, l^\varepsilon r^\varepsilon_f + \beta^\varepsilon\, r^\varepsilon_b - d^\varepsilon_f\, r^\varepsilon_f\big)|l^\varepsilon|^2 d\sigma_x.
$$

Applying the trace estimate (8.1) and using the assumptions on α^ε and β^ε, along with the nonnegativity of l^ε and r^ε_j, the boundedness of r^ε_j, uniform in ε, and the estimate $\varepsilon\|\partial_t r^\varepsilon_b\|^2_{L^2(\Gamma^\varepsilon_T)} \leq C$, we obtain

$$
\begin{aligned}
\varepsilon \int_0^\tau \int_{\Gamma^\varepsilon} (\beta^\varepsilon\, r^\varepsilon_b - \alpha^\varepsilon\, r^\varepsilon_f\, l^\varepsilon)\partial_t l^\varepsilon d\sigma_x dt &\leq C_1\big[\|l^\varepsilon(\tau)\|^2_{L^2(\Omega^*_{\varepsilon,K})} + \varepsilon^2\|\nabla l^\varepsilon(\tau)\|^2_{L^2(\Omega^*_{\varepsilon,K})}\big]\\
&\quad + C_2\big[\|l^\varepsilon\|^2_{L^2((0,\tau)\times\Omega^*_{\varepsilon,K})} + \varepsilon^2\|\nabla l^\varepsilon\|^2_{L^2((0,\tau)\times\Omega^*_{\varepsilon,K})}\big] + C_3
\end{aligned}
$$

for $\tau \in (0, T]$. Standard arguments pertaining to the difference of two solutions $l_1^\varepsilon - l_2^\varepsilon$, $r_{j,1}^\varepsilon - r_{j,2}^\varepsilon$, with $j = f, b$, imply the uniqueness of a weak solution of the microscopic problem (9.1)–(9.2). In particular, the nonnegativity of α^ε, r_j^ε, and l^ε along with the boundedness of r_j^ε, where $j = f, b$, ensures that

$$\partial_t \|r_{f,1}^\varepsilon - r_{f,2}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} \leq C\bigg(\sum_{j=f,b} \|r_{j,1}^\varepsilon - r_{j,2}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \|l_1^\varepsilon - l_2^\varepsilon\|^2_{L^2(\hat\Gamma^\varepsilon)}\bigg). \tag{9.6}$$

Testing the difference of the equations for $r_{f,1}^\varepsilon + r_{b,1}^\varepsilon$ and $r_{f,2}^\varepsilon + r_{b,2}^\varepsilon$ by $r_{f,1}^\varepsilon + r_{b,1}^\varepsilon - r_{f,2}^\varepsilon - r_{b,2}^\varepsilon$ yields

$$\|r_{b,1}^\varepsilon(\tau) - r_{b,2}^\varepsilon(\tau)\|^2_{L^2(\Gamma^\varepsilon)} \leq C \int_0^\tau \sum_{j=f,b} \|r_{j,1}^\varepsilon - r_{j,2}^\varepsilon\|^2_{L^2(\Gamma^\varepsilon)} + \|l_1^\varepsilon - l_2^\varepsilon\|^2_{L^2(\hat\Gamma^\varepsilon)} dt. \tag{9.7}$$

Applying Gronwall's lemma yields the estimate for $\|r_{j,1}^\varepsilon(\tau) - r_{j,2}^\varepsilon(\tau)\|^2_{L^2(\Gamma^\varepsilon)}$, with $\tau \in (0, T]$ and $j = f, b$, in terms of $\|l_1^\varepsilon - l_2^\varepsilon\|^2_{L^2(\hat\Gamma^\varepsilon_\tau)}$. Taking $(l^\varepsilon - S)^+$ as a test function in (9.3) and using the boundedness of r_j^ε, we obtain

$$\|(l^\varepsilon - S)^+\|_{L^\infty(0,T;L^2(\Omega^*_{\varepsilon,K}))} + \|\nabla(l^\varepsilon - S)^+\|_{L^2((0,T)\times\Omega^*_{\varepsilon,K})} \leq 2S\bigg(\int_0^T |\Omega^{*,S}_{\varepsilon,K}(t)| dt\bigg)^{\frac12},$$

where $S \geq \max\{\sup_\Omega l_0(x),\ \sup_{\Omega\times Y_x} |\beta(x,y)|,\ \sup_{\Omega\times Y_x} |\alpha(x,y)|,\ \|r_j^\varepsilon\|_{L^\infty(\Gamma_T^\varepsilon)}\}$ and $\Omega^{*,S}_{\varepsilon,K}(t) = \{x \in \Omega^*_{\varepsilon,K} : l^\varepsilon(t,x) > S\}$ for a.a. $t \in (0,T)$. Then, applying Theorem II.6.1 in [33] yields the boundedness of l^ε for every fixed $\varepsilon > 0$. Considering (9.3) for l_1^ε and l_2^ε, we obtain the estimate for $\|l_1^\varepsilon - l_2^\varepsilon\|_{L^2(0,\tau;H^1(\Omega^*_{\varepsilon,K}))}$ in terms of $\varepsilon^{1/2}\|r_{j,1}^\varepsilon - r_{j,2}^\varepsilon\|_{L^2(\Gamma^\varepsilon_\tau)}$, with $j = f, b$ and $\tau \in (0, T]$. Then, using the estimates for $\|r_{j,1}^\varepsilon(\tau) - r_{j,2}^\varepsilon(\tau)\|_{L^2(\Gamma^\varepsilon)}$ in (9.6) and (9.7) yields that $r_{j,1}^\varepsilon = r_{j,2}^\varepsilon$ a.e. in Γ_T^ε, where $j = f, b$, and hence $l_1^\varepsilon = l_2^\varepsilon$ a.e. in $(0,T) \times \Omega^*_{\varepsilon,K}$.

To show (9.5), we consider $(l^\varepsilon - Me^{Bt})^+$ as a test function in (9.3). Using the boundedness of r_j^ε, uniform in ε, and the trace estimate (8.1), we obtain for $\tau \in (0,T)$

$$\begin{aligned}\|(l^\varepsilon(\tau) - Me^{B\tau})^+\|^2_{L^2(\Omega^*_{\varepsilon,K})} &+ \|\nabla(l^\varepsilon - Me^{Bt})^+\|^2_{L^2((0,\tau)\times\Omega^*_{\varepsilon,K})} \\ &\leq C_1\|(l^\varepsilon - Me^{Bt})^+\|^2_{L^2((0,\tau)\times\Omega^*_{\varepsilon,K})} + C_2\varepsilon,\end{aligned}$$

where $M \geq \sup_\Omega l_0(x)$, $MB \geq \big(\sup_{\Omega\times Y_x} |F(x,y,0)| + \mu_\Gamma \sup_{\Omega\times Y_x} \beta(x,y)\|r_b^\varepsilon\|_{L^\infty(\hat\Gamma_T^\varepsilon)}\big)$, with μ_Γ as in (8.1). Then, applying Gronwall's lemma in the last inequality yields (9.5). ∎

Notice that in the case of a perforated domain where the periodicity and the shape of perforations vary in space, i.e., $K \neq \mathbf{I}$, we cannot apply the l-p unfolding operator to functions defined on $\Omega^*_{\varepsilon,K}$ directly. To overcome this problem we consider a local extension of a function from $\hat\Omega^{*,\varepsilon}_{n,K}$ to $\hat\Omega^\varepsilon_n$ and then apply the l-p unfolding operator $\mathcal{T}^\varepsilon_{\mathcal{L}}$, determined for functions defined on $\hat\Omega^\varepsilon$. Applying the assumptions on the microstructure of $\Omega^*_{\varepsilon,K}$ considered here, i.e., $K_x\overline{Y_0} \subset Y$ or a fibrous microstructure, we obtain the following lemma.

LEMMA 9.3. *For $x_n^\varepsilon \in \hat\Omega_n^\varepsilon$, where $1 \leq n \leq N_\varepsilon$, and $u \in W^{1,p}(Y^*_{x_n^\varepsilon,K})$, with $p \in (1, +\infty)$, there exists an extension $\hat u \in W^{1,p}(Y_{x_n^\varepsilon})$ such that*

$$\|\hat u\|_{L^p(Y_{x_n^\varepsilon})} \leq \mu\|u\|_{L^p(Y^*_{x_n^\varepsilon,K})}, \qquad \|\nabla \hat u\|_{L^p(Y_{x_n^\varepsilon})} \leq \mu\|\nabla u\|_{L^p(Y^*_{x_n^\varepsilon,K})}, \tag{9.8}$$

where μ depends on Y, Y_0, D, and K and is independent of ε and n.

For $u \in W^{1,p}(\Omega^*_{\varepsilon,K})$ we have an extension $\hat{u} \in W^{1,p}(\hat{\Omega}^\varepsilon)$ from $\hat{\Omega}^*_{\varepsilon,K}$ to $\hat{\Omega}^\varepsilon$ such that

$$\|\hat{u}\|_{L^p(\hat{\Omega}^\varepsilon)} \le \mu \|u\|_{L^p(\hat{\Omega}^*_{\varepsilon,K})}, \qquad \|\nabla \hat{u}\|_{L^p(\hat{\Omega}^\varepsilon)} \le \mu \|\nabla u\|_{L^p(\hat{\Omega}^*_{\varepsilon,K})}, \tag{9.9}$$

where μ depends on Y, Y_0, D, and K and is independent of ε.

Sketch of the proof. The proof follows along the same lines as in the periodic case; see, e.g., [14, 22, 32]. The only difference is that the extension depends on the Lipschitz continuity of K and D and the uniform boundedness from above and below of $|\det K(x)|$ and $|\det D(x)|$. To show (9.9), we first consider an extension from $D_{x_n^\varepsilon}(\widetilde{Y}^*_{K_{x_n^\varepsilon}} + \xi)$ to $D_{x_n^\varepsilon}(Y + \xi)$ satisfying the estimates (9.8), where $\xi \in \hat{\Xi}^\varepsilon_n$. Then, scaling by ε and summing up over $\xi \in \hat{\Xi}^\varepsilon_n$ and $n = 1, \dots, N_\varepsilon$ implies the estimates (9.9). $\square$

Remark. Notice that the definition of $\Omega^*_{\varepsilon,K}$ implies that there are no perforations in $\big(\Omega^{*,\varepsilon}_{n,K} \setminus \overline{\hat{\Omega}^{*,\varepsilon}_{n,K}}\big) \cap \widetilde{\Omega}_{\varepsilon/2}$, with $\widetilde{\Omega}_{\varepsilon/2} = \{x \in \Omega : \operatorname{dist}(x, \partial\Omega) > 2\,\varepsilon \max_{x \in \partial\Omega} \operatorname{diam}(D(x)Y)\}$. Also in the case of a plywood-like structure the fibers are orthogonal to the boundaries of Ω^ε_n, and near $\partial\Omega^\varepsilon_n$ we need to extend l^ε only in the directions parallel to $\partial\Omega^\varepsilon_n$. Thus, applying Lemma 9.3 we can extend l^ε from $\Omega^{*,\varepsilon}_{n,K}$ into $\hat{\Omega}^\varepsilon_n \cup \big(\Omega^\varepsilon_n \cap \widetilde{\Omega}_{\varepsilon/2}\big)$ for $n = 1, \dots, N_\varepsilon$.

THEOREM 9.4. *A sequence of solutions of the microscopic problem* (9.1)–(9.2) *converges to a solution (l, r_f, r_b) with $l \in L^2(0,T; H^1(\Omega)) \cap H^1(0,T; L^2(\Omega))$ and $r_j \in H^1(0,T; L^2(\Omega; L^2(\Gamma_x))) \cap L^\infty(\Omega_T; L^\infty(\Gamma_x))$ of the macroscopic equations*

$$\begin{aligned}
&\frac{|Y^*_{x,K}|}{|Y_x|}\partial_t l - \operatorname{div}(\mathcal{A}(x)\nabla l) = \frac{1}{|Y_x|}\int_{Y^*_{x,K}} F(x,y,l)\,dy \\
&\qquad\qquad\qquad\qquad + \frac{1}{|Y_x|}\int_{\Gamma_x} (\beta(x,y)\, r_b - \alpha(x,y)\, r_f\, l)\, d\sigma_y \quad \text{in } \Omega_T, \\
&\mathcal{A}(x)\nabla l \cdot \boldsymbol{n} = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{on } \partial\Omega, \\
&\partial_t r_f = p(x,y,r_b) - \alpha(x,y)\, l\, r_f + \beta(x,y) r_b - d_f(x,y)\, r_f \quad \text{for } y \in \Gamma_x, \\
&\partial_t r_b = \qquad\qquad \alpha(x,y)\, l\, r_f - \beta(x,y)\, r_b - d_b(x,y)\, r_b \quad \text{for } y \in \Gamma_x,
\end{aligned} \tag{9.10}$$

and for $(t,x) \in \Omega_T$, where $Y^*_{x,K} = D_x(Y \setminus K_x Y_0)$ and the macroscopic diffusion matrix is defined as

$$\mathcal{A}_{ij}(x) = \frac{1}{|Y_x|}\int_{Y^*_{x,K}} \big[A_{ij}(x,y) + (A(x,y)\nabla_y \omega^j(x,y))_i\big]\, dy \qquad \text{for } x \in \Omega,$$

for $i, j = 1, \dots, d$, with

$$\begin{aligned}
&\operatorname{div}_y(A(x,y)(\nabla_y \omega^j + e_j)) = 0 \quad \text{in } Y^*_{x,K}, \\
&A(x,y)(\nabla_y \omega^j + e_j) \cdot \boldsymbol{n} = 0 \qquad \text{on } \Gamma_x, \quad \omega^j \text{ is } Y_x\text{-periodic.}
\end{aligned} \tag{9.11}$$

We have that $\hat{l}^\varepsilon \to l$ in $L^2(\Omega_T)$, $\partial_t l^\varepsilon \rightharpoonup \partial_t l$ and $\partial_t r^\varepsilon_j \rightharpoonup \partial_t r_j$ l-t-s, $r^\varepsilon_j \to r_j$ strongly l-t-s, $j = f, b$, and

$$\begin{aligned}
&\nabla l^\varepsilon \rightharpoonup \nabla l + \nabla_y l_1 \qquad \text{l-t-s}, \quad l_1 \in L^2(\Omega_T; H^1_{\mathrm{per}}(Y^*_{x,K})), \\
&\lim_{\varepsilon \to 0} \langle A^\varepsilon \nabla l^\varepsilon, \nabla l^\varepsilon \rangle_{\Omega^*_{\varepsilon,K},T} = \langle |Y_x|^{-1} A(x,y)(\nabla l + \nabla_y l_1), \nabla l + \nabla_y l_1 \rangle_{\Omega_T, Y^*_{x,K}},
\end{aligned}$$

where $l_1(t,x,y) = \sum_{j=1}^d \frac{\partial l}{\partial x_j}(t,x)\,\omega^j(x,y)$. *Here* $\hat\phi$ *denotes the extension as in Lemma 9.3 from* $(0,T)\times\Omega^*_{\varepsilon,K}$ *to* $(0,T)\times(\widetilde\Omega_{\varepsilon/2}\cup\Omega^*_{\varepsilon,K})$ *and then by zero to* Ω_T.

Proof. Applying Lemma 9.3, we can extend l^ε from $\Omega^*_{\varepsilon,K}$ into $\hat\Omega^\varepsilon\cup\Lambda^*_{\varepsilon,K}$. We shall use the same notation for original functions and their extensions. The a priori estimates in Lemma 9.2 imply

$$\|l^\varepsilon\|_{L^2(0,T;H^1(\hat\Omega^\varepsilon\cup\Lambda^*_{\varepsilon,K}))} + \|\partial_t l^\varepsilon\|_{L^2((0,T)\times(\hat\Omega^\varepsilon\cup\Lambda^*_{\varepsilon,K}))} \le C, \tag{9.12}$$

where the constant C depends on D and K and is independent of ε. Then the sequences $\{l^\varepsilon\}$, $\{\nabla l^\varepsilon\}$, and $\{\partial_t l^\varepsilon\}$ are defined on $\hat\Omega^\varepsilon$, and we can determine $\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$, $\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)$, and $\partial_t\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$. The properties of $\mathcal{T}^\varepsilon_{\mathcal{L}}$ together with (9.12) ensure that

$$\|\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)\|_{L^2(\Omega_T\times Y)} + \|\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)\|_{L^2(\Omega_T\times Y)} + \|\partial_t\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)\|_{L^2(\Omega_T\times Y)} \le C.$$

The a priori estimates in Lemma 9.2 yield the estimates for the l-p boundary unfolding operator

$$\|\mathcal{T}^{\varepsilon,b}_{\mathcal{L}}(l^\varepsilon)\|_{L^2(\Omega_T\times\Gamma)} + \|\mathcal{T}^{\varepsilon,b}_{\mathcal{L}}(r^\varepsilon_f)\|_{H^1(0,T;L^2(\Omega\times\Gamma))} + \|\mathcal{T}^{\varepsilon,b}_{\mathcal{L}}(r^\varepsilon_b)\|_{H^1(0,T;L^2(\Omega\times\Gamma))} \le C.$$

Notice that due to the assumptions on $\Omega^*_{\varepsilon,K}$ we have that $\widetilde\Omega_{\varepsilon/2}\subset\hat\Omega^\varepsilon\cup\Lambda^*_{\varepsilon,K}$.

Then, the convergence results in Theorems 4.2, 4.4, 4.5, and 4.6 imply that there exist subsequences (denoted again by l^ε, r^ε_f, r^ε_b) and the functions $l\in L^2(0,T;H^1(\Omega))\cap H^1(0,T;L^2(\Omega))$, $l_1\in L^2(\Omega_T;H^1_{\rm per}(Y_x))$, and $r_j\in H^1(0,T;L^2(\Omega;L^2(\Gamma_x)))$ such that

$$\begin{aligned}
&\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)\rightharpoonup l && \text{weakly in } L^2(\Omega_T;H^1(Y)),\\
&\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)\to l && \text{strongly in } L^2(0,T;L^2_{\rm loc}(\Omega;H^1(Y))),\\
&\partial_t\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)\rightharpoonup \partial_t l && \text{weakly in } L^2(\Omega_T\times Y),\\
&\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)\rightharpoonup \nabla l + D_x^{-T}\nabla_{\tilde y} l_1(\cdot,D_x\cdot) && \text{weakly in } L^2(\Omega_T\times Y),\\
&\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(l^\varepsilon)\rightharpoonup l && \text{weakly in } L^2(\Omega_T\times\Gamma),\\
&\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(l^\varepsilon)\to l && \text{strongly in } L^2(0,T;L^2_{\rm loc}(\Omega;L^2(\Gamma))),\\
&r^\varepsilon_j\rightharpoonup r_j, \quad \partial_t r^\varepsilon_j\rightharpoonup \partial r_j && \text{l-t-s}, \quad r_j,\ \partial_t r_j\in L^2(\Omega_T;L^2(\Gamma_x)),\\
&\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r^\varepsilon_j)\rightharpoonup r_j(\cdot,D_xK_x\cdot) && \text{weakly in } L^2(\Omega_T\times\Gamma),\\
&\partial_t\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r^\varepsilon_j)\rightharpoonup \partial_t r_j(\cdot,D_xK_x\cdot) && \text{weakly in } L^2(\Omega_T\times\Gamma), \quad j=f,b.
\end{aligned}\tag{9.13}$$

Notice that for l^ε we have a priori estimates only in $L^2(0,T;H^1(\hat\Omega^\varepsilon\cup\Lambda^*_{\varepsilon,K}))$ and not in $L^2(0,T;H^1(\Omega))$ and cannot apply the convergence results in Theorem 4.2 directly. However, using $\|l^\varepsilon\|_{L^2(0,T;H^1(\widetilde\Omega_{\varepsilon/2}))} + \|\partial_t l^\varepsilon\|_{L^2((0,T)\times\widetilde\Omega_{\varepsilon/2})}\le C$, ensured by (9.12), applying Lemmas 7.2 and 7.3 to $\mathcal{Q}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$ and $\mathcal{R}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$, respectively, and considering the proof of Theorem 4.4, we obtain the convergences for $\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$, $\partial_t\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$, and $\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)$ in (9.13). Lemma 5.4 implies that $\nabla l^\varepsilon\rightharpoonup\nabla l+\nabla_y l_1$ l-t-s and $\partial_t l^\varepsilon\rightharpoonup\partial_t l$ l-t-s. The local strong convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$ together with the estimate $\|(l^\varepsilon - Me^{Bt})^+\|_{L^\infty(0,T;L^2(\Omega^*_{\varepsilon,K}))}\le C\varepsilon^{1/2}$, shown in Lemma 9.2, yields the strong convergence of $\hat l^\varepsilon$ in $L^2(\Omega_T)$.

To derive macroscopic equations for l^ε, we consider $\psi^\varepsilon(x)=\psi_1(x)+\varepsilon\mathcal{L}^\varepsilon_\rho(\psi_2)(x)$ with $\psi_1\in C^1(\overline\Omega)$ and $\psi_2\in C^1_0(\Omega;C^1_{\rm per}(Y_x))$ as a test function in (9.3). Applying the l-p unfolding operator and the l-p boundary unfolding operator implies

$$\frac{1}{|Y|}\Big[\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\partial_t \mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon), \mathcal{T}^\varepsilon_{\mathcal{L}}(\psi^\varepsilon)\rangle_{\Omega_T\times Y} + \langle \mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon), \mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla\psi^\varepsilon)\rangle_{\Omega_T\times Y}\Big]$$
$$= |Y|^{-1}\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\,\hat F^\varepsilon(x,\tilde y, \mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)), \mathcal{T}^\varepsilon_{\mathcal{L}}(\psi^\varepsilon)\rangle_{\Omega_T\times Y}$$
$$+\left\langle \sum_{n=1}^{N_\varepsilon}\frac{\sqrt{g_{x_n^\varepsilon}}}{\sqrt{g}|Y_{x_n^\varepsilon}|}\big[\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\beta^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r_b^\varepsilon) - \mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\alpha^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(l^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r_f^\varepsilon)\big]\chi_{\Omega_n^\varepsilon}, \mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\psi^\varepsilon)\right\rangle_{\Omega_T\times\Gamma}$$
$$-\langle \partial_t l^\varepsilon, \psi^\varepsilon\rangle_{\Lambda^*_{\varepsilon,K},T} - \langle A^\varepsilon(x)\nabla l^\varepsilon, \nabla\psi^\varepsilon\rangle_{\Lambda^*_{\varepsilon,K},T} + \langle F^\varepsilon(x,l^\varepsilon),\psi^\varepsilon\rangle_{\Lambda^*_{\varepsilon,K},T},$$

where $\hat F^\varepsilon(x,\tilde y,\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)) = \sum_{n=1}^{N_\varepsilon} F(x_n^\varepsilon, D_{x_n^\varepsilon}\tilde y, \mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon))\chi_{\hat\Omega_n^\varepsilon}(x)$ for $\tilde y\in Y$, $x\in\Omega$, and $\chi^\varepsilon_{\Omega^*_{\varepsilon,K}} = \mathcal{L}^\varepsilon_0(\chi_{Y^*_{x,K}})$. Here $\chi_{Y^*_{x,K}}$ is the characteristic function of $Y^*_{x,K} = D_x(Y\setminus K_xY_0)$, extended Y_x-periodically to $\mathbb{R}^d$. We notice that $\hat F^\varepsilon(x,\tilde y,\xi) = \mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon_0(F(x,y,\xi)))$.

Applying Lemma 5.3 yields $\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})(x,\tilde y)\to\chi_{Y^*_{x,K}}(x,D_x\tilde y)$, $\mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)(x,\tilde y)\to A(x,D_x\tilde y)$, and $\hat F^\varepsilon(x,\tilde y,l)\to F(x,D_x\tilde y,l)$ in $L^p(\Omega_T\times Y)$ for $p\in(1,+\infty)$ as $\varepsilon\to 0$. Lemma 8.3 ensures that $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\phi^\varepsilon)(x,\hat y)\to\phi(x,D_xK_x\hat y)$ in $L^p(\Omega\times\Gamma)$ as $\varepsilon\to 0$, where $\phi^\varepsilon(x) = \beta^\varepsilon(x),\alpha^\varepsilon(x)$, or $d_j^\varepsilon(x)$ and $\phi(x,y) = \alpha(x,y),\beta(x,y)$, or $d_j(x,y)$, with $j = f,b,l$, respectively.

For an arbitrary $\delta>0$ we consider $\Omega^\delta = \{x\in\Omega : \operatorname{dist}(x,\partial\Omega)>\delta\}$ and rewrite the boundary integral in the form

$$\left\langle \sum_{n=1}^{N_\varepsilon}\frac{\sqrt{g_{x_n^\varepsilon}}}{\sqrt{g}|Y_{x_n^\varepsilon}|}\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\alpha^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(l^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r_f^\varepsilon)\chi_{\Omega_n^\varepsilon}, \mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\psi^\varepsilon)\right\rangle_{\Omega^\delta\times\Gamma_T}$$
$$+\left\langle \sum_{n=1}^{N_\varepsilon}\frac{\sqrt{g_{x_n^\varepsilon}}}{\sqrt{g}|Y_{x_n^\varepsilon}|}\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\alpha^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(l^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r_f^\varepsilon)\chi_{\Omega_n^\varepsilon}, \mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\psi^\varepsilon)\right\rangle_{(\Omega\setminus\Omega^\delta)\times\Gamma_T} = I_1 + I_2.$$

Using the a priori estimates for l^ε and r_j^ε, the weak convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$ in $L^2(\Omega_T;H^1(Y))$, and the strong convergence in $L^2(0,T;L^2_{\text{loc}}(\Omega;H^1(Y)))$, we obtain

$$\begin{aligned}&\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_1 = \left\langle \frac{\sqrt{g_x}}{\sqrt{g}|Y_x|}\alpha(x,D_xK_x\hat y)\,r_f(x,D_xK_x\hat y)\,l(x),\psi_1(x)\right\rangle_{\Omega_T\times\Gamma},\\ &\lim_{\delta\to 0}\lim_{\varepsilon\to 0} I_2 = 0.\end{aligned}\tag{9.14}$$

To obtain (9.14), we also use the strong convergence and boundedness of $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\alpha^\varepsilon)$, the weak convergence and boundedness of $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r_f^\varepsilon)$, the regularity of D and K, and the strong convergence of $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\psi^\varepsilon)$. Similar arguments, along with the Lipschitz continuity of F and the strong convergence of $\hat F^\varepsilon(x,\tilde y,l)$ and $\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}}) = \mathcal{T}^\varepsilon_{\mathcal{L}}(\mathcal{L}^\varepsilon_0(\chi_{Y^*_{x,K}}))$, ensure that

$$\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\,\hat F^\varepsilon(x,\tilde y,\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)),\mathcal{T}^\varepsilon_{\mathcal{L}}(\psi^\varepsilon)\rangle_{\Omega_T\times Y}\to\langle \chi_{Y^*_{x,K}}(x,D_x\tilde y)F(x,D_x\tilde y,l),\psi_1\rangle_{\Omega_T\times Y}$$

as $\varepsilon\to 0$ and $\delta\to 0$. Using the convergence results (9.13), the strong convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(\psi^\varepsilon)$ and $\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla\psi^\varepsilon)$, and the fact that $|\Lambda^*_{\varepsilon,K}|\to 0$ as $\varepsilon\to 0$, taking the limit as $\varepsilon\to 0$, and considering the transformation of variables $y = D_x\tilde y$ for $\tilde y\in Y$ and $y = D_xK_x\hat y$ for $\hat y\in\Gamma$ yield

$$\langle |Y_x|^{-1}l,\psi_1\rangle_{Y^*_{x,K}\times\Omega_T} + \langle |Y_x|^{-1}A(x,y)(\nabla l+\nabla_y l_1),\nabla\psi_1+\nabla_y\psi_2\rangle_{Y^*_{x,K}\times\Omega_T}$$
$$+\langle |Y_x|^{-1}\big[\alpha(x,y)\,r_f\,l - \beta(x,y)\,r_b\big],\psi_1\rangle_{\Gamma_x\times\Omega_T} = \langle |Y_x|^{-1}F(x,y,l),\psi_1\rangle_{Y^*_{x,K}\times\Omega_T}.$$

Considering $\psi_1(t,x) = 0$ for $(t,x) \in \Omega_T$, we obtain $l_1(t,x,y) = \sum_{j=1}^d \partial_{x_j} l(t,x)\omega^j(x,y)$, where ω^j are solutions of the unit cell problems (9.11). Choosing $\psi_2(t,x,y) = 0$ for $x \in \Omega_T$ and $y \in Y_x$ implies the macroscopic equation for l. Applying the l-p boundary unfolding operator to the equations on Γ^ε, we obtain

$$\begin{aligned} \partial_t \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_f^\varepsilon) &= \hat{p}^\varepsilon(x,\hat{y},\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon)) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\alpha^\varepsilon)\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(l^\varepsilon)\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_f^\varepsilon) \\ &\quad + \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\beta^\varepsilon)\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(d_f^\varepsilon)\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_f^\varepsilon), \\ \partial_t \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon) &= \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\alpha^\varepsilon)\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(l^\varepsilon)\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_f^\varepsilon) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\beta^\varepsilon)\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(d_b^\varepsilon)\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon) \end{aligned} \tag{9.15}$$

in $\Omega_T \times \Gamma$, where $\hat{p}^\varepsilon(x,\hat{y},\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon)) = \sum_{n=1}^{N_\varepsilon} p(x_n^\varepsilon, D_{x_n^\varepsilon} K_{x_n^\varepsilon}\hat{y}, \mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon))\chi_{\hat{\Omega}_n^\varepsilon}(x)$ for $\hat{y} \in \Gamma$ and $x \in \Omega$. In order to pass to the limit in the nonlinear function $\hat{p}^\varepsilon(x,\hat{y},\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon))$, we have to show the strong convergence of $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon)$. We consider the difference of the equations for $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_f^{\varepsilon_k})$ and $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_f^{\varepsilon_m})$ and use $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_f^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_f^{\varepsilon_m})$ as a test function. Applying the Lipschitz continuity of p along with the strong convergence of $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\alpha^\varepsilon)$, $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(\beta^\varepsilon)$, and $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(d_j^\varepsilon)$, and the nonnegativity of l^ε and α^ε yields

$$\begin{aligned} \frac{d}{dt}\|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_f^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_f^{\varepsilon_m})\|^2_{L^2(\Omega\times\Gamma)} \leq C\Bigg[&\sum_{j=f,b} \|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_j^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_j^{\varepsilon_m})\|^2_{L^2(\Omega\times\Gamma)} \\ + \|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(l^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(l^{\varepsilon_m})\|^2_{L^2(\Omega^\delta\times\Gamma)} &+ \delta^{\frac12}\|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(l^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(l^{\varepsilon_m})\|_{L^2((\Omega\setminus\Omega^\delta)\times\Gamma)} \\ &+ \sigma(\varepsilon_k,\varepsilon_m)\Bigg], \end{aligned}$$

where $\sigma(\varepsilon_k,\varepsilon_m) \to 0$ as $\varepsilon_k,\varepsilon_m \to 0$. Considering the sum of the equations for $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_j^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_j^{\varepsilon_m})$, with $j = f,b$, using $\sum_{j=f,b}\big(\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_j^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_j^{\varepsilon_m})\big)$ as a test function, and applying the Lipschitz continuity of p imply

$$\begin{aligned} \|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_b^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_b^{\varepsilon_m})\|^2_{L^2(\Omega\times\Gamma)} &\leq C_1\int_0^\tau \|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(l^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(l^{\varepsilon_m})\|^2_{L^2(\Omega^\delta\times\Gamma)}dt \\ &+ C_2\int_0^\tau \sum_{j=f,b}\|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_j^{\varepsilon_k}) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_j^{\varepsilon_m})\|^2_{L^2(\Omega\times\Gamma)}dt + \sigma(\varepsilon_k,\varepsilon_m) + C_3\delta^{\frac12}. \end{aligned}$$

Using the a priori estimates for l^ε and the local strong convergence of $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(l^\varepsilon)$, collecting the estimates from above, and applying the Gronwall inequality, we obtain

$$\|\mathcal{T}_{\mathcal{L}}^{b,\varepsilon_k}(r_j^{\varepsilon_k})(\tau) - \mathcal{T}_{\mathcal{L}}^{b,\varepsilon_m}(r_j^{\varepsilon_m})(\tau)\|_{L^2(\Omega\times\Gamma)} \leq C\big(\sigma(\varepsilon_k,\varepsilon_m) + \delta^{\frac14}\big) \qquad \text{for} \quad j = f,b,$$

where $\sigma(\varepsilon_k,\varepsilon_m) \to 0$ as $\varepsilon_k,\varepsilon_m \to 0$ and $\delta > 0$ is arbitrary. Thus, we conclude that $\{\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_j^\varepsilon)\}$, for $j = f,b$, are Cauchy sequences in $L^2(\Omega_T\times\Gamma)$. Using the strong convergence of $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon)$ and the Lipschitz continuity of p, we obtain $\hat{p}^\varepsilon(x,\hat{y},\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_b^\varepsilon)) \rightharpoonup p(x, D_xK_x\hat{y}, r_b)$ in $L^2(\Omega_T\times\Gamma)$. Then, passing in the weak formulation of (9.15) to the limit as $\varepsilon \to 0$ implies the macroscopic equations (9.10) for r_f and r_b. This concludes the proof of the convergence up to subsequences. The strong convergence of $\mathcal{T}_{\mathcal{L}}^{b,\varepsilon}(r_j^\varepsilon)$ together with the estimates in Lemma 8.2, the boundedness of r_j^ε, with $j = f,b$, and the regularity of D and K ensures the strong l-t-s convergence of r_j^ε; i.e.,

$$\lim_{\varepsilon\to0}\varepsilon\|r_j^\varepsilon\|^2_{L^2(\Gamma_T^\varepsilon)} = \int_{\Omega_T}\frac{1}{|Y_x|}\int_{\Gamma_x}|r_j(t,x,y)|^2 d\sigma_x dx dt \qquad \text{for} \quad j = f,b.$$

The nonnegativity of l^ε and r^ε_j and the uniform boundedness of r^ε_j, with $j = f, b$ (see Lemma 9.2), along with the weak convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(r^\varepsilon_j)$ and l^ε ensure the nonnegativity of r_j and l and the boundedness of $r_j(t,x,y)$ for a.a. $(t,x) \in \Omega_T$ and $y \in \Gamma_x$. Considering $(l - M_1 e^{M_2 t})^+$ as a test function in the weak formulation of the macroscopic model (9.10) and using the boundedness of r_f and r_b, we obtain

$$\|(l - M_1 e^{M_2 t})^+\|_{L^\infty(0,T;L^2(\Omega))} + \|\nabla(l - M_1 e^{M_2 t})^+\|_{L^2(\Omega_T)} \leq 0.$$

Hence, $0 \leq l(t,x) \leq M_1 e^{M_2 T}$ for a.a. $(t,x) \in \Omega_T$, where $M_1 \geq \sup_\Omega l_0(x)$ and $M_1 M_2 \geq (\|F(x,y,0)\|_{L^\infty(\Omega;L^\infty(Y_x))} + |Y^*_{x,K}|^{-1}\|\beta(x,y)\|_{L^\infty(\Omega;L^\infty(Y_x))}\|r_b\|_{L^\infty(\Omega;L^1(\Gamma_x))})$.

Considering equations for the difference of two solutions of (9.10), taking $l_1 - l_2$, $r_{f,1} - r_{f,2}$, and $r_{b,1} - r_{b,2}$ as test functions in the weak formulation of the macroscopic problem, and using the Lipschitz continuity of F and p along with boundedness of r_j and l, we obtain uniqueness of a weak solution of the problem (9.10). Thus, we have that the entire sequence of weak solutions $(l^\varepsilon, r^\varepsilon_f, r^\varepsilon_b)$ of the microscopic problem (9.1)–(9.2) convergences to the weak solution of the macroscopic equations (9.10).

Applying the lower semicontinuity of a norm, the ellipticity of A, and the strong convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)$ and $\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})$ in $L^p(\Omega_T \times Y)$ for any $p \in (1, +\infty)$ yields

$$\begin{aligned}
&\langle |Y_x|^{-1} A(x,y)(\nabla l + \nabla_y l_1), \nabla l + \nabla_y l_1\rangle_{\Omega_T, Y^*_{x,K}} \\
&\leq \liminf_{\varepsilon\to 0} |Y|^{-1}\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon), \mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)\rangle_{\Omega_T,Y} \\
&\leq \limsup_{\varepsilon\to 0} |Y|^{-1}\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon), \mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)\rangle_{\Omega_T,Y} \\
&\leq \limsup_{\varepsilon\to 0}\langle A^\varepsilon \nabla l^\varepsilon, \nabla l^\varepsilon\rangle_{\Omega^*_{\varepsilon,K},T} = \limsup_{\varepsilon\to 0}[I_1 + I_2 + I_3],
\end{aligned}$$

where

$$I_1 = |Y|^{-1}\big\langle \hat{F}^\varepsilon(x, \tilde{y}, \mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)) - \partial_t \mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon), \mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)\big\rangle_{\Omega_T,Y},$$

$$I_2 = \int_{\Omega_T\times\Gamma} \sum_{n=1}^{N_\varepsilon} \frac{\sqrt{g_{x^\varepsilon_n}}}{\sqrt{g}|Y_{x^\varepsilon_n}|}\Big[\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\beta^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r^\varepsilon_b) - \mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(\alpha^\varepsilon)\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(l^\varepsilon\, r^\varepsilon_f)\Big]\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(l^\varepsilon)\chi_{\Omega^\varepsilon_n} d\sigma_y dx dt,$$

$$I_3 = \langle F^\varepsilon(x, l^\varepsilon) - \partial_t l^\varepsilon, l^\varepsilon\rangle_{\Lambda^*_{\varepsilon,K},T}.$$

Using the estimates in Lemma 9.2, together with $0 \leq l^\varepsilon \leq M + (l^\varepsilon - M)^+$ and the definition of $\Lambda^*_{\varepsilon,K}$, we obtain $\lim_{\varepsilon\to 0} I_3 = 0$.

Considering the strong convergence $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(r^\varepsilon_j)$, with $j = f, b$, and the local strong convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(l^\varepsilon)$ and $\mathcal{T}^{b,\varepsilon}_{\mathcal{L}}(l^\varepsilon)$, together with (9.5), taking l as a test function in (9.3), and using the fact that l_1 is a solution of the unit cell problem yields

$$\lim_{\varepsilon\to 0}[I_1 + I_2] = \langle |Y_x|^{-1} A(x,y)(\nabla l + \nabla_y l_1), \nabla l + \nabla_y l_1\rangle_{\Omega_T, Y^*_{x,K}}.$$

Hence, we conclude the convergence of the energy

$$\lim_{\varepsilon\to 0}\langle A^\varepsilon \nabla l^\varepsilon, \nabla l^\varepsilon\rangle_{\Omega^*_{\varepsilon,K},T} = \langle |Y_x|^{-1} A(x,y)(\nabla l + \nabla_y l_1), \nabla l + \nabla_y l_1\rangle_{\Omega_T, Y^*_{x,K}}, \tag{9.16}$$

as well as

$$\begin{aligned}
&\lim_{\varepsilon\to 0} |Y|^{-1}\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\,\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon), \mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)\rangle_{\Omega_T,Y} \\
&\qquad = \langle |Y_x|^{-1} A(x,y)(\nabla l + \nabla_y l_1), \nabla l + \nabla_y l_1\rangle_{\Omega_T, Y^*_{x,K}}.
\end{aligned}$$

This implies also the strong convergence of the unfolded gradient

$$\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi_{\Omega^*_{\varepsilon,K}})\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon) \to \chi_{Y^*_{x,K}}(D_x\cdot)(\nabla l + D_x^{-T}\nabla_{\tilde y} l_1(\cdot, D_x\cdot)) \quad \text{in } L^2(\Omega_T \times Y). \tag{9.17}$$

To show the strong convergence in (9.17), we consider

$$\begin{aligned}
&\big\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})(\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon) - \nabla l - D_x^{-T}\nabla_{\tilde y} l_1), \mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon) - \nabla l - D_x^{-T}\nabla_{\tilde y} l_1\big\rangle_{\Omega_T\times Y}\\
&= \big\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon), \mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)\big\rangle_{\Omega_T\times Y}\\
&- \big\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon), \nabla l + D_x^{-T}\nabla_{\tilde y} l_1\big\rangle_{\Omega_T\times Y}\\
&- \big\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})(\nabla l + D_x^{-T}\nabla_y l_1), \mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)\big\rangle_{\Omega_T\times Y}\\
&+ \big\langle \mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})(\nabla l + D_x^{-T}\nabla_y l_1), \nabla l + D_x^{-T}\nabla_{\tilde y} l_1\big\rangle_{\Omega_T\times Y}.
\end{aligned}$$

Applying the strong convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(A^\varepsilon)$ and $\mathcal{T}^\varepsilon_{\mathcal{L}}(\chi^\varepsilon_{\Omega^*_{\varepsilon,K}})$ along with the weak convergence of $\mathcal{T}^\varepsilon_{\mathcal{L}}(\nabla l^\varepsilon)$, the convergence of the energy (9.16), and the uniform ellipticity of $A(x, y)$ implies the convergence (9.17). □

Remark. Since in $\Omega^*_{\varepsilon,K}$ we have spatial changes both in the periodicity of the microstructure and in the shape of perforations, the l-p unfolding operator $\mathcal{T}^{*,\varepsilon}_{\mathcal{L}}$ is not defined on $\Omega^*_{\varepsilon,K}$ directly, and in the derivation of the macroscopic equations we use a local extension of l^ε from $\hat\Omega^{*,\varepsilon}_K$ to $\hat\Omega^\varepsilon$. The local extension allows us to apply the l-p unfolding operator $\mathcal{T}^\varepsilon_{\mathcal{L}}$ to l^ε. If we have changes only in the periodicity and no additional changes in the shape of perforations, then we can apply the l-p unfolding operator defined in a perforated domain Ω^*_ε directly, without considering an extension from $\hat\Omega^*_\varepsilon$ to $\hat\Omega_\varepsilon$, and derive macroscopic equations in the same way as in the proof of Theorem 9.4.

10. Discussions. The macroscopic model (9.10) derived from the microscopic description of a signaling process in a domain with l-p perforations reflects spatial changes in the microscopic structure of a cell tissue. The effective coefficients of the macroscopic model describe the impact of changes in the microstructure on the movement (diffusion) of signaling molecules (ligands) and on interactions between ligands and receptors in a biological tissue. The multiscale analysis also allows us to consider the influence of nonhomogeneous distribution of receptors in a cell membrane, as well as nonhomogeneous membrane properties (e.g., cells with top-bottom and front-back polarities) on the signaling process. The dependence of the coefficients on the macroscopic variables represents the difference in the signaling properties of cells depending on the size and/or position. For example, the changes in the size and shape of cells in epithelium tissues are caused by the maturation process and, hence, cells of different age may show different activity in a signaling process. Expanding the microscopic model by including equations for cell biomechanics and using the proposed multiscale analysis techniques, we can also consider the impact of mechanical properties of a biological tissue with a nonperiodic microstructure on signaling processes.

Techniques of l-p homogenization allow us to consider a wider range of composite and perforated materials than the methods of periodic homogenization allow. The structures of macroscopic equations obtained for microscopic problems posed in domains with periodic and l-p microstructures are similar. If we consider the microscopic model (9.1)–(9.2) in a domain with periodic microstructure, i.e., $D(x) = \mathbf{I}$ and $K(x) = \mathbf{I}$, where $\mathbf{I}$ denotes the identity matrix, then the macroscopic equations (9.10) with $D(x) = \mathbf{I}$ and $K(x) = \mathbf{I}$ correspond to the macroscopic equations obtained in [34] by considering the periodic distribution of cells and applying methods of periodic homogenization. For some l-p microstructures, e.g., domains consisting of periodic cells with smoothly changing perforations, it is possible to derive the same macroscopic equations by applying periodic and l-p homogenization techniques; see, e.g., [36, 37, 46]. However, as mentioned in the introduction, for the microscopic description and homogenization of processes defined in domains with, e.g., plywood-like microstructures or on oscillating surfaces of l-p microstructures the techniques of l-p homogenization are essential. Notice that methods of l-p homogenization are applied to analyze microscopic problems posed in domains with nonperiodic but deterministic microstructures, in contrast to stochastic homogenization techniques used to derive macroscopic equations for problems posed in domains with random microstructures.

The corrector function l_1 and the macroscopic diffusion coefficient in the macroscopic problem (9.10) are determined by solutions of the unit cell problems (9.11), which depend on the macroscopic variables x. This dependence corresponds to spatial changes in the structure of the microscopic domains. To compute solutions of the unit cell problems (9.11) (and hence the effective macroscopic coefficients and the corrector l_1) numerically, approaches from the two-scale finite element method [38] or the heterogeneous multiscale method [1, 2, 24, 25] can be applied. Using heterogeneous multiscale methods, one would have to compute the solutions of (9.11) only at the grid points of a discretization of the macroscopic domain, which requires much lower spatial resolution than computing the microscopic problem on the scale of a single cell. A similar approach can be applied for numerical simulations of the ordinary differential equations determining the dynamics of receptor densities, which depend on the macroscopic x and the microscopic y variables as parameters.

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