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Lin, Ping

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A SEQUENTIAL REGULARIZATION METHOD FOR TIME-DEPENDENT INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

PING LIN†

Abstract. The objective of the paper is to present a method, called the sequential regularization method (SRM), for the nonstationary incompressible Navier–Stokes equations from the viewpoint of regularization of differential-algebraic equations (DAEs), and to provide a way to apply a DAE method to partial differential-algebraic equations (PDAEs). The SRM is a functional iterative procedure. It is proved that its convergence rate is $O(\epsilon^m)$, where m is the number of the SRM iterations and ϵ is the regularization parameter. The discretization and implementation issues of the method are considered. In particular, a simple explicit-difference scheme is analyzed and its stability is proved under the usual step-size condition of explicit schemes. It appears that the SRM formulation is new in the Navier–Stokes context. Unlike other regularizations or pseudocompressibility methods in the Navier–Stokes context, the regularization parameter ϵ in the SRM need not be very small and the regularized problem in the sequence may be essentially nonstiff in time direction for any ϵ. Hence the stability condition is independent of ϵ even for explicit time discretization. Numerical experiments are given to verify our theoretical results.

Key words. Navier–Stokes equations, regularization method, asymptotic analysis, finite difference, energy estimates

AMS subject classifications. 65M12, 76D05, 35Q30, 35B40

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1. Introduction.

Many methods have been proposed for the numerical solution of the nonstationary, incompressible Navier–Stokes equations. Direct discretizations include finite-difference and finite-volume techniques on staggered grids (e.g., [20, 7, 21]), finite-element methods using conformal and nonconformal elements (e.g., [13, 34, 19]), and spectral methods (e.g., [10]). Another approach yielding many methods has involved some initial reformulation and/or regularization of the equations, to be followed by a discretization of the (hopefully) simplified system of equations. Examples of such methods include pseudocompressibility methods, projection, and pressure-Poisson reformulations (e.g., [11, 15, 23, 29, 31]).

Another topic of great recent interest is the numerical solution of differential-algebraic equations (DAEs). In their most popular special form, these are ordinary differential equations with some equality constraints (e.g., [9, 16]). Recall that an important concept for measuring the difficulty in solving DAEs is given by the (differential) *index*, which is defined by the minimal number of analytical constraint differentiations such that the DAE can be transformed by algebraic manipulations into an explicit first-order differential system for all original unknowns. For instance,

$$\frac{dx}{dt} = f(x,t) - B(t)y, \tag{1.1a}$$

$$0 = C(t)x + r(t) \equiv g(x,t) \tag{1.1b}$$

is an index-2 DAE if the matrix CB is invertible for all t.

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†Department of Mathematics/Institute of Applied Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2, Canada. Current address: Scientific Computing and Computational Math (SCCM) Program, Division of Mechanics and Computation, Stanford University, Durand 262, Stanford, CA 94305-4040 (plin@sccm.stanford.edu). The work of this author was partially supported under a Killam postgraduate grant.

While a significant body of knowledge about the theory and numerical methods for DAEs has been accumulated, not much has been extended to partial differential-algebraic equations (PDAEs). The incompressible Navier–Stokes equations form, in fact, an example of a PDAE: to recall, these equations read

$$\mathbf{u}_t + (\mathbf{u} \cdot \mathbf{grad})\mathbf{u} = \mu\Delta\mathbf{u} - \mathbf{grad}p + \mathbf{f}, \tag{1.2a}$$

$$\mathrm{div}\mathbf{u} = 0, \tag{1.2b}$$

$$\mathbf{u}|_{\partial\Omega} = \mathbf{b}, \quad \mathbf{u}|_{t=0} = \mathbf{a} \tag{1.2c}$$

in a bounded two- or three-dimensional domain Ω and the time interval $0 \leq t \leq T$. Here $\mathbf{u}(\mathbf{x}, t)$ represents the velocity of a viscous incompressible fluid, $p(\mathbf{x}, t)$ the pressure, $\mathbf{f}$ the prescribed external force, $\mathbf{a}(\mathbf{x})$ the prescribed initial velocity, and $\mathbf{b}(t)$ the prescribed velocity boundary values. The system (1.2c) can be seen as a partial differential equation (PDE) with constraint (1.2b) with respect to the time variable t. Comparing with the DAE form (1.1), it is easily verified that (1.2c) has index-2, since the operator div$\mathbf{grad} = \Delta$ is invertible (under appropriate boundary conditions). Indeed, the pressure-Poisson reformulation of (1.2c) (see, e.g., [15]) corresponds to a direct index reduction of the PDAE, i.e., a differentiation of the constraint with respect to t followed by substitution into the momentum equations.

In this paper we propose and analyze a *sequential regularization method* (SRM) for solving the incompressible Navier–Stokes equations. The method is defined as follows: with $p_0(\mathbf{x}, t)$ an initial guess, for $s = 1, 2, \ldots$, solve the problem

$$\begin{aligned} &\epsilon(\mathbf{u}_s)_t - \mathbf{grad}(\alpha_1(\mathrm{div}\mathbf{u}_s)_t + \alpha_2\mathrm{div}\mathbf{u}_s) + \epsilon(\mathbf{u}_s \cdot \mathbf{grad})\mathbf{u}_s \\ &\quad = \epsilon\mu\Delta\mathbf{u}_s - \epsilon\mathbf{grad}p_{s-1} + \epsilon\mathbf{f}, \end{aligned} \tag{1.3a}$$

$$\mathbf{u}_s|_{\partial\Omega} = \mathbf{b}, \mathbf{u}_s|_{t=0} = \mathbf{a}, \tag{1.3b}$$

$$p_s = p_{s-1} - \frac{1}{\epsilon}(\alpha_1(\mathrm{div}\mathbf{u}_s)_t + \alpha_2\mathrm{div}\mathbf{u}_s). \tag{1.3c}$$

This method is an extension of the SRM that was proposed and analyzed in [3, 4] for DAEs with possible isolated singularities (i.e., when the matrix CB in (1.1) may become singular at isolated times t). In that DAE case the method reads

$$\frac{dx_s}{dt} = f(x_s, t) + By_s, \tag{1.4a}$$

$$By_s = By_{s-1} - \frac{1}{\epsilon}BE\left(\alpha_1\frac{d}{dt}g(x_s, t) + \alpha_2 g(x_s, t)\right), s = 1, 2, \ldots \tag{1.4b}$$

with the same initial or boundary conditions of (1.1) and initial conditions $g(x(0), 0) = 0$, where By_0 is given and the matrix E is chosen to make the differential equation (plugging (1.4b) into (1.4a)) for x_s stable. For example, E could be the unit matrix I if $B = C^T$, which is the case in (1.2c)–(1.3). Also, B can be dropped in (1.4b) if the DAE is without singularity. The SRM was motivated by Bayo and Avello's augmented Lagrangian method for constrained mechanical systems [6], and also bears a relationship to Uzawa's algorithm [2] in the context of optimization theory and economics and to the augmented Lagrangian method of [12] in the Navier–Stokes context.

It is well known that direct-index reduction via differentiation may lead to the drift-off problem; i.e., the constraints (1.1b) need not be satisfied when the reformulated problem is integrated in time. Baumgarte's stabilization is the most popular method to remedy the drift difficulty [5]. The SRM is derived by combining a modified penalty-regularization method with Baumgarte's stabilization formulation. It is not difficult to see that the one-step SRM iteration becomes a usual-penalty method (cf. [27] or [22]) for problem (1.1) if we take $\alpha_1 = 0$, $\alpha_2 = 1$, and $y_0 = 0$. Also, one-step SRM can be seen as a usual-penalty method for Baumgarte's formulation if we let $y_0 = 0$. In [3, 4] we proved that the difference between the exact solution of a DAE and the corresponding SRM iterate becomes $O(\epsilon^m)$ in magnitude after the mth iteration (away from the starting value of the independent variable t if $\alpha_1 = 0$). Hence, unlike usual regularizations, the perturbation parameter ϵ does not have to be chosen very small, so the regularized problems can be less stiff and/or more stable. Also, from (1.4b) we can see that the constraints are enforced in the iteration procedure.

The SRM with $\alpha_1 = 0$ is especially useful for DAEs with singularities since in this case Baumgarte's stabilization does not work [3, 4]. However, for DAEs without singularities it is much better to take $\alpha_1 \neq 0$ because certain restrictions on choosing y_0 do not apply and, more importantly, the equation for x_s is essentially not stiff if the original problem (1.1a) with given y is not. Hence a nonstiff time integrator can be used for any regularization parameter ϵ. For the Navier–Stokes application (1.3) we therefore choose $\alpha_1 > 0$ so that we can still take ϵ to be very small even when we use an explicit time discretization. So one SRM iteration is often good enough. However, we cannot ignore the choice $\alpha_1 = 0$. In the case of $\alpha_1 > 0$, although we use explicit time discretization, a symmetric positive definite system relevant to the discretization of the operator $I + \frac{\alpha_1}{\epsilon}\mathbf{grad}\mathrm{div}$ still needs to be inverted. If we take $\alpha_1 = 0$, then we do not need to solve any system to obtain the discrete solution. In this case, (1.3) is not stiff only for relatively large ϵ. So more than one SRM iteration is required generally. In what follows, the convergence proof in section 3 is mainly for the case of $\alpha_1 > 0$. The discussion for the case of $\alpha_1 = 0$ is a bit more complicated but can essentially be carried out in a similar way. We will give a remark about the convergence for this case in section 3 and a numerical verification in section 4.

The importance of the treatment of the incompressibility constraint has long been recognized in the Navier–Stokes context. A classical approach is the projection method of [11], where one has to solve a Poisson equation for the pressure p with zero Neumann boundary condition which is, however, an unphysical boundary condition. Recently a reinterpretation of the projection method in the context of the so-called *pressure stabilization methods* or, more generally, "pseudocompressibility methods" has been given in [29]. Some convergence estimates for the pressure can be obtained (cf. [30, 28]). In his review paper [29] Rannacher lists some best-known examples of "pseudocompressibility methods" (which are actually sorts of regularization methods):

$$\begin{array}{llll} \mathrm{div}\mathbf{u} + \epsilon p_t & = 0, \text{ in } \Omega \times [0,T), & p|_{t=0} = p_0 & \text{(artificial compressibility)}, \\ \mathrm{div}\mathbf{u} + \epsilon p & = 0, \text{ in } \Omega \times [0,T) & & \text{(penalty method)}, \\ \mathrm{div}\mathbf{u} - \epsilon \Delta p & = 0, \text{ in } \Omega \times [0,T), & \frac{\partial p}{\partial n}|_{\partial\Omega} = p_0 & \text{(pressure stabilization)}. \end{array}$$

If we generalize Baumgarte's stabilization to this PDAE example (1.2c), we get

$$\mathbf{u}_t + (\mathbf{u}\cdot\mathbf{grad})\mathbf{u} = \mu\Delta\mathbf{u} - \mathbf{grad}p + \mathbf{f}, \tag{1.5a}$$

$$(\mathrm{div}\mathbf{u})_t + \gamma\mathrm{div}\mathbf{u} = 0. \tag{1.5b}$$

Eliminating $\mathbf{u}_t$ from (1.5), we obtain an equation for p. We then find that this stabilization can be seen as a kind of pressure stabilization with $\gamma = \epsilon^{-1}$. Although it works, since we do not have singularity here, it still suffers from the problem of setting up an unphysical boundary condition for the Poisson equation for p. Also, in this formulation equations for $\mathbf{u}$ and p are not uncoupled.

In the SRM formulation (1.3) we do not need to set up boundary conditions for p, so it should be more natural than various pressure-Poisson formulations. This method relates to the idea of penalty methods but, unlike the penalty method, the parameter ϵ can be large here. Hence more convenient methods (nonstiff) can be used for time integration, and then nonlinear terms can be treated easily. We will indicate in section 4 that ϵ has little to do with the stability of the discretization there; i.e., the stability restriction is satisfied for a wide range of ϵ. We also indicate there that, in the case of small viscosity, the usual time-step restrictions for the explicit schemes can be loosened.

A similar procedure following [2] (Uzawa's iterative algorithm) in the framework of optimization theory and economics has actually appeared in the Navier–Stokes context for the stationary Stokes equations (i.e., without the nonlinear term and the time-dependent term in (1.2c)) with $\alpha_1 = 0$ using the augmented Lagrangian idea; see Fortin and Glowinski [12]. (Also see [13] for some related discussion.) Note that in their procedure ϵ^{-1} in (1.3c) is replaced by a parameter ρ. They prove that $\rho = \epsilon^{-1}$ is approximately optimal. For the nonlinear case, they combine Uzawa's algorithm with a linearization iteration. They claim convergence but find it hard to analyze the convergence rate because their analysis depends on the spectrum of an operator which is nonsymmetric in the nonlinear case. For the nonstationary case (1.2c), the augmented Lagrangian method cannot be applied directly. Therefore [12] first discretizes (1.2a) with respect to the time t (an implicit scheme is used). Then the problem becomes a stationary one in each time step. Hence Uzawa's algorithm can be applied and converges in each time step. So, for the nonstationary case, their iterative procedure is in essence to provide a method to solve the time-discretized problem. Thus their iterative procedure has little to do with the time discretization or, in other words, they still do time discretization directly for the problem (1.2c). Consequently an implicit scheme is always suggested because of the constraints (1.2b), and then a linearization is always needed to treat the nonlinear case.

These properties are not shared by our method. We will prove that the convergence results of [3, 4] still hold for the PDAE case (1.3). Hence the solution sequence of (1.3) converges to the solution of (1.2c) with the error estimate of $O(\epsilon^m)$ after the mth iteration. Therefore, roughly speaking, the rate is about $O(\epsilon)$. We prove the convergence results using the method of asymptotic expansions which is independent of the optimization theory and is also applicable to the steady-state case. In addition, when the finite-element method is used, the difficulty of constructing test functions in a divergence-free space can be avoided by using the formulation of the SRM.

We indicate here that, as many others do, we include the viscosity parameter μ in the error estimates; i.e., the estimates could deteriorate when μ is very small. This is because we have here an unresolved technical difficulty associated with our inability to obtain an appropriate upper bound for the nonlinear term and with the weaker elliptic operator $\mu\Delta\mathbf{u}$ (which is a dissipative term) as $\mu \to 0$. In the SRM formulation a supplementary dissipative term, $-\alpha_2 \mathbf{grad}\mathrm{div}\mathbf{u}_s$, is introduced without perturbing the solution. As indicated in [12] for the stationary case, the relative advantage of such methods may therefore become more apparent for small values of the viscosity.

The paper is organized as follows: in section 2 we define some preliminaries and discuss regularity properties of the solution of (1.3). The convergence of the SRM for Navier–Stokes equations is proved in section 3. Finally, in section 4 a simple difference scheme is discussed and some numerical experiments are presented. These numerical experiments are only exploratory in nature.

To summarize, our objective in this paper is to present a method for the nonstationary Navier–Stokes equations from the viewpoint of DAE regularization and to provide a way to apply a DAE method to PDAEs. It appears that such a formulation is new in the Navier–Stokes context and it is worthwhile because of the following points:

- Since ϵ need not be taken very small, the regularized problems in the sequence (1.3) are more stable/less stiff and then more convenient difference schemes, e.g., explicit schemes in time, can be used under theoretical assurance. If we take $\alpha_1 > 0$ then this is also true for small ϵ.
- The problem of additional boundary conditions which arises in the pressure-Poisson formulation and projection methods does not arise here. Finite-element methods can be used easily and the elements do not have to conform to the incompressibility condition to separate the variables $\mathbf{u}$ and p.

We intend in the near future to do a more thorough study and in particular to carry out more thorough numerical experiments to verify the appeal of the SRM.

2. Preliminaries. Before we begin our analysis, we first describe some notation and assumptions. As usual, we use $\mathbf{L}^p(\Omega)$, or simply $\mathbf{L}^p$, to denote the space of functions defined and pth-power integrable in Ω, and

$$\|\mathbf{u}\|_p = \left(\int_\Omega \sum_{i=1}^n u_i^p \, d\mathbf{x} \right)^{\frac{1}{p}}$$

its norm, where $\mathbf{u} = (u_1, \ldots, u_n)$. We denote the inner product in $\mathbf{L}^2$ by $(\cdot, \cdot)$ and let $\|\cdot\| \equiv \|\cdot\|_2$. $\mathbf{C}^\infty$ is the space of functions continuously differentiable any number of times in Ω, and $\mathbf{C}_0^\infty$ consists of those members of $\mathbf{C}^\infty$ with compact support in Ω. $\mathbf{H}^m$ is the completion in the norm

$$\|\mathbf{u}\|_{\mathbf{H}^m} = \left(\sum_{0 \le |\alpha| \le m} \|D^\alpha \mathbf{u}\|^2 \right)^{\frac{1}{2}}.$$

We will consider the boundary conditions to be homogeneous, i.e., $\mathbf{b} \equiv \mathbf{0}$ in (1.2c), to simplify the analysis. Nevertheless, through the inclusion of a general forcing term, the results may be generalized to the case of nonhomogeneous boundary values. We are interested in the case that (1.2c) has a unique solution and the solution belongs to $\mathbf{H}^2$, where the arbitrary constant which the pressure p is up to is determined by

$$\int_\Omega p(\mathbf{x}, \cdot) \, d\mathbf{x} = 0. \tag{2.1}$$

Hence some basic compatibility condition is assumed (cf. [19]):

$$\mathbf{a}|_{\partial\Omega} = \mathbf{0}, \ \mathrm{div}\mathbf{a} \equiv 0. \tag{2.2}$$

Furthermore, we assume

$$\sup_{t \in [0,T]} \|\mathbf{f}\| \le M_1, \ \|\mathbf{a}\|_{\mathbf{H}^2} \le M_1, \tag{2.3}$$

where M_1 is a positive constant.

We take p_0 in (1.3) satisfying (2.1). Hence it is easy to see that p_s satisfies (2.1) for all s.

For simplicity, we only consider the two-dimensional case in this paper. We can treat the three-dimensional case in the same way, possibly with some more assumptions. Throughout the paper M represents a generic constant which may depend on μ as we have explained in the introduction. We will also allow that M depends on the finite time-interval length T since we are not going to discuss very long-time behavior of the method in this paper.

First we write down some inequalities.

- Poincaré's inequality:

$$\|\mathbf{u}\| \le \gamma \|\mathbf{grad\ u}\|, \quad \text{if } \mathbf{u}|_\Omega = 0 \tag{2.4}$$

  or, more generally (see [26]), for $\mathbf{u} \in \mathbf{H}^1(\Omega)$

$$\|\mathbf{u}\| \le C_\Omega \left(\|\mathbf{grad\ u}\| + \left| \int_\Omega \mathbf{u}\, d\mathbf{x} \right| \right). \tag{2.5}$$

- Young's inequality:

$$abc \le \frac{1}{p} a^p + \frac{1}{q} b^q + \frac{1}{r} c^r \tag{2.6}$$

  if $a, b, c > 0,\ p, q, r > 1$, and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

- Hölder's inequality:

$$\int_\Omega |f||g||h| d\mathbf{x} \le \|f\|_p \|g\|_q \|h\|_r \tag{2.7}$$

  if $p, q, r > 1$ and $\frac{1}{p} + \frac{1}{q} + \frac{1}{r} = 1$.

- Sobolev's inequality in two-dimensional space:

$$\|\mathbf{u}\|_4 \le \gamma_1^{\frac{1}{4}} \|\mathbf{u}\|^{\frac{1}{2}} \|\mathbf{grad\ u}\|^{\frac{1}{2}}, \tag{2.8}$$

  where $\gamma_1 = 2$ if $\Omega = \mathbf{R}^2$.

Suppose that $\mathbf{w}$ stands for the difference of two solutions of the SRM (1.3). Then $\mathbf{w}$ satisfies a homogeneous problem of (3.1) (see next section). Hence, using the estimate in Lemma 3.1, uniqueness of the solution of the SRM (1.3) is easy to discuss. The existence can be analyzed by following the standard existence argument of Navier–Stokes equations (e.g., [33, 18]) and that of penalized Navier–Stokes equations (e.g., [8]). In this paper we assume the existence of the solution of the SRM and concentrate on the proof of the convergence of the method. Before we do that, we derive the following regularity results of the solution of the SRM (1.3).

LEMMA 2.1. *For the solution* $\{\mathbf{u}_s, p_s\}$ *of* (1.3)*, we have the following estimates:*

$$\begin{aligned} &\|\mathbf{u}_s\|^2_{\mathbf{H}^1} + \int_0^T \left(\frac{\alpha_1}{\epsilon^2} \|(\mathrm{div}\mathbf{u}_s)_t\|^2_{\mathbf{H}^1} + \frac{\alpha_2}{\epsilon^2} \|\mathrm{div}\mathbf{u}_s\|^2_{\mathbf{H}^1} + \|(\mathbf{u}_s)_t\|^2 + \|\Delta \mathbf{u}_s\|^2 + \|p_s\|^2_{H^1} \right) dt \\ &\le M \left[\|a\|^2_{\mathbf{H}^1} + \int_0^T \left(\|f\|^2 + \|\mathbf{grad} p_{s-1}\|^2 \right) dt \right]. \end{aligned} \tag{2.9}$$

Proof. For simplicity of notation we denote $\mathbf{u}_s$ as $\mathbf{v}$ here. The proof for the case of $\alpha_1 = 0$ is just the same as that in [8]. So we only consider the case $\alpha_1 > 0$. Hence, without loss of generality, we take $\alpha_1 = 1$ and $\alpha_2 = \alpha$. We then write (1.3) as

$$\mathbf{v}_t - \frac{1}{\epsilon}\mathbf{grad}((\mathrm{div}\mathbf{v})_t + \alpha\mathrm{div}\mathbf{v}) + (\mathbf{v}\cdot\mathbf{grad})\mathbf{v} \tag{2.10a}$$
$$= \mu\Delta\mathbf{v} - \mathbf{grad}p_{s-1} + \mathbf{f},$$

$$\mathbf{v}|_{\partial\Omega} = \mathbf{0}, \mathbf{v}|_{t=0} = \mathbf{a}, \tag{2.10b}$$

$$p_s = p_{s-1} - \frac{1}{\epsilon}((\mathrm{div}\mathbf{v})_t + \alpha\mathrm{div}\mathbf{v}). \tag{2.10c}$$

The proof follows the ideas in [8]. Multiplying (2.10a) by $\mathbf{v}$ and integrating with respect to the space variables on the domain Ω, we get

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\|\mathbf{v}\|^2 + \frac{1}{2\epsilon}\frac{d}{dt}\|\mathrm{div}\mathbf{v}\|^2 + \frac{\alpha}{\epsilon}\|\mathrm{div}\mathbf{v}\|^2 + \mu\|\mathbf{grad}\mathbf{v}\|^2\\
&= -((\mathbf{v}\cdot\mathbf{grad})\mathbf{v},\mathbf{v}) - (\mathbf{grad}p_{s-1},\mathbf{v}) + (\mathbf{f},\mathbf{v})\\
&\le \frac{\alpha}{2\epsilon}\|\mathrm{div}\mathbf{v}\|^2 + \frac{\gamma_1\epsilon}{2\alpha}\|\mathbf{v}\|^2\|\mathbf{grad}\mathbf{v}\|^2 + \frac{\mu}{2\gamma}\|\mathbf{v}\|^2 + \frac{\gamma}{\mu}\|\mathbf{grad}p_{s-1}\|^2 + \frac{\gamma}{\mu}\|\mathbf{f}\|^2,
\end{aligned}$$

where we use $-((\mathbf{v}\cdot\mathbf{grad})\mathbf{v},\mathbf{v}) = \frac{1}{2}((\mathrm{div}\mathbf{v})\mathbf{v},\mathbf{v})$. Then let $c = \min(\frac{\alpha}{\epsilon}, \frac{\mu}{2\gamma})$ and $Y = \|\mathbf{v}\|^2 + \frac{1}{\epsilon}\|\mathrm{div}\mathbf{v}\|^2$. Using Poincaré's inequality (2.4), we obtain

$$\frac{d}{dt}Y + cY + \frac{1}{2}\left(\mu - \frac{\gamma_1\epsilon}{\alpha}Y\right)\|\mathbf{grad}\mathbf{v}\|^2 \le \frac{\gamma}{\mu}(\|\mathbf{f}\|^2 + \|\mathbf{grad}p_{s-1}\|^2). \tag{2.11}$$

Note $Y(0) = \|\mathbf{a}\|^2$. Write (2.11) as

$$\frac{d}{dt}\left(\mu - \frac{\gamma_1\epsilon}{\alpha}Y\right) - \frac{\gamma_1\epsilon}{2\alpha}\|\mathbf{grad}\,\mathbf{v}\|^2\left(\mu - \frac{\gamma_1\epsilon}{\alpha}Y\right) \ge -\frac{\gamma\gamma_1\epsilon}{\mu\alpha}(\|\mathbf{grad}p_{s-1}\|^2 + \|\mathbf{f}\|^2).$$

Applying a standard technique for solving linear differential equations and taking ϵ appropriately small so that

$$\mu - \frac{\gamma_1\epsilon}{\alpha}Y(0) \ge \frac{\mu}{2} \quad\text{and}\quad \frac{\gamma\gamma_1\epsilon}{\mu\alpha}\int_0^T (\|\mathbf{f}\|^2 + \|\mathbf{grad}p_{s-1}\|^2)\,dt \le \frac{\mu}{4},$$

we get

$$\mu - \frac{\gamma_1\epsilon}{\alpha}Y(t) \ge \frac{\mu}{4} \quad \forall t\in[0,T]. \tag{2.12}$$

Then, using the same technique and (2.11), we have

$$\begin{aligned}
Y &\le \|\mathbf{a}\|^2\exp(-ct) + M\exp(-ct)\int_0^t (\|\mathbf{f}\|^2 + \|\mathbf{grad}p_{s-1}\|^2)\exp(cz)\,dz\\
&\le M\left[\|\mathbf{a}\|^2 + \int_0^t (\|\mathbf{f}\|^2 + \|\mathbf{grad}p_{s-1}\|^2)\,dz\right].
\end{aligned} \tag{2.13}$$

That is, (2.9) holds for $\|\mathbf{u}\|^2$. Integrating (2.11) directly and using (2.12) yields

$$\int_0^t \|\mathbf{grad}\mathbf{u}\|^2\,dz \le M\left[\|\mathbf{a}\|^2 + \int_0^t (\|\mathbf{f}\|^2 + \|\mathbf{grad}p_{s-1}\|^2)\,dz\right]. \tag{2.14}$$

To prove other estimates of (2.9) we define an operator

$$A\mathbf{w} = -\frac{1}{\epsilon}\mathbf{grad}((\mathrm{div}\mathbf{w})_t + \alpha \mathrm{div}\mathbf{w}) - \mu\Delta\mathbf{w} = \mathbf{g}, \tag{2.15}$$

where $\mathbf{w}$ satisfies $\mathbf{w}|_{\partial\Omega} = \mathbf{0}$ and $\mathbf{w}|_{t=0} = \mathbf{a}$. Let

$$q = -\frac{1}{\epsilon}((\mathrm{div}\mathbf{w})_t + \alpha \mathrm{div}\mathbf{w}).$$

Then we have (noting $\mathrm{div}\mathbf{w}|_{t=0} = 0$)

$$-\mu\Delta\mathbf{w} + \mathbf{grad}q = \mathbf{g}, \tag{2.16a}$$

$$\mathrm{div}\mathbf{w} = -\epsilon \int_0^t q \exp(-\alpha(t-z))\, dz. \tag{2.16b}$$

This is a general nonhomogeneous Stokes problem. Using the results described in [8] (or cf. [33]), we get

$$\|\Delta\mathbf{w}\| + \|\mathbf{grad}q\| \le M\left[\|\mathbf{g}\| + \epsilon\int_0^t \|\mathbf{grad}q\| \exp(-\alpha(t-z))\, dz\right]. \tag{2.17}$$

Applying Gronwall's inequality technique it is easy to obtain

$$\|\mathbf{grad}q\| \le M\left(\|\mathbf{g}\| + \epsilon\int_0^t \|\mathbf{grad}q\|\, dz\right) \tag{2.18}$$

and

$$\int_0^t \|\mathbf{grad}q\|^2\, dz \le M\int_0^t \|\mathbf{g}\|^2\, dz = M\int_0^t \|A\mathbf{w}\|^2\, dz. \tag{2.19}$$

It thus follows that

$$\|\Delta\mathbf{w}\| \le M\left(\|\mathbf{g}\| + \epsilon\int_0^t \|\mathbf{g}\|\, dz\right) = M\left(\|A\mathbf{w}\| + \epsilon\int_0^t \|A\mathbf{w}\|\, dz\right) \tag{2.20}$$

and then

$$\int_0^t \|\Delta\mathbf{w}\|^2\, dz \le M\int_0^t \|\mathbf{g}\|^2\, dz = M\int_0^t \|A\mathbf{w}\|^2\, dz. \tag{2.21}$$

From (2.16b) and (2.19) we thus have

$$\frac{1}{\epsilon^2}\int_0^t \|\mathbf{grad}\mathrm{div}\mathbf{w}\|^2\, dz = \int_0^t \|\mathbf{grad}q\|^2\, dz \le M\int_0^t \|A\mathbf{w}\|^2\, dz. \tag{2.22}$$

Then

$$\frac{1}{\epsilon^2}\int_0^t \|\mathbf{grad}(\mathrm{div}\mathbf{w})_t\|^2\, dz \le M\int_0^t \|A\mathbf{w}\|^2\, dz \tag{2.23}$$

follows from (2.15).

Now taking the scalar product of (2.10a) with $A\mathbf{v}$, we have

$$\frac{1}{2\epsilon}\|(\mathrm{div}\mathbf{v})_t\|^2 + \frac{\alpha}{2\epsilon}\frac{d}{dt}\|\mathrm{div}\mathbf{v}\|^2 + \frac{\mu}{2}\frac{d}{dt}\|\mathbf{grad}\mathbf{v}\|^2 + \|A\mathbf{v}\|^2$$
$$= -((\mathbf{v}\cdot\mathbf{grad})\mathbf{v}, A\mathbf{v}) - (\mathbf{grad}p_{s-1}, A\mathbf{v}) + (\mathbf{f}, A\mathbf{v}).$$

Note that

$$-((\mathbf{v}\cdot\mathbf{grad})\mathbf{v}, A\mathbf{v}) \le \|\mathbf{v}\|_4\|\mathbf{grad}\mathbf{v}\|_4\|A\mathbf{v}\| \le \gamma_1^{\frac{1}{2}}\|\mathbf{v}\|^{\frac{1}{2}}\|\mathbf{grad}\mathbf{v}\|\|\Delta\mathbf{v}\|^{\frac{1}{2}}\|A\mathbf{v}\|$$
$$\le \delta(\|A\mathbf{v}\|^2 + \|\Delta\mathbf{v}\|^2) + \frac{\gamma_1^2}{16\delta^3}(\|\mathbf{v}\|^2\|\mathbf{grad}\mathbf{v}\|^2)\|\mathbf{grad}\mathbf{v}\|^2$$
$$\le \delta\left[\|A\mathbf{v}\|^2 + M^2\|A\mathbf{v}\|^2 + M^2\epsilon^2\left(\int_0^t \|A\mathbf{v}\|\,dz\right)^2\right]$$
$$+\frac{\gamma_1^2}{16\delta^3}(\|\mathbf{v}\|^2\|\mathbf{grad}\mathbf{v}\|^2)\|\mathbf{grad}\mathbf{v}\|^2,$$

where we use (2.20) for the last inequality. Recall that we have gotten the estimates for $\|\mathbf{v}\|^2$ and $\int_0^t \|\mathbf{grad}\mathbf{v}\|^2\,dz$. Therefore, taking $\delta(1+M^2) < \frac{1}{4}$, it is not difficult to obtain

$$\|\mathbf{grad}\mathbf{v}\| + \frac{1}{\epsilon}\int_0^t \|(\mathrm{div}\mathbf{v})_t\|^2\,dz + \int_0^t \|A\mathbf{v}\|^2\,dz$$
$$\le M\left[\|\mathbf{grad}\mathbf{a}\|^2 + \int_0^t (\|\mathbf{f}\|^2 + \|\mathbf{grad}p_{s-1}\|^2)\,dz + \epsilon^2\int_0^t \|A\mathbf{v}\|^2\,dz\right].$$

Taking ϵ such that $M\epsilon^2 < 1$, we then get (2.9) for $\int_0^t \|A\mathbf{v}\|^2\,dz$ and $\|\mathbf{grad}\mathbf{v}\|$. Noting (2.22), (2.23), and from (1.3c), (2.9) is true for $\int_0^t \|\mathbf{grad}p_s\|^2\,dz$ also. Applying the inequality (2.5) and noting that p_s satisfies (2.1) yields the bound for $\int_0^t \|p_s\|^2\,dz$. Hence from (1.3c) we can obtain (2.9) for $\int_0^t \|\mathrm{div}\mathbf{v}\|^2\,dz$ and then $\int_0^t \|(\mathrm{div}\mathbf{v})_t\|^2\,dz$. We thus complete the proof. □

From this lemma we see that if we choose p_0 such that $\int_0^t \|\mathbf{grad}p_0\|^2\,dz$ is bounded then by induction all terms in the left of (2.9) are bounded for any given s.

3. Convergence of the SRM. In this section we estimate the error of the SRM (1.3) toward the solution of (1.2c) by using the technique of asymptotic expansion as in the appendix of [3]. Note that in the Navier–Stokes context the method of asymptotic expansion was used in [13] to get a more precise estimate for a penalty method for the stationary Stokes equations and in [33] to calculate a slightly compressible steady-state flow. We will mainly consider the case of $\alpha_1 > 0$. Hence we take $\alpha_1 = 1$ and $\alpha_2 = \alpha$ for convenience. The result for the case of $\alpha_1 = 0$ will be described in Remark 3.3. At first we discuss a couple of linear auxiliary problems. Then we go to the proof.

3.1. A couple of linear auxiliary problems. We discuss two linear problems in this section. One is

$$\epsilon\mathbf{w}_t - \mathbf{grad}(\mathrm{div}\mathbf{w})_t - \alpha\mathbf{grad}\mathrm{div}\mathbf{w} + \epsilon(\mathbf{w}\cdot\mathbf{grad})\mathbf{U}$$
$$+\,\epsilon(\mathbf{V}\cdot\mathbf{grad})\mathbf{w} = \epsilon\mu\Delta\mathbf{w} - \epsilon\mathbf{grad}q + \epsilon\mathbf{f}, \tag{3.1a}$$
$$\mathbf{w}|_{\partial\Omega} = \mathbf{0},\ \mathbf{w}|_{t=0} = \mathbf{0}, \tag{3.1b}$$

where $\mathbf{U}$, $\mathbf{V}$, and q are given functions. The other is

$$\mathbf{w}_t + (\mathbf{V}\cdot\mathbf{grad})\mathbf{w} + (\mathbf{w}\cdot\mathbf{grad})\mathbf{V} = \mu\Delta\mathbf{w} - \mathbf{grad}p + f, \tag{3.2a}$$

$$(\mathrm{div}\mathbf{w})_t + \alpha\mathrm{div}\mathbf{w} = g, \tag{3.2b}$$

$$\mathbf{w}|_{\partial\Omega} = \mathbf{0}, \mathbf{w}|_{t=0} = \mathbf{a}, \tag{3.2c}$$

where $\mathbf{V}$, g, and $\mathbf{a}$ are given functions, $\mathbf{a}$ satisfies the compatibility conditions (2.2), and g satisfies (2.1). Now we show some properties of these two problems which will be used in the proof of the convergence of SRM later.

LEMMA 3.1. *For the solution of problem* (3.1), *if* $\mathbf{U}$ *and* $\mathbf{V}$ *satisfy*

$$\|\cdot\|^2_{\mathbf{H}^1} + \int_0^T \|\cdot\|^2_{\mathbf{H}^2}\,dt \le M, \tag{3.3}$$

then we have the following estimate:

$$\epsilon\|\mathbf{w}\|^2 + \|\mathrm{div}\mathbf{w}\|^2 \le M\epsilon\int_0^t (\|\mathbf{f}\|^2 + \|q\|^2)\,ds, \tag{3.4a}$$

$$\epsilon\|\mathbf{grad}\mathbf{w}\|^2 + \int_0^t (\epsilon\|\mathbf{w}_t\|^2 + \|\mathrm{div}\mathbf{w}_t\|^2)\,ds \le M\epsilon\int_0^t (\|\mathbf{f}\|^2 + \|q\|^2)\,ds. \tag{3.4b}$$

Proof. Multiplying (3.1a) by $\mathbf{w}$ and then integrating on the domain Ω yields

$$\begin{aligned}
&\frac{1}{2}\epsilon\frac{d}{dt}\|\mathbf{w}\|^2 + \frac{1}{2}\frac{d}{dt}\|\mathrm{div}\mathbf{w}\|^2 + \alpha\|\mathrm{div}\mathbf{w}\|^2 + \epsilon\mu\|\mathbf{grad}\mathbf{w}\|^2\\
&= -\epsilon((\mathbf{w}\cdot\mathbf{grad})\mathbf{U},\mathbf{w}) - \epsilon((\mathbf{V}\cdot\mathbf{grad})\mathbf{w},\mathbf{w}) + \epsilon(q,\mathrm{div}\mathbf{w}) + \epsilon(\mathbf{f},\mathbf{w})\\
&\le \epsilon\|\mathbf{grad}\mathbf{U}\|\|\mathbf{w}\|_4^2 + \frac{\epsilon}{2}\|\mathrm{div}\mathbf{V}\|\|\mathbf{w}\|_4^2 + \epsilon(q,\mathrm{div}\mathbf{w}) + \epsilon(\mathbf{f},\mathbf{w}) \text{ (using (2.7))}\\
&\le \epsilon\gamma_1^{\frac{1}{2}}\left(\|\mathbf{grad}\mathbf{U}\| + \frac{1}{2}\|\mathrm{div}\mathbf{V}\|\right)\|\mathbf{w}\|\|\mathbf{grad}\mathbf{w}\| + \epsilon(q,\mathrm{div}\mathbf{w}) + \epsilon(\mathbf{f},\mathbf{w}) \text{ (using (2.8))}\\
&\le \frac{1}{2}\epsilon\mu\|\mathbf{grad}\mathbf{w}\|^2 + \frac{\epsilon\gamma_1}{2\mu}\left(\|\mathbf{grad}\mathbf{U}\| + \frac{1}{2}\|\mathrm{div}\mathbf{w}\|\right)^2\|\mathbf{w}\|^2 + \epsilon(q,\mathrm{div}\mathbf{w}) + \epsilon(\mathbf{f},\mathbf{w}),
\end{aligned}$$

where we have used $-\epsilon((\mathbf{V}\cdot\mathbf{grad})\mathbf{w},\mathbf{w}) = \frac{\epsilon}{2}((\mathrm{div}\mathbf{V})\mathbf{w},\mathbf{w})$. Therefore we have

$$\begin{aligned}
&\frac{d}{dt}(\epsilon\|\mathbf{w}\|^2 + \|\mathrm{div}\mathbf{w}\|^2) - C(t)(\epsilon\|\mathbf{w}\|^2 + \|\mathrm{div}\mathbf{w}\|^2)\\
&\qquad\le -\epsilon\mu\|\mathbf{grad}\mathbf{w}\|^2 - (\alpha + C(t))\|\mathrm{div}\mathbf{w}\|^2 + 2\epsilon(q,\mathrm{div}\mathbf{w}) + 2\epsilon(\mathbf{f},\mathbf{w})\\
&\qquad\le -\epsilon\mu\|\mathbf{grad}\mathbf{w}\|^2 + \epsilon\frac{\mu}{\gamma^2}\|\mathbf{w}\|^2 + \epsilon\frac{\gamma^2}{\mu}\|\mathbf{f}\|^2 + \epsilon\frac{1}{\alpha}\|q\|^2\\
&\qquad\le \epsilon\frac{\gamma^2}{\mu}\|\mathbf{f}\|^2 + \epsilon\frac{1}{\alpha}\|q\|^2, \qquad (3.5)
\end{aligned}$$

where

$$C(t) = \frac{\gamma_1}{\mu}\left(\|\mathbf{grad}\mathbf{U}\| + \frac{1}{2}\|\mathrm{div}\mathbf{V}\|\right)^2.$$

Noting that $\mathbf{w}|_{t=0} = 0$ and $\mathrm{div}\mathbf{w}|_{t=0} = 0$, we thus get (3.4a).

Now, multiplying (3.1) by $\mathbf{w}_t$, then integrating with respect to $\mathbf{x}$ over Ω, we get

$$\begin{aligned}&\epsilon\|\mathbf{w}_t\|^2 + \|\mathrm{div}\mathbf{w}_t\|^2 + \frac{\alpha}{2}\frac{d}{dt}\|\mathrm{div}\mathbf{w}\|^2 + \epsilon\mu\frac{d}{dt}\|\mathbf{grad}\mathbf{w}\|^2\\ &= \epsilon((\mathbf{w}\cdot\mathbf{grad})\mathbf{U},\mathbf{w}_t) + \epsilon((\mathbf{V}\cdot\mathbf{grad})\mathbf{w},\mathbf{w}_t) + \epsilon(q,\mathrm{div}\mathbf{w}_t) + \epsilon(\mathbf{f},\mathbf{w}_t).\end{aligned} \tag{3.6}$$

We use the inequalities listed in the previous section to estimate the right-hand side of (3.6) and have the following:

$$\begin{aligned}\epsilon((\mathbf{w}\cdot\mathbf{grad})\mathbf{U},\mathbf{w}_t) &\le \frac{\epsilon}{4}\|\mathbf{w}_t\|^2 + M\epsilon(\|\mathbf{grad}\mathbf{w}\|^2 + \|\mathbf{w}\|^2),\\ \epsilon((\mathbf{V}\cdot\mathbf{grad})\mathbf{w},\mathbf{w}_t) &\le \frac{\epsilon}{4}\|\mathbf{w}_t\|^2 + M\epsilon\sup_\Omega|\mathbf{V}|^2\|\mathbf{grad}\mathbf{w}\|^2,\\ \epsilon(\mathbf{f},\mathbf{w}_t) &\le \frac{\epsilon}{4}\|\mathbf{w}_t\|^2 + M\epsilon\|\mathbf{f}\|^2,\\ \epsilon(q,\mathrm{div}\mathbf{w}_t) &\le \frac{\epsilon}{2}\|\mathrm{div}\mathbf{w}_t\|^2 + M\epsilon\|q\|^2,\end{aligned}$$

where the bounds of $\sup_\Omega|\mathbf{U}|$ and $\sup_\Omega|\mathbf{V}|$ can be obtained by using the inequality

$$\sup_\Omega|\cdot| \le M\|\Delta\cdot\|$$

(see, e.g., [35]). Then, similarly to the procedure of getting (3.4a), we obtain (3.4b). ◻

Next we consider problem (3.2).

LEMMA 3.2. *There exists a solution for problem* (3.2) *for which we have the following estimate:*

$$\|\mathbf{w}\|_{\mathbf{H}^1} + \int_0^T (\|\mathbf{w}\|^2_{\mathbf{H}^2} + \|\mathbf{w}_t\|^2 + \|p\|^2_{H^1})\,dt \le M \tag{3.7}$$

if $\int_0^T \|\mathbf{f}\|^2\,dt$ *and* $\int_0^T \|g\|^2_{H^1}\,dt$ *are bounded.*

Proof. First we can solve $\mathrm{div}\mathbf{w}$ from (3.2b) (noting that $\mathrm{div}\mathbf{w}|_{t=0} = 0$):

$$\mathrm{div}\mathbf{w} = g_1, \tag{3.8}$$

where

$$g_1 = \exp(-\alpha t)\int_0^t g\exp(\alpha s)ds \tag{3.9}$$

and g_1 satisfies (2.1) since g does. By applying Corollary 2.4 in [13, p. 23], the problem

$$\mathrm{div}\mathbf{w} = g_1, \tag{3.10a}$$

$$\mathbf{w}|_{\partial\Omega} = \mathbf{0} \tag{3.10b}$$

has many solutions. We pick up one and denote it as $\mathbf{w}_p$. Then $\bar{\mathbf{w}} := \mathbf{w} - \mathbf{w}_p$ satisfies the linearized Navier–Stokes equations in the form of (3.2a) with a proper force term (denoted by $\bar{\mathbf{f}}$) and

$$\mathrm{div}\bar{\mathbf{w}} = 0, \bar{\mathbf{w}}|_{\partial\Omega} = \mathbf{0} \text{ and } \bar{\mathbf{w}}|_{t=0} = \mathbf{a} - \mathbf{w}_p|_{t=0}.$$

Noting that $\mathrm{div}\mathbf{w}_p|_{t=0} = g_1|_{t=0} = 0$, we thus know that the basic compatibility conditions like (2.2) for $\bar{\mathbf{w}}$ are satisfied. From (3.9) and the assumption for g we know that $\int_0^T (\|g_1\|_{H^1}^2 + \|(g_1)_t\|^2)\, dt$ is bounded. Hence, based on the estimates for the solution of (3.10) (see [1] and [13]), it is not difficult to get

$$\|\mathbf{w}_p\|_{\mathbf{H}^1} + \int_0^T (\|(\mathbf{w}_p)_t\|^2 + \|\mathbf{w}_p\|_{\mathbf{H}^2}^2)\, dt \le M. \tag{3.11}$$

We thus obtain that $\int_0^T \|\bar{\mathbf{f}}\|^2\, dt$ is bounded. Simulating the regularity argument of [18] or [19] (multiplying the linearized Navier–Stokes equations by $\bar{\mathbf{w}}$, $\bar{\mathbf{w}}_t$, and $\mathbf{P}\Delta\bar{\mathbf{w}}$, where $\mathbf{P}$ is a projection operator (cf. [18]), respectively), we can obtain

$$\|\bar{\mathbf{w}}\|_{\mathbf{H}^1} + \int_0^T (\|\bar{\mathbf{w}}\|_{\mathbf{H}^2}^2 + \|\bar{\mathbf{w}}_t\|^2 + \|p\|_{H^1})\, dt \le M. \tag{3.12}$$

Therefore (3.7) follows from (3.11) and (3.12). Using the estimate (3.12) and following the global existence argument (e.g., [18] or [33]), the existence of the solution for $\bar{\mathbf{w}}$ can be obtained. We thus have the results of the lemma. □

Remark 3.1. The uniqueness of the solution of (3.2) follows from the standard argument for Navier–Stokes equations (cf. [33]).

3.2. The error estimate of SRM. In this section we prove the convergence of iteration (1.3) based on the same procedure described in the appendix of [3]. We describe our results in the following theorem.

THEOREM 3.1. *Let* $\mathbf{u}$ *and* p *be the solution of problem* (1.2c) *and* $\mathbf{u}_m$ *and* p_m *the solution of problem* (1.3) *at the* m*th iteration. Then we have the following error estimates:*

$$\|\mathbf{u} - \mathbf{u}_m\|_{\mathbf{H}^1} \le M\epsilon^m, \tag{3.13a}$$

$$\left(\int_0^T \|p - p_m\|^2\, dt\right)^{\frac{1}{2}} \le M\epsilon^m, \tag{3.13b}$$

where $m = 1, 2, \ldots$.

Proof. First consider the case $s = 1$ of (1.3). Let

$$\mathbf{u}_1 = \mathbf{u}_{10} + \epsilon\mathbf{u}_{11} + \cdots + \epsilon^m\mathbf{u}_{1m} + \cdots.$$

Comparing the coefficients of like powers of ϵ, we thus have

$$\mathbf{grad}((\mathrm{div}\mathbf{u}_{10})_t + \alpha\mathrm{div}\mathbf{u}_{10}) = 0, \tag{3.14a}$$

$$\begin{aligned}\mathbf{grad}((\mathrm{div}\mathbf{u}_{11})_t + \alpha\mathrm{div}\mathbf{u}_{11}) = {}& (\mathbf{u}_{10})_t + (\mathbf{u}_{10}\cdot\mathbf{grad})\mathbf{u}_{10}\\ & -\mu\Delta\mathbf{u}_{10} + \mathbf{grad}p_0 - \mathbf{f},\end{aligned} \tag{3.14b}$$

$$\begin{aligned}\mathbf{grad}((\mathrm{div}\mathbf{u}_{1i})_t + \alpha\mathrm{div}\mathbf{u}_{1i}) = {}& (\mathbf{u}_{1i-1})_t + \sum_{j=1}^{i-1}(\mathbf{u}_{1j}\cdot\mathbf{grad})\mathbf{u}_{1i-1-j}\\ & -\mu\Delta\mathbf{u}_{1i-1}, \quad 2 \le i \le m+1,\end{aligned} \tag{3.14c}$$

where (3.14a) satisfies (1.3b) and (3.14b) and (3.14c) satisfy the homogeneous initial and boundary conditions corresponding to (1.3b). Now (3.14a) has infinitely many solutions in general. We should choose $\mathbf{u}_{10}$ not only to satisfy (3.14a) but also to ensure that the solution of (3.14b) exists. A choice of $\mathbf{u}_{10}$ is the exact solution $\mathbf{u}$ of

(1.2c); i.e.,

$$(\mathbf{u}_{10})_t + (\mathbf{u}_{10}\cdot\mathbf{grad})\mathbf{u}_{10} = \mu\Delta\mathbf{u}_{10} - \mathbf{grad}p + \mathbf{f}, \tag{3.15a}$$

$$(\mathrm{div}\mathbf{u}_{10})_t + \alpha\mathrm{div}\mathbf{u}_{10} = 0, \tag{3.15b}$$

$$\mathbf{u}_{10}|_{\partial\Omega} = 0,\ \mathbf{u}_{10}|_{t=0} = \mathbf{a}. \tag{3.15c}$$

Note that $\mathrm{div}\mathbf{u}_{10}|_{t=0} = \mathrm{div}\mathbf{a} = 0$ and p is taken to satisfy (2.1). So $\mathbf{u}_{10} \equiv \mathbf{u}$ and (3.14b) has the following form:

$$\mathbf{grad}((\mathrm{div}\mathbf{u}_{11})_t + \alpha\mathrm{div}\mathbf{u}_{11}) = \mathbf{grad}(p_0 - p). \tag{3.16}$$

Now we choose $\mathbf{u}_{11}$ and a corresponding p_{11} to satisfy

$$(\mathbf{u}_{11})_t + (\mathbf{u}_{10}\cdot\mathbf{grad})\mathbf{u}_{11} + (\mathbf{u}_{11}\cdot\mathbf{grad})\mathbf{u}_{10} = \mu\Delta\mathbf{u}_{11} - \mathbf{grad}p_{11}, \tag{3.17a}$$

$$(\mathrm{div}\mathbf{u}_{11})_t + \alpha\mathrm{div}\mathbf{u}_{11} = p_0 - p, \tag{3.17b}$$

$$\mathbf{u}_{11}|_{\partial\Omega} = 0,\ \mathbf{u}_{11}|_{t=0} = 0. \tag{3.17c}$$

Again we have $\mathrm{div}\mathbf{u}_{11}|_{t=0} = 0$ and let p_{11} satisfy (2.1). According to Lemma 3.2, $\mathbf{u}_{11}$ and p_{11} exist.

Generally, supposing we have gotten $\mathbf{u}_{1i-1}$, p_{1i-1} for $i \geq 2$, we choose $\mathbf{u}_{1i}$, p_{1i} satisfying

$$\begin{aligned}&(\mathbf{u}_{1i})_t + (\mathbf{u}_{10}\cdot\mathbf{grad})\mathbf{u}_{1i} + (\mathbf{u}_{1i}\cdot\mathbf{grad})\mathbf{u}_{10}\\ &= \mu\Delta\mathbf{u}_{1i} - \mathbf{grad}p_{1i} - \sum_{j=1}^{i-1}(\mathbf{u}_{1j}\cdot\mathbf{grad})\mathbf{u}_{1i-1-j},\end{aligned} \tag{3.18a}$$

$$(\mathrm{div}\mathbf{u}_{1i})_t + \alpha\mathrm{div}\mathbf{u}_{1i} = -p_{1i-1}, \tag{3.18b}$$

$$\mathbf{u}_{1i}|_{\partial\Omega} = 0,\ \mathbf{u}_{1i}|_{t=0} = 0, \tag{3.18c}$$

where we note that $\mathrm{div}\mathbf{u}_{1i}|_{t=0} = 0$ and p_{1i} satisfies (2.1). Applying Lemma 3.2, we obtain that all $\mathbf{u}_{1i}$ and p_{1i} $i = 0, 1, \dots$ exist and satisfy (3.7).

Next we estimate the remainder of the asymptotic expansion to the $(m+1)$th power of ϵ. Denote

$$\bar{\mathbf{u}}_{1m} = \mathbf{u}_{10} + \epsilon\mathbf{u}_{11} + \cdots + \epsilon^{m+1}\mathbf{u}_{1m+1} \tag{3.19}$$

($\bar{\mathbf{u}}_{1m}$ also satisfies (3.7)) and

$$\mathbf{w}_{1m} = \mathbf{u}_1 - \bar{\mathbf{u}}_{1m}. \tag{3.20}$$

Then $\mathbf{w}_{1m}$ satisfies

$$\begin{aligned}&\epsilon(\mathbf{w}_{1m})_t - \mathbf{grad}(\mathrm{div}\mathbf{w}_{1m})_t - \alpha\mathbf{grad}\mathrm{div}\mathbf{w}_{1m}\\ &+ \epsilon(\mathbf{w}_{1m}\cdot\mathbf{grad})\mathbf{u}_1 + \epsilon(\bar{\mathbf{u}}_{1m}\cdot\mathbf{grad})\mathbf{w}_{1m} = \epsilon\mu\Delta\mathbf{w}_{1m}\\ &- \epsilon^{m+2}\left\{(\mathbf{u}_{1m+1})_t + \sum_{i=0}^{m+1}[(\mathbf{u}_{1i}\cdot\mathbf{grad})\mathbf{u}_{1m+1-i}] - \mu\Delta\mathbf{u}_{1m+1}\right\},\end{aligned} \tag{3.21a}$$

$$\mathbf{w}_{1m}|_{\partial\Omega} = 0,\ \mathbf{w}_{1m}|_{t=0} = 0. \tag{3.21b}$$

Then, using regularity we have gotten for $\mathbf{u}_{1i}$, $\bar{\mathbf{u}}_{1m}$, and $\mathbf{u}_1$ (see (2.9)) and Lemma 3.1, we obtain $\|\mathbf{w}_{1m}\| = O(\epsilon^{m+1})$ and $\|\mathbf{grad}\mathbf{w}_{1m}\| = O(\epsilon^{m+1})$. Therefore

$$\mathbf{u}_1 = \mathbf{u}_{10} + \epsilon\mathbf{u}_{11} + \cdots + \epsilon^m\mathbf{u}_{1m} + O(\epsilon^{m+1}) \tag{3.22}$$

in the sense of $\mathbf{H}^1$-norm for spatial variables. Noting $\mathbf{u}_{10} \equiv \mathbf{u}$, we thus obtain

$$\mathbf{u}_1 - \mathbf{u} = O(\epsilon). \tag{3.23}$$

Furthermore, according to Lemma 3.1, we can get

$$\|\mathrm{div}\mathbf{w}_{1m}\| = O(\epsilon^{m+\frac{3}{2}}), \ \left(\int_0^T \|(\mathrm{div}\mathbf{w}_{1m})_t\|^2 \, dt\right)^{\frac{1}{2}} = O(\epsilon^{m+\frac{3}{2}}).$$

Then, by using (1.3c), (3.22), (3.15b), (3.17b), (3.18b), and the estimates for $\mathrm{div}\mathbf{w}_{1m}$ and $(\mathrm{div}\mathbf{w}_{1m})_t$, it follows that

$$p_1 = p + \epsilon p_{11} + \cdots + \epsilon^m p_{1m} + O(\epsilon^{m+\frac{1}{2}}) \tag{3.24}$$

or

$$p_1 - p = O(\epsilon) \tag{3.25}$$

in the sense of $\mathbf{L}^2$-norm for both spatial and time variables; i.e., $(\int_0^T \|\cdot\|^2 \, dt)^{\frac{1}{2}}$.

Now we look at the second iteration $s = 2$ of (1.3). Let

$$\mathbf{u}_2 = \mathbf{u}_{20} + \epsilon \mathbf{u}_{21} + \cdots + \epsilon^m \mathbf{u}_{2m} + \cdots .$$

Noting that (3.24) gives us a series expansion for p_1 we obtain

$$\mathbf{grad}((\mathrm{div}\mathbf{u}_{20})_t + \alpha \mathrm{div}\mathbf{u}_{20}) = 0, \tag{3.26a}$$

$$\begin{aligned}\mathbf{grad}((\mathrm{div}\mathbf{u}_{21})_t + \alpha \mathrm{div}\mathbf{u}_{21}) &= (\mathbf{u}_{20})_t + (\mathbf{u}_{20} \cdot \mathbf{grad})\mathbf{u}_{20} \\ &\quad -\mu \Delta \mathbf{u}_{20} + \mathbf{grad} p - \mathbf{f},\end{aligned} \tag{3.26b}$$

$$\begin{aligned}\mathbf{grad}((\mathrm{div}\mathbf{u}_{2i})_t + \alpha \mathrm{div}\mathbf{u}_{2i}) &= (\mathbf{u}_{2i-1})_t + \sum_{j=1}^{i-1} (\mathbf{u}_{2j} \cdot \mathbf{grad})\mathbf{u}_{2i-1} \\ &\quad -\mu \Delta \mathbf{u}_{2i-1} - \mathbf{grad} p_{1i-1}, 2 \le i \le m+1.\end{aligned} \tag{3.26c}$$

Again, (3.26a) is combined with initial and boundary conditions (1.3b), and (3.26b) and (3.26c) are combined with the corresponding homogeneous ones. As in the case of $s = 1$, we choose $\mathbf{u}_{20} = \mathbf{u}$ again. We thus have

$$\mathbf{grad}((\mathrm{div}\mathbf{u}_{21})_t + \alpha \mathrm{div}\mathbf{u}_{21}) = 0. \tag{3.27}$$

Then $\mathbf{u}_{21}$ is constructed to satisfy

$$(\mathbf{u}_{21})_t + (\mathbf{u}_{20} \cdot \mathbf{grad})\mathbf{u}_{21} + (\mathbf{u}_{21} \cdot \mathbf{grad})\mathbf{u}_{20} = \mu \Delta \mathbf{u}_{21} - \mathbf{grad} p_{21}, \tag{3.28a}$$

$$(\mathrm{div}\mathbf{u}_{21})_t + \alpha \mathrm{div}\mathbf{u}_{21} = 0, \tag{3.28b}$$

$$\mathbf{u}_{21}|_{\partial\Omega} = 0, \ \mathbf{u}_{21}|_{t=0} = 0. \tag{3.28c}$$

Obviously $\mathbf{u}_{21} = 0$, $p_{21} = 0$ is the solution of (3.28) and (2.1).

In general, similarly to the case of $s = 1$, we choose $\mathbf{u}_{2i}, p_{2i}$ to satisfy

$$\begin{aligned}&(\mathbf{u}_{2i})_t + (\mathbf{u}_{20} \cdot \mathbf{grad})\mathbf{u}_{2i} + (\mathbf{u}_{2i} \cdot \mathbf{grad})\mathbf{u}_{20} \\ &= \mu \Delta \mathbf{u}_{2i} - \mathbf{grad} p_{2i} - \sum_{j=1}^{i-1} (\mathbf{u}_{2j} \cdot \mathbf{grad})\mathbf{u}_{2i-1-j},\end{aligned} \tag{3.29a}$$

$$(\mathrm{div}\mathbf{u}_{2i})_t + \alpha \mathrm{div}\mathbf{u}_{2i} = p_{1i-1} - p_{2i-1}, \tag{3.29b}$$

$$\mathbf{u}_{2i}|_{\partial\Omega} = 0, \ \mathbf{u}_{2i}|_{t=0} = 0 \tag{3.29c}$$

for $2 \le i \le m+1$, where p_{2i} satisfies (2.1). By the same procedure as in the case of $s = 1$ we obtain the error equations similar to (3.21) with an addition of a remainder term $\mathbf{grad}(p_1 - \bar{p}_{1m})$ in the right-hand side, where $\bar{p}_{1m}$ stands for the asymptotic expansion (3.24) of p_1. Applying Lemma 3.1 again, we get

$$\mathbf{u}_2 = \mathbf{u}_{20} + \epsilon \mathbf{u}_{21} + \cdots + \epsilon^m \mathbf{u}_{2m} + O(\epsilon^{m+1}). \tag{3.30}$$

Noting $\mathbf{u}_{21} \equiv 0$, hence

$$\mathbf{u}_2 - \mathbf{u} = O(\epsilon^2). \tag{3.31}$$

Then, using (1.3b), (3.30), (3.28b), and (3.29b), we conclude

$$p_2 = p + \epsilon p_{21} + \cdots + \epsilon^m p_{2m} + O(\epsilon^{m+\frac{1}{2}}) \tag{3.32}$$

or

$$p_2 - p = O(\epsilon^2) \tag{3.33}$$

by noting $p_{21} \equiv 0$.

We can repeat this procedure and, by induction, conclude the results of the theorem. $\square$

Remark 3.2. Corresponding to [3, Theorem 3.1], we expect that the error estimates (3.13) hold also for the SRM (1.3) with $\alpha_1 = 0$, at least, away from $t = 0$. In section 4 a numerical example verifies the convergence for this case.

Remark 3.3. In Theorem 3.1 we can find that the result for p is in a weaker norm $\int_0^T \|\cdot\|^2\,dt$. This is because we have difficulty in estimating the first-order time derivative of the right-hand side of (3.21) or, concretely, the term $\int_0^T \|(\mathbf{u}_{1m+1})_{tt}\|^2\,ds$. In [19, Corollary 2.1], it is shown that $\int_0^T \|(\mathbf{u}_{1m+1})_{tt}\|^2\,ds$ may be unbounded as $t \to 0$ if we only assume the local compatibility conditions (2.2). In the case that this integral is bounded for $0 < t < T$, we can get

$$\|p - p_m\| + \left(\int_0^t \|(p-p_m)_t\|^2\,ds\right)^{\frac{1}{2}} \le M\epsilon^m. \tag{3.34}$$

Otherwise, we only can expect that (3.34) holds away from $t = 0$ by following the argument in [19].

Remark 3.4. Multiplying (3.1) by $A\mathbf{w}$, where A is the operator defined by (2.15), and following the later steps of the proof of Lemma 2.1, we can get

$$\begin{aligned}&\frac{1}{\epsilon^2}\int_0^T (\|\mathbf{grad}(\mathrm{div}\mathbf{w})_t\|^2 + \alpha\|\mathbf{grad}\mathrm{div}\mathbf{w}\|^2)\,dt\\ &\le M\int_0^T (\|\mathbf{f}\|^2 + \|\mathbf{grad}\,q\|^2)\,dt.\end{aligned}$$

Using this result to estimate the remainders of the asymptotic solutions in the proof of Theorem 3.1, we can actually prove

$$\left(\int_0^T \|p - p_m\|^2_{\mathbf{H}^1}\,dt\right)^{\frac{1}{2}} \le M\epsilon^m. \qquad \square \tag{3.35}$$

4. Discretization issues and numerical experiments. In previous sections we have proposed the SRM and performed some basic analysis on it. The SRM yields a sequence of PDEs which are to be solved numerically. The problem at the sth iteration can be written as follows:

$$\epsilon(\mathbf{u}_s)_t - \mathbf{grad}(\alpha_1(\mathrm{div}\mathbf{u}_s)_t + \alpha_2\mathrm{div}\mathbf{u}_s) + \epsilon(\mathbf{u}_s\cdot\mathbf{grad})\mathbf{u}_s = \epsilon\mu\Delta\mathbf{u}_s + \epsilon\mathbf{r}_s, \tag{4.1a}$$

$$\mathbf{u}_s|_{\partial\Omega} = \mathbf{0}, \mathbf{u}_s|_{t=0} = \mathbf{a}, \tag{4.1b}$$

where $\mathbf{r}_s(t)$ is a known inhomogeneity

$$\mathbf{r}_s = -\mathbf{grad}p_{s-1} + \mathbf{f}. \tag{4.2}$$

A variational formulation of (4.1) gives the following: find $\mathbf{u}_s \in \mathbf{H}_0^1$ such that

$$\epsilon\frac{d}{dt}(\mathbf{u}_s,\phi) + \alpha_1\frac{d}{dt}(\mathrm{div}\mathbf{u}_s,\mathrm{div}\phi) + \alpha_2(\mathrm{div}\mathbf{u}_s,\mathrm{div}\phi) + \epsilon\mu(\mathbf{gradu}_s,\mathbf{grad}\phi) + b(\mathbf{u}_s,\mathbf{u}_s,\phi) = \epsilon(\mathbf{r}_s,\phi)\ \forall\phi\in\mathbf{H}_0^1, \tag{4.3a}$$

$$\mathbf{u}_s|_{t=0} = \mathbf{a}, \quad \mathrm{div}\mathbf{u}_s|_{t=0} = 0, \tag{4.3b}$$

where the trilinear form

$$b(\mathbf{u},\mathbf{v},\mathbf{w}) = ((\mathbf{u}\cdot\mathbf{grad})\mathbf{v},\mathbf{w}).$$

From (4.3) we see that finite-element methods in spatial variables combined with time discretizations can be easily adopted. Note that we do not need to construct divergence-free test functions at all. Nevertheless, in this paper we are not going to discuss finite-element methods further. As an initial test of the sequential regularization method for the PDAE, we would like to make everything as simple as possible. We consider a very simple first-order difference scheme (forward Euler scheme in the time direction) in two-dimensional space. Concretely, we consider a rectangular domain such that an equidistant mesh can be used. Let $(u,v)^T$ stand for the approximation of $\mathbf{u}_s$, and let k, h_x, h_y denote step sizes in time and spatial direction, respectively. Without loss of generality we assume that $h_x = h_y = h$ and that the domain is a unit square. Thus mesh points can be expressed as

$$x_i = ih, i = 0,1,\dots,I;\ y_j = jh, j = 0,1,,\dots,J;\ t_n = nk, n = 0,1,\dots,N,\ N = [T/k].$$

The difference scheme reads

$$\epsilon u_{\dot t} - \alpha_1(u_{\bar x\dot x} + v_{\bar y\dot x})_{\dot t} = \alpha_2(u_{\bar x\dot x} + u_{\bar y\dot x}) - \epsilon(uu_{\bar x} + vu_{\bar y}) + \epsilon\mu(u_{\bar x\dot x} + u_{\bar y\dot y}) + \epsilon r_u, \tag{4.4a}$$

$$\epsilon v_{\dot t} - \alpha_1(u_{\bar x\dot y} + u_{\bar y\dot y})_{\dot t} = \alpha_2(u_{\bar x\dot y} + u_{\bar y\dot y}) - \epsilon(uv_{\bar x} + vv_{\bar y}) + \epsilon\mu(v_{\bar x\dot x} + v_{\bar y\dot y}) + \epsilon r_v, \tag{4.4b}$$

$$u|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0, \qquad u|_{t=0} = a_u,\ v|_{t=0} = a_v, \tag{4.4c}$$

where

$$u = u_{i,j}^n,$$
$$u_{\dot t} = \frac{u_{i,j}^{n+1} - u_{i,j}^n}{k},$$
$$u_{\dot x} = \frac{u_{i+1,j}^n - u_{i,j}^n}{h},$$
$$u_{\bar x} = \frac{u_{i,j}^n - u_{i-1,j}^n}{h}.$$

$u_{\dot y}$ and $u_{\bar y}$ can be defined accordingly and the definitions for v are similar.

Obviously, this is a first-order scheme explicit in time, where the nonlinear term is discretized somewhat arbitrarily. The scheme is easy to implement. Next we discuss its stability. For simplicity, we analyze the linear case (corresponding to the Stokes equations) first and consider the full nonlinear equations (4.4) in Remark 4.3 afterwards.

We write the linear case of (4.4) as follows:

$$\epsilon u_{\dot{t}} - \alpha_1(u_{\bar{x}\dot{x}} + v_{\bar{y}\dot{x}})_{\dot{t}} = \alpha_2(u_{\bar{x}\dot{x}} + u_{\bar{y}\dot{x}}) + \epsilon\mu(u_{\bar{x}\dot{x}} + u_{\bar{y}\dot{y}}) + \epsilon r_u, \tag{4.5a}$$

$$\epsilon v_{\dot{t}} - \alpha_1(u_{\bar{x}\dot{y}} + u_{\bar{y}\dot{y}})_{\dot{t}} = \alpha_2(u_{\bar{x}\dot{y}} + u_{\bar{y}\dot{y}}) + \epsilon\mu(v_{\bar{x}\dot{x}} + v_{\bar{y}\dot{y}}) + \epsilon r_v, \tag{4.5b}$$

$$u|_{\partial\Omega} = 0,\ v|_{\partial\Omega} = 0, \qquad u|_{t=0} = a_u,\ v|_{t=0} = a_v. \tag{4.5c}$$

Here we take $\alpha_1 = 1$ and $\alpha_2 = \alpha$. The result for the case of $\alpha_1 = 0$ will be given in Remark 4.2.

The following theorem gives the stability estimate for (4.5) in the sense of discrete $\mathbf{L}^2$-norm:

$$\|w^h\|_h^2 = h^2 \sum_{i=0}^{I-1} \sum_{j=0}^{J-1} (w_{i,j})^2, \tag{4.6}$$

where $w^h = (w_{i,j}),\ i = 0, 1, \ldots, I-1,\ j = 0, 1, \ldots, J-1$.

THEOREM 4.1. *Let u and v be the solution of* (4.5) *and*

$$A = \epsilon(\|u\|_h^2 + \|v\|_h^2) + \|u_{\bar{x}} + v_{\bar{y}}\|_h^2 + \epsilon\mu(\|u_{\bar{x}}\|_h^2 + \|u_{\bar{y}}\|_h^2 + \|v_{\bar{x}}\|_h^2 + \|v_{\bar{y}}\|_h^2). \tag{4.7}$$

If $\frac{\mu k}{h^2} \le 1 - c$, where c is any constant in $(0,1)$, *then*

$$A + k \sum_{n=0}^{N-1} \|(u_{\bar{x}} + v_{\bar{y}})_{\dot{t}}\|_h^2 \le M\epsilon \max_{0 \le t_n \le T} (\|r_u\|_h^2 + \|r_v\|_h^2), \tag{4.8}$$

where M is a generic constant dependent on μ and c.

Proof. We first write down some difference identities and inequalities that will be used in the proof:

- some difference identities [24]

$$(\phi\psi)_{\bar{x}} = \phi\psi_{\bar{x}} + \phi_{\bar{x}} E_x^{-1}\psi, \tag{4.9a}$$

$$(\phi\psi)_{\dot{x}} = \phi\psi_{\dot{x}} + \phi_{\dot{x}} E_x^{1}\psi, \tag{4.9b}$$

$$2\phi\phi_{\dot{t}} = (\phi^2)_{\dot{t}} - k(\phi_{\dot{t}})^2, \tag{4.9c}$$

$$\phi\phi_{\dot{x}\bar{x}} = (\phi\phi_{\dot{x}})_{\bar{x}} - (\phi_{\bar{x}})^2, \tag{4.9d}$$

  where the translation operator $E_x^i\phi(x,y,t) = \phi(x+ih,y,t)$;
- a difference inequality [24]

$$h\|\phi_{\bar{x}}\|_h \le 2\|\phi\|_h; \tag{4.10}$$

- a discrete version of the Poincaré inequality (cf. [21])

$$\|\phi\|_h^2 \le \|\phi_{\bar{x}}\|_h^2 + \|\phi_{\bar{y}}\|_h^2 \tag{4.11}$$

  if ϕ satisfies homogeneous boundary conditions.

Now multiplying (4.5a) by $au+bu_{\bar{t}}$ and (4.5b) by $av+bv_{\bar{t}}$ and adding the two expressions together, then summing for all $(i,j),\ i=1,\dots,I-1,\ j=1,\dots,J-1$, where we use difference identities (4.9) (omitting lots of tedious algebraic manipulations), we obtain

$$\epsilon a(\|u\|_h^2+\|v\|_h^2)_{\bar{t}}+\frac{1}{2}(\alpha b+a)(\|u_{\bar{x}}+v_{\bar{y}}\|_h^2)_{\bar{t}}+\frac{1}{2}\epsilon\mu b(\|u_{\bar{x}}\|_h^2+\|u_{\bar{y}}\|_h^2+\|v_{\bar{x}}\|_h^2+\|v_{\bar{y}}\|_h^2)_{\bar{t}}$$
$$+\epsilon(b-ak)(\|u_{\bar{t}}\|_h^2+\|v_{\bar{t}}\|_h^2)+a\alpha\|u_{\bar{x}}+v_{\bar{y}}\|_h^2+\left(b-\frac{1}{2}(b\alpha+a)k\right)\|(u_{\bar{x}}+v_{\bar{y}})_{\bar{t}}\|_h^2$$
$$+\epsilon\mu a(\|u_{\bar{x}}\|_h^2+\|u_{\bar{y}}\|_h^2+\|v_{\bar{x}}\|_h^2+\|v_{\bar{y}}\|_h^2)-\frac{1}{2}\epsilon\mu bk(\|u_{\bar{x}\bar{t}}\|_h^2+\|u_{\bar{y}\bar{t}}\|_h^2+\|v_{\bar{x}\bar{t}}\|_h^2+\|v_{\bar{y}\bar{t}}\|_h^2)$$
$$\le M\epsilon(\|r_u\|_h^2+\|r_v\|_h^2)+\frac{1}{2}\epsilon\mu a(\|u\|_h^2+\|v\|_h^2)+\frac{1}{2}\epsilon b\delta(\|u_{\bar{t}}\|_h^2+\|v_{\bar{t}}\|_h^2),$$

where $\delta>0$ can be chosen to be less than c/b. Applying (4.10) and (4.11), we get

$$h^2((\|u_{\bar{x}\bar{t}}\|_h^2+\|u_{\bar{y}\bar{t}}\|_h^2+\|v_{\bar{x}\bar{t}}\|_h^2+\|v_{\bar{y}\bar{t}}\|_h^2)\le 4(\|u_{\bar{t}}\|_h^2+\|v_{\bar{t}}\|_h^2)$$

and

$$\|u\|_h^2+\|v\|_h^2\le\|u_{\bar{x}}\|_h^2+\|u_{\bar{y}}\|_h^2+\|v_{\bar{x}}\|_h^2+\|v_{\bar{y}}\|_h^2,$$

respectively. Then we can choose a and b such that

$$b-ak-\mu\frac{k}{h^2}-\frac{1}{2}b\delta>0,\ b-\frac{1}{2}(b\alpha+a)k>0$$

and obtain the following:

$$(A)_{\bar{t}}+d\epsilon\mu A+\|(u_{\bar{x}}+v_{\bar{y}})_{\bar{t}}\|_h^2\le M\epsilon(\|r_u\|_h^2+\|r_v\|_h^2),$$

where d is a constant independent of k, h, ϵ, and μ. From this inequality, it is not difficult to see that (4.8) holds. ◻

Remark 4.1. From (4.8) of Theorem 4.1, we find that the value of ϵ will not affect the stability of the difference scheme. This means that the forward Euler scheme in time direction works for any value of ϵ. Also, the time step restriction $k\le(1-c)h^2/\mu$ is actually loosened in the case of small viscosity (or large Reynolds number), which people are often interested in. This implies that the explicit scheme (4.5) to which an appropriate discretization of the nonlinear term (see next remark) is added works very well. It enables us not only to avoid the complicated iteration procedure of nonlinear equations but also to choose the time step fairly widely in the case of small viscosity.

Remark 4.2. We have mentioned before that sometimes we may like to take $\alpha_1=0$ to avoid solving any algebraic system. Following the same procedure as the proof of Theorem 4.1, we can get the stability condition for the case of $\alpha_1=0$; that is, $k\le m\epsilon h^2$, where m is a positive constant independent of ϵ, h, and μ. We thus see that the stability of (4.5) with $\alpha_1=0$ depends on the parameter ϵ. This coincides with our experience with stiff problems discretized by explicit schemes. Fortunately, using SRM, we do not need to take ϵ very small. So the time-step restriction is not much worse than the usual one corresponding to an explicit scheme applied to a nonstiff problem.

Remark 4.3. For the nonlinear case (4.4), when the viscosity μ is not small, we expect similar results since the nonlinear term can be dominated by the viscous

TABLE 4.1
SRM errors for $\mu = 0.1$ without upwinding.

ϵ	Iteration	Error at $\rightarrow$	$t = k$	$t = 1.0$	$t = 2.0$	$t = 3.0$	$t = 4.0$	$t = 5.0$
$5e{-}1$	1	eu	$4.65e{-}3$	$2.69e{-}1$	$1.55e{-}1$	$1.31e{-}1$	$1.15e{-}1$	$1.08e{-}1$
		ep	$2.49e{-}1$	$1.96e{-}1$	$1.57e{-}1$	$1.36e{-}1$	$1.25e{-}1$	$1.21e{-}1$
	2	eu	$2.16e{-}3$	$2.53e{-}2$	$3.12e{-}2$	$3.18e{-}2$	$3.12e{-}2$	$3.06e{-}2$
		ep	$1.80e{-}1$	$9.28e{-}2$	$7.37e{-}2$	$6.74e{-}2$	$6.48e{-}2$	$6.35e{-}2$
	3	eu	$2.15e{-}3$	$1.77e{-}2$	$2.28e{-}2$	$2.48e{-}2$	$2.55e{-}2$	$2.57e{-}2$
		ep	$1.80e{-}1$	$8.81e{-}2$	$6.69e{-}2$	$6.10e{-}2$	$5.91e{-}2$	$5.83e{-}2$
$1e{-}3$	1	eu	$2.14e{-}3$	$1.73e{-}2$	$2.21e{-}2$	$2.41e{-}2$	$2.48e{-}2$	$2.50e{-}2$
		ep	$1.80e{-}1$	$8.78e{-}2$	$6.61e{-}2$	$6.01e{-}2$	$5.82e{-}2$	$5.75e{-}2$

term. When the viscosity is small, however, the scheme (4.4) is unstable. Although numerical computations indicate that we do get better stability if we increase α_2, i.e., some kind of dissipation effect is obtained (we have to note that such a dissipation becomes small when the incompressibility condition is close to being satisfied), we suggest using spatial discretizations with better stability properties, e.g., upwinding schemes (cf. [31]) in the case of small viscosity.

Remark 4.4. Applying corresponding difference identities for a nonuniform mesh (see, e.g., [32]), the results of Theorem 4.1 may be generalized to difference schemes (4.5) on a nonuniform mesh. Hence the difference scheme may be used for problems defined on more general domains.

Next we verify our theoretical results by calculating an example. Some relevant numerical experiments regarding the SRM and its implementation can be found in [25] for a problem in the reservoir simulation.

Numerical example. Consider the Navier–Stokes equations (1.2c) with exact solution $\mathbf{u} = (u, v)$:

$$\begin{aligned} u &= 50x^2(1-x)^2 y(1-y)(1-2y)[1+\exp(-t)], \\ v &= -50y^2(1-y)^2 x(1-x)(1-2x)[1+\exp(-t)], \\ p &= \left[-x\left(\frac{x}{2}+2\right) - y\left(\frac{y}{2}-2\right) + \frac{1}{3}\right][1+\exp(-t)]^2 . \end{aligned}$$

As indicated in [3], to carry out the SRM iterations, we do not need to store the entire approximation of p_{s-1} on $[0, T]$ for calculating $\mathbf{u}_s$. Assuming that the number of the SRM iterations is chosen in advance, we can rearrange the computation order to make the storage requirements independent of N, where N represents the number of the mesh lines in t direction. We use constant steps $k = 0.01$ and $h = 0.1$ first. At a given time t, we use "eu" to denote the absolute discrete L^2-error in $\mathbf{u}_s$ while "ep" denotes the absolute discrete L^2-error in p_s. Table 4.1 summarizes the computational results of the difference scheme (4.4) with $\alpha_1 = \alpha_2 = 1$ and the viscosity $\mu = 0.1$.

We notice that the errors improve as the iteration proceeds until ϵ^s reaches the discretization accuracy $O(h)$, where s is the number of iterations.

For the case of small viscosity, say $\mu = 0.001$, the difference scheme (4.4) does not work. The errors blow up around $t = 1$. When we increase α_2, say to $\alpha_2 = 50$, we do get pretty good results around $t = 1$; however, the errors still blow up at a later time. This suggests that the scheme is not stable for small viscosity μ. So next we discretize the nonlinear term using an upwinding scheme given in [31]. For the case of small viscosity, e.g., $\mu = 0.001$, we get good results (see Table 4.2).

Recall that according to Remark 4.1, in the case of small viscosity, the time-step size can be increased to some extent without adverse stability effects. To demonstrate

TABLE 4.2
SRM errors for $\mu = 0.001$ with upwinding.

ϵ	Iteration	Error at $\rightarrow$	$t = k$	$t = 1.0$	$t = 2.0$	$t = 3.0$	$t = 4.0$	$t = 5.0$
$5e{-}1$	1	eu	$4.66e{-}3$	$2.26e{-}1$	$2.50e{-}1$	$2.29e{-}1$	$2.13e{-}1$	$2.03e{-}1$
		ep	$2.58e{-}1$	$1.06e{-}1$	$6.67e{-}2$	$5.45e{-}2$	$5.12e{-}2$	$5.04e{-}2$
	2	eu	$2.16e{-}3$	$7.74e{-}2$	$8.78e{-}2$	$9.13e{-}2$	$9.34e{-}2$	$9.53e{-}2$
		ep	$1.84e{-}1$	$8.81e{-}2$	$6.22e{-}2$	$5.39e{-}2$	$5.11e{-}2$	$5.02e{-}2$
	3	eu	$2.14e{-}3$	$7.69e{-}2$	$8.71e{-}2$	$9.06e{-}2$	$9.29e{-}2$	$9.48e{-}2$
		ep	$1.83e{-}1$	$8.78e{-}2$	$6.21e{-}2$	$5.39e{-}2$	$5.11e{-}2$	$5.01e{-}2$
$1e{-}3$	1	eu	$2.14e{-}3$	$7.69e{-}2$	$8.72e{-}2$	$9.07e{-}2$	$9.29e{-}2$	$9.49e{-}2$
		ep	$1.83e{-}1$	$8.78e{-}2$	$6.21e{-}2$	$5.39e{-}2$	$5.11e{-}2$	$5.01e{-}2$

TABLE 4.3
SRM errors for $\mu = 0.001$ with a fairly large time step $k = h = 0.1$.

ϵ	Iteration	Error at $\rightarrow$	$t = k$	$t = 1.0$	$t = 2.0$	$t = 3.0$	$t = 4.0$	$t = 5.0$
$1e{-}3$	1	eu	$2.18e{-}2$	$8.61e{-}2$	$9.43e{-}2$	$9.70e{-}2$	$9.86e{-}2$	$9.99e{-}2$
		ep	$1.83e{-}1$	$8.83e{-}2$	$6.26e{-}2$	$5.42e{-}2$	$5.13e{-}2$	$5.03e{-}2$

TABLE 4.4
SRM errors for $\mu = 0.1$ with $\alpha_1 = 0$.

ϵ	Iteration	Error at $\rightarrow$	$t = k$	$t = 1.0$	$t = 2.0$	$t = 3.0$	$t = 4.0$	$t = 5.0$
$5e{-}1$	2	eu	$5.64e{-}3$	$3.57e{-}2$	$2.94e{-}2$	$2.71e{-}2$	$2.62e{-}2$	$2.60e{-}2$
		ep	$2.92e{-}0$	$9.70e{-}2$	$7.03e{-}2$	$6.17e{-}2$	$5.87e{-}2$	$5.77e{-}2$

this we take $k = h = 0.1$ and still $\mu = 0.001$. The numerical results in Table 4.3 show that it is true.

Although we use explicit schemes for SRM (1.3) with $\alpha_1 > 0$, we still have to solve a banded symmetric positive definite system. An alternative is to take $\alpha_1 = 0$ to avoid solving any algebraic systems. Table 4.4 shows the computational results of the difference scheme (4.4) with $\alpha_1 = 0$ and $\alpha_2 = 1$. We take viscosity $\mu = 0.1$, $h = 0.1$, and $k = 0.0005$. Good results are obtained except for the pressure near $t = 0$ (cf. Remark 3.4).

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