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Davidson, Fordyce A.; Rynne, Bryan P.

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## SELF-ADJOINT BOUNDARY-VALUE PROBLEMS ON TIME-SCALES

FORDYCE A. DAVIDSON, BRYAN P. RYNNE

ABSTRACT. In this paper we consider a second order, Sturm-Liouville-type boundary-value operator of the form

$$Lu := -[pu^\nabla]^\Delta + qu,$$

on an arbitrary, bounded time-scale  $\mathbb{T}$ , for suitable functions  $p, q$ , together with suitable boundary conditions. We show that, with a suitable choice of domain, this operator can be formulated in the Hilbert space  $L^2(\mathbb{T}_\kappa)$ , in such a way that the resulting operator is self-adjoint, with compact resolvent (here, ‘self-adjoint’ means in the standard functional analytic meaning of this term). Previous discussions of operators of this, and similar, form have described them as ‘self-adjoint’, but have not demonstrated self-adjointness in the standard functional analytic sense.

### 1. INTRODUCTION

Over the past decade a large number of papers on second order, Sturm-Liouville-type boundary value problems on bounded time-scales  $\mathbb{T}$  have appeared. Most of these deal with a  $\Delta\Delta$  formulation of the corresponding differential operator, viz.

$$Lu := -(pu^\Delta)^\Delta + qu^\sigma, \quad u \in D(L), \quad (1.1)$$

for suitable functions  $p, q$ , on a suitable domain  $D(L)$  (the specification of the domain  $D(L)$  includes suitable boundary conditions on  $u$ ; in this introductory section we omit details of spaces and domains). Much of the basic theory of such operators is described in, for example, [3, Chapter 4]. Such operators have often been termed ‘self-adjoint’. However, it was shown in [6] that expressions of this form do not, in general, yield self-adjoint operators, in the standard functional-analytic meaning of the term ‘self-adjoint’. Indeed, it is shown in [6] that a fundamental property of self-adjoint operators can fail for operators of the form (1.1), so that the standard theory of self-adjoint operators cannot readily be applied to such operators.

More recently, in an attempt to obtain self-adjointness, differential operators in the following  $\nabla\Delta$  form

$$Lu := -(pu^\nabla)^\Delta + qu, \quad u \in D(L) \quad (1.2)$$

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have been considered, see for example, [2, 9] and the references therein. Such ‘mixed’ operators result in a symmetric Green’s function, which is taken to indicate that the corresponding operators possess some of the features of self-adjoint operators. However, the operators constructed in these papers map between (different) Banach spaces of continuously differentiable functions on  $\mathbb{T}$ , whereas, in the standard functional-analytic definition, a self-adjoint operator is defined on a subspace of a Hilbert space  $H$ , and maps this subspace of  $H$  into  $H$  itself. This Hilbert space formulation is necessary to obtain many of the desirable properties of such operators.

In this paper our goal is to formulate the  $\nabla\Delta$  operator in (1.2) in the setting of the Hilbert space  $L^2(\mathbb{T}_\kappa)$  defined in [11]. This formulation is based on the Sobolev-type spaces defined in [11] consisting of functions on  $\mathbb{T}$  having  $L^2$ -type generalised derivatives. We then show that the resulting operator  $L$ , in  $L^2(\mathbb{T}_\kappa)$ , is an unbounded, self-adjoint operator, with compact resolvent (in the standard functional-analytic sense). The extensive functional-analytic theory of such operators is then available for this operator, although, for brevity, we will not discuss any applications of this general theory to this operator.

**Remark 1.1.** We consider the  $\nabla\Delta$  operator in (1.2), but operators involving  $\Delta\nabla$  combinations (see e.g. [2, 4, 9]) could be treated similarly, there is no essential difference in these formulations. Using the  $\nabla\Delta$  form allows us to apply the results in [11] (based on a Lebesgue-type ‘ $\Delta$ -integral’) unaltered. A corresponding Lebesgue-type ‘ $\nabla$ -integral’ could be constructed using the methods in [11], and this would then allow  $\Delta\nabla$  operators to be considered in a similar manner.

## 2. PRELIMINARIES

Papers on time-scales usually go through a set of standard definitions of integration and differentiation on time-scales. For brevity we will omit this and simply refer to [11, Section 2] for this standard material (which is, of course, also discussed in most other time-scales papers). In particular, we will use the Lebesgue-type  $\Delta$ -integral defined in [11]. A similar Lebesgue-type  $\nabla$ -integral could readily be defined, but will not be required here. However, we will need to use spaces of  $\nabla$ -differentiable functions, in addition to the spaces of  $\Delta$ -differentiable functions discussed in [11]. To distinguish between these spaces will require some slight modifications to the notation used for various spaces and norms in [11], so we briefly discuss time-scale differentiation, and the notation we will use.

Recall that a function  $u : \mathbb{T} \rightarrow \mathbb{R}$  is  $\nabla$ -differentiable on  $\mathbb{T}$  if, at each  $t \in \mathbb{T}_\kappa$ , there exists  $u^\nabla(t)$  such that, for any  $\epsilon > 0$  there exists  $\delta > 0$  such that

$$s \in \mathbb{T} \text{ and } |t - s| < \delta \implies |u(\rho(t)) - u(s) - u^\nabla(t)(\rho(t) - s)| \leq \epsilon|\rho(t) - s|,$$

see, for example, [4, Ch. 3]; the  $\Delta$ -derivative is defined similarly, by replacing  $\rho(t)$  with  $\sigma(t)$  throughout.

We let  $C^0(\mathbb{T})$  (respectively  $C_{\text{rd}}^0(\mathbb{T})$ ,  $C_{\text{ld}}^0(\mathbb{T})$ ) denote the set of continuous (respectively rd-continuous, ld-continuous) functions on  $\mathbb{T}$ ; with the norm

$$\|u\|_{\mathbb{T}} := \sup_{t \in \mathbb{T}} |u(t)|, \quad u \in C_{\text{rd}}(\mathbb{T}) \cup C_{\text{ld}}(\mathbb{T}),$$

all these spaces are Banach spaces. We now let  $C^1(\mathbb{T}, \Delta)$  (respectively  $C_{\text{rd}}^1(\mathbb{T}, \Delta)$ ) denote the set of functions  $u \in C^0(\mathbb{T})$  which are  $\Delta$ -differentiable and for which

$u^\Delta \in C^0(\mathbb{T}^\kappa)$  (respectively  $u^\Delta \in C_{\text{rd}}^0(\mathbb{T}^\kappa)$ ); with the norm

$$|u|_{\mathbb{T}, \Delta} := |u|_{\mathbb{T}} + |u^\Delta|_{\mathbb{T}^\kappa}, \quad u \in C_{\text{rd}}^1(\mathbb{T}, \Delta),$$

these spaces are Banach spaces. Similarly, we define the Banach spaces  $C^1(\mathbb{T}, \nabla)$  and  $C_{\text{id}}^1(\mathbb{T}, \nabla)$  with norm

$$|u|_{\mathbb{T}, \nabla} := |u|_{\mathbb{T}} + |u^\nabla|_{\mathbb{T}^\kappa}, \quad u \in C_{\text{id}}^1(\mathbb{T}, \nabla).$$

The spaces  $C^1(\mathbb{T}, \nabla)$  and  $C^1(\mathbb{T}, \Delta)$  need not be equal. For example, let  $\mathbb{T} = [-1, 0] \cup [1, 2]$  and define the function  $u \equiv 0$  on  $[-1, 0]$ ,  $u(t) = t$  on  $[1, 2]$ . It can be verified that  $u \in C^1(\mathbb{T}, \nabla)$ , but  $u \notin C^1(\mathbb{T}, \Delta)$ . However, the following result gives a simple relationship between these spaces

**Lemma 2.1** ([9, Theorem 6]).  $C^1(\mathbb{T}, \nabla) \subset C_{\text{rd}}^1(\mathbb{T}, \Delta)$ . If  $u \in C^1(\mathbb{T}, \nabla)$  then  $u^\Delta = (u^\nabla)^\sigma$ .

It will also be necessary to  $\nabla$ -differentiate indefinite  $\Delta$ -integrals, for which we will require the following lemma.

**Lemma 2.2** ([1, Theorem 2.10]). If  $u \in C^0(\mathbb{T})$ ,  $t_0 \in \mathbb{T}$ , and

$$U_{t_0}(t) := \int_{t_0}^t u \Delta, \quad t \in \mathbb{T},$$

then  $U_{t_0} \in C_{\text{id}}^1(\mathbb{T}, \nabla)$  and  $U_{t_0}^\nabla = u^\rho$  on  $\mathbb{T}^\kappa$ .

We will also require the Sobolev-type space of functions with generalised  $\Delta$ -derivatives defined in [11], which we will denote here by  $H^1(\mathbb{T}, \Delta)$  with associated norm

$$\|u\|_{\mathbb{T}, \Delta} := \|u\|_{\mathbb{T}} + \|u^\Delta\|_{\mathbb{T}}, \quad u \in H^1(\mathbb{T}, \Delta),$$

where

$$\|u\|_{\mathbb{T}}^2 := \int_{\mathbb{T}} |u|^2 \Delta, \quad u \in L^2(\mathbb{T}).$$

Note that the integral used here is the Lebesgue-type  $\Delta$ -integral constructed in [11]. We also note that [11, Lemma 3.5] shows that  $C_{\text{rd}}^1(\mathbb{T}, \Delta) \subset H^1(\mathbb{T}, \Delta)$ , so Lemma 2.1 has the following simple corollary, which will be required below.

**Corollary 2.3.**  $C^1(\mathbb{T}, \nabla) \subset H^1(\mathbb{T}, \Delta)$ .

Finally, in this preliminary section, we recall some basic functional-analytic definitions, see for example [10, Ch. 13]. Let  $T : D(T) \subset H \rightarrow H$  be a linear operator in a Hilbert space  $H$ , with inner product  $\langle \cdot, \cdot \rangle$ . Then  $T$  is *symmetric* if

$$\langle Tx, y \rangle = \langle x, Ty \rangle, \quad \forall x, y \in D(T),$$

and  $T$  is *self-adjoint* if  $D(T)$  is dense in  $H$  and

$$\langle Tx, y \rangle = \langle x, z \rangle, \quad \forall x \in D(T) \implies y \in D(T) \text{ and } z = Ty.$$

## 3. A BOUNDARY VALUE LINEAR OPERATOR

3.1. **Definition of  $L$ .** Let  $a = \inf \mathbb{T}$ ,  $b = \sup \mathbb{T}$ . We are interested in the class of functions defined on  $\mathbb{T}$  which satisfy the boundary conditions

$$u^\nabla(\sigma(a)) = \gamma_a u(\sigma(a)), \quad u^\nabla(b) = -\gamma_b u(b), \quad (3.1)$$

with arbitrary constants  $\gamma_a \in (-\infty, \infty]$ ,  $\gamma_b \in (-\infty, \infty]$ , and we define the following set of functions

$$\mathcal{D} := \{u \in C^1(\mathbb{T}, \nabla) : u^\nabla \in H^1(\mathbb{T}_\kappa, \Delta) \text{ and } u \text{ satisfies (3.1)}\},$$

$$D(L) := \{w \in L^2(\mathbb{T}_\kappa) : w = u|_{\mathbb{T}_\kappa} \text{ for some } u \in \mathcal{D}\}$$

(in the definition of  $D(L)$ ,  $w = u|_{\mathbb{T}_\kappa}$  denotes the restriction of  $u$  to the set  $\mathbb{T}_\kappa^\kappa$ , and we recall from [11] that the point  $b$  has  $\mu_\mathbb{T}$ -measure zero, so in the setting of equivalence classes of  $L^2(\mathbb{T}_\kappa)$  functions, the value  $u(b)$  is not well-defined).

Throughout, we impose the following additional assumption on  $\gamma_a, \gamma_b$ .

**Assumption 3.1.** (i) If  $a$  is right-scattered then  $\gamma_a < \infty$ . (ii) If  $b$  is left-scattered then  $1 + \gamma_b(b - \rho(b)) \neq 0$ .

These constructions require some further explanation and remarks.

- (a) In the above notation the cases  $\gamma_a = \infty$  or  $\gamma_b = \infty$  are taken to mean the conditions  $u(\sigma(a)) = 0$  or  $u(b) = 0$ , and in these cases it is the latter form that would be used in the calculations below. Furthermore, if  $a$  is right-dense, these cases correspond to the Dirichlet-type conditions  $u(a) = 0$  or  $u(b) = 0$ . It will be seen in Remark 3.1 below that when  $a$  is right-scattered the Dirichlet-type condition at  $a$  arises from a different value of  $\gamma_a$ .
- (b) Assumption 3.1 precludes the boundary conditions  $u(\sigma(a)) = 0$  (when  $a$  is right-scattered) or  $u(\rho(b)) = 0$  (when  $b$  is left-scattered). Either of these conditions lead to certain pathological properties of the operator  $L$  which we wish to avoid.
- (c) If  $a$  is right-scattered it is natural, in view of the definition of the  $\nabla$ -derivative, to formulate the first boundary condition in (3.1) in terms of  $u^\nabla(\sigma(a))$ , but the use of  $u(\sigma(a))$ , rather than  $u(a)$ , may seem slightly strange. This formulation is chosen, primarily, to simplify certain formulae arising from various integrations by parts below. The following remark shows that  $u(a)$  could be used in (3.1) simply by changing the value of  $\gamma_a$ .
- (d) If  $a$  is right-scattered or  $b$  is left-scattered, the corresponding boundary conditions in (3.1) can be rewritten in the alternative forms

$$\begin{aligned} u(a) - (1 + (\sigma(a) - a)\gamma_a)u(\sigma(a)) &= 0, \\ u(\rho(b)) - (1 + (b - \rho(b))\gamma_b)u(b) &= 0. \end{aligned} \quad (3.2)$$

Hence, by (3.2) and Assumption 3.1, if  $u \in \mathcal{D}$  then  $u(a)$  and  $u(b)$  are determined by  $u(\sigma(a))$  and  $u(\rho(b))$ , that is,  $u$  is determined entirely by its restriction  $w = u|_{\mathbb{T}_\kappa} \in D(L)$ . Conversely, by using (3.2), any function  $w \in D(L)$  can be extended to  $\mathbb{T}$  to yield a function  $u \in \mathcal{D}$ . Thus, the sets  $\mathcal{D}$  and  $D(L)$  are (algebraically) isomorphic, and can be naturally identified with each other. In our discussion of  $L$  below we will make use of this identification, and we will generally use the symbol  $u$  interchangeably for an element of either  $\mathcal{D}$  or  $D(L)$ .

- (e) If  $a$  is right-scattered then it follows from (3.2) that we obtain the Dirichlet-type condition  $u(a) = 0$  by choosing  $\gamma_a$  such that  $1 + (\sigma(a) - a)\gamma_a = 0$ .

Having dealt with the boundary conditions, we now define the desired differential operator  $L$ . Suppose that  $p \in H^1(\mathbb{T}_\kappa, \Delta)$ ,  $q \in L^2(\mathbb{T}_\kappa)$ , with

$$p_{\min} := \min\{p(t) : t \in \mathbb{T}_\kappa\} > 0,$$

and define the linear operator  $L : D(L) \subset L^2(\mathbb{T}_\kappa) \rightarrow L^2(\mathbb{T}_\kappa)$  by

$$Lu := -[pu^\nabla]^{\Delta_g} + qu, \quad u \in D(L),$$

where  $\Delta_g$  is the generalised  $\Delta$ -derivative constructed in [11]. The definition  $L$  also requires some further explanation and remarks.

- (a) The set  $\mathcal{D}$  would be a natural domain for the operator  $L$ . However, to obtain a self-adjoint operator it is necessary that the domain and range of  $L$  lie in the same Hilbert space (which we take to be  $L^2(\mathbb{T}_\kappa)$ ). For this reason we introduce the domain  $D(L) \subset L^2(\mathbb{T}_\kappa)$ , isomorphic to  $\mathcal{D}$ .
- (b) In light of the identification of  $\mathcal{D}$  and  $D(L)$  described in Remark 3.1 above, we regard the calculation of  $Lu \in L^2(\mathbb{T}_\kappa)$  from  $u \in D(L)$  as proceeding in the following manner: use (3.2) to extend  $u$  from the set  $\mathbb{T}_\kappa$  to  $\mathbb{T}$  (yielding an element of  $\mathcal{D}$ , which we still write as  $u$ ), and then construct  $u^\nabla \in H^1(\mathbb{T}_\kappa, \Delta)$  and  $(u^\nabla)^{\Delta_g} \in L^2(\mathbb{T}_\kappa)$  in the usual manner (by the definition of  $\mathcal{D}$ , these are well-defined for  $u \in \mathcal{D}$ ).
- (c) The operator  $Lu = -[pu^\Delta]^{\Delta_g} + qu^\sigma$ , on a similar domain, was considered in [11]. However, we will see that the above operator is self-adjoint, (in the functional-analytic sense), whereas the operator in [11] is not. Despite this difference, the comments in [11, Remarks 5.1 and 5.2] regarding the definition of  $L$  there apply equally well to the above operator.

**3.2. Properties of  $L$ .** We now obtain various basic properties of  $L$ .

**Lemma 3.2.** *The operator  $L$  is symmetric with respect to the inner product  $\langle \cdot, \cdot \rangle_{\mathbb{T}_\kappa}$  on  $L^2(\mathbb{T}_\kappa)$ , that is,*

$$\langle Lu, v \rangle_{\mathbb{T}_\kappa} = \langle u, Lv \rangle_{\mathbb{T}_\kappa}, \quad u, v \in D(L). \quad (3.3)$$

*Proof.* By definition, we can regard  $u, v$  as belonging to  $\mathcal{D}$ , that is  $u, v \in C^1(\mathbb{T}, \nabla)$ . Thus, by Lemma 2.1,  $u, v \in C_{\text{rd}}^1(\mathbb{T}, \Delta)$ , and hence, by [11, Corollary 4.6 (f)],

$$\begin{aligned} \langle Lu, v \rangle_{\mathbb{T}_\kappa} &= \int_{\sigma(a)}^b (pu^\nabla)^\sigma v^\Delta \Delta - [pu^\nabla v]_{\sigma(a)}^b + \int_{\sigma(a)}^b quv \Delta \\ &= \int_{\sigma(a)}^b (p^\sigma u^\Delta v^\Delta + quv) \Delta + B(u, v) \end{aligned} \quad (3.4)$$

where, by (3.1),

$$B(u, v) = \gamma_a p(\sigma(a))u(\sigma(a))v(\sigma(a)) + \gamma_b p(b)u(b)v(b)$$

(if  $\gamma_a = \infty$  or  $\gamma_b = \infty$  then we omit the corresponding term in this formula). The result now follows from the symmetry in  $u$  and  $v$  of the right hand side of (3.4).  $\square$

**Lemma 3.3.** *There exists a constant  $C_L$  such that*

$$\langle Lu, u \rangle_{\mathbb{T}_\kappa} \geq \frac{1}{2} p_{\min} \|u\|_{\mathbb{T}_\kappa, \Delta}^2 + C_L \|u\|_{\mathbb{T}_\kappa}^2, \quad u \in D(L). \quad (3.5)$$

**Remark 3.4.** The constant  $C_L$  in Lemma 3.3 need not be positive.

*Proof.* Suppose that  $u \in D(L)$ . Then we can regard  $u$  as belonging to  $\mathcal{D}$ , and it follows from (3.4) and the Cauchy-Schwarz inequality that

$$\langle Lu, u \rangle_{\mathbb{T}_\kappa} \geq p_{\min} \|u^\Delta\|_{\mathbb{T}_\kappa}^2 - C_1 |u|_{\mathbb{T}_\kappa}^2, \quad (3.6)$$

for some constant  $C_1 \geq 0$  (independent of  $u$ ), and by (3.2) and Assumption 3.1,

$$|u|_{\mathbb{T}_\kappa}^2 \leq C_2 |u|_{\mathbb{T}_\kappa}^2, \quad (3.7)$$

for some constant  $C_2 > 0$ . Also, a straightforward modification of the proof of [11, Theorem 4.16] shows that for any  $\epsilon > 0$  there exists a constant  $C_3(\epsilon) > 0$  such that

$$|w|_{\mathbb{T}_\kappa} \leq \epsilon \|w^\Delta\|_{\mathbb{T}_\kappa} + C_3(\epsilon) \|w\|_{\mathbb{T}_\kappa}, \quad w \in H^1(\mathbb{T}_\kappa, \Delta). \quad (3.8)$$

By Corollary 2.3 and the definition of  $\mathcal{D}$ ,  $u \in H^1(\mathbb{T}_\kappa, \Delta)$ , so putting  $\epsilon$  sufficiently small and  $w = u$  in (3.8) and combining this with (3.6) and (3.7) yields (3.5).  $\square$

Invertibility of  $L$  will be important below, and it will be seen that invertibility follows from injectivity of  $L$ , so we now consider this. In general,  $L$  need not be injective, but the following result shows that we can obtain injectivity by adding to  $L$  a sufficiently large scalar multiple of the identity operator  $I : L^2(\mathbb{T}_\kappa) \rightarrow L^2(\mathbb{T}_\kappa)$ . In many situations, if  $L$  itself is not injective then it is possible, with no loss of generality, to replace  $L$  with the injective operator  $L_c$  given by the following result.

**Theorem 3.5.** *If  $c + C_L > 0$  then the operator  $L_c := L + cI$  is injective.*

*Proof.* It follows from (3.5) that

$$\langle L_c u, u \rangle_{\mathbb{T}_\kappa} \geq \frac{1}{2} p_{\min} \|u\|_{\mathbb{T}_\kappa, \Delta}^2 + (c + C_L) \|u\|_{\mathbb{T}_\kappa}^2 > 0, \quad 0 \neq u \in D(L),$$

which proves that  $L_c$  is injective.  $\square$

The following result gives simple criteria under which  $L$  itself is injective.

**Theorem 3.6.** *Suppose that  $q \geq 0$  on  $\mathbb{T}_\kappa$  and  $\gamma_a, \gamma_b \geq 0$ . Then  $L$  is injective under either of the hypotheses:*

- (i)  $\gamma_a + \gamma_b > 0$ ;
- (ii)  $\|q\|_{\mathbb{T}_\kappa} > 0$ .

*Proof.* We consider hypothesis (i), a similar proof holds for hypothesis (ii). Suppose that  $0 \neq u \in D(L)$  and  $Lu = 0$ . It follows from this and (3.4) that

$$0 = \langle Lu, u \rangle_{\mathbb{T}_\kappa} \geq p_{\min} \|u^\Delta\|_{\mathbb{T}_\kappa}^2 + B(u, u),$$

and hence, by [11, Corollary 4.6],  $u \equiv 0$  on  $\mathbb{T}_\kappa$ .  $\square$

We will also need the following result regarding solutions of the corresponding initial value problem. This result can be proved in a similar manner to that of [11, Theorem 5.8].

**Theorem 3.7.** *For any  $h \in L^2(\mathbb{T}_\kappa)$  and  $\tau \in \mathbb{T}_\kappa$ ,  $\eta_1, \eta_2 \in \mathbb{R}$ , the initial value problem*

$$\begin{aligned} -(pu^\nabla)^{\Delta_g} + qu &= h, \\ u(\tau) &= \eta_1, \quad u^\nabla(\tau) = \eta_2, \end{aligned} \quad (3.9)$$

*has a unique solution  $u \in C^1(\mathbb{T}, \nabla)$ , with  $u^\nabla \in H^1(\mathbb{T}_\kappa, \Delta)$ .*

Let  $\phi, \psi$  be the solutions of (3.9) given by Theorem 3.7, with  $h = 0$  and the ‘initial’ conditions

$$\begin{aligned} \phi(\sigma(a)) &= 1, & \phi^\nabla(\sigma(a)) &= \gamma_a, \\ \psi(b) &= 1, & \psi^\nabla(b) &= -\gamma_b, \end{aligned} \quad (3.10)$$

with the obvious modification, here and below, when  $\gamma_a = \infty$  or  $\gamma_b = \infty$ . Also, let

$$W := p(\phi^\nabla\psi - \psi^\nabla\phi) \in H^1(\mathbb{T}_\kappa, \Delta)$$

(it follows from the properties of  $\phi, \psi$  given by Theorem 3.7, together with Corollary 2.3 and [11, Corollary 4.6], that  $W \in H^1(\mathbb{T}_\kappa, \Delta)$ ).

**Lemma 3.8.**  *$W$  is constant on  $\mathbb{T}_\kappa$ . The operator  $L$  is injective if and only if  $W \neq 0$ .*

*Proof.* From the definitions of  $\phi, \psi$ , Corollary 2.3 and [11, Corollary 4.6],

$$\begin{aligned} W^{\Delta_g} &= (p\phi^\nabla)^{\Delta_g}\psi + (p\phi^\nabla)^\sigma\psi^\Delta - (p\psi^\nabla)^{\Delta_g}\phi - (p\psi^\nabla)^\sigma\phi^\Delta \\ &= q\phi\psi + (p\phi^\nabla\psi^\nabla)^\sigma - q\psi\phi - (p\psi^\nabla\phi^\nabla)^\sigma = 0, \end{aligned}$$

so by [11, Corollary 4.6],  $W \equiv \text{const}$ . Moreover, by (3.10),

$$W = p(b)(\phi^\nabla(b) + \gamma_b\phi(b)),$$

so  $W = 0$  if and only if  $\phi$  satisfies the boundary conditions (3.1). Clearly, if  $\phi$  satisfies (3.1) then  $L$  is not injective, and the converse follows immediately from linearity and the uniqueness of the solution of the initial value problem for  $\phi$ .  $\square$

We can now begin the construction of the inverse of  $L$  (when  $L$  is injective). Equivalently, we construct a solution of the boundary value problem

$$Lu = h, \quad h \in L^2(\mathbb{T}_\kappa), \quad u \in D(L), \quad (3.11)$$

for any  $h \in L^2(\mathbb{T}_\kappa)$ .

**Definition 3.9.** Suppose that  $L$  is injective. For  $(t, s) \in \mathbb{T} \times \mathbb{T}$  let

$$g(t, s) := \begin{cases} W^{-1}\psi(t)\phi(s), & \text{if } t \geq s, \\ W^{-1}\phi(t)\psi(s), & \text{if } t \leq s. \end{cases}$$

Clearly,  $g$  is continuous on  $\mathbb{T} \times \mathbb{T}$ . For any  $h \in L^2(\mathbb{T}_\kappa)$ , let

$$Gh(t) := \int_{\sigma(a)}^b g(t, \cdot)h \Delta, \quad t \in \mathbb{T}_\kappa^\kappa. \quad (3.12)$$

**Theorem 3.10.** *Suppose that  $L$  is injective. Then:*

- (i) *for any  $h \in L^2(\mathbb{T}_\kappa)$  the function  $u = Gh \in D(L)$ , and  $u$  is the unique solution of (3.11);*
- (ii) *the operators  $L : D(L) \subset L^2(\mathbb{T}_\kappa) \rightarrow L^2(\mathbb{T}_\kappa)$ ,  $G : L^2(\mathbb{T}_\kappa) \rightarrow D(L) \subset L^2(\mathbb{T}_\kappa)$ , are invertible, linear operators and  $L^{-1} = G$ ,  $G^{-1} = L$ . The operator  $G$  is compact, while if  $\dim L^2(\mathbb{T}_\kappa) = \infty$  then  $L$  is unbounded.*

**Remark 3.11.** We call  $g$  the *Green’s function* and  $G$  the *Green’s operator* for the operator  $L$ .



*Proof.* The uniqueness follows immediately from the injectivity of  $L$ . Now suppose that  $h \in C^0(\mathbb{T}_\kappa)$ . To simplify the following calculations we will suppose that  $p \equiv 1$ ; the general proof is similar.

It is clear that the formula for  $u = Gh$  in (3.12) can be extended to define a function on the whole of  $\mathbb{T}$ , which we continue to denote by  $u$ . Now suppose that  $a$  is right-dense. Then by direct calculation (using Lemma 2.2 above, the product rule for nabla derivatives, see [3], and for generalised derivatives, see Corollary 4.6 in [11], and (3.3) in [11]),

$$\begin{aligned} Wu(t) &= \psi(t) \int_{\sigma(a)}^t \phi h \Delta + \phi(t) \int_t^b \psi h \Delta, \quad t \in \mathbb{T}, \\ Wu^\nabla(t) &= \psi^\rho(t) \phi^\rho(t) h^\rho(t) + \psi^\nabla(t) \int_{\sigma(a)}^t \phi h \Delta \\ &\quad - \phi^\rho(t) \psi^\rho(t) h^\rho(t) + \phi^\nabla(t) \int_t^b \psi h \Delta \\ &= \psi^\nabla(t) \int_{\sigma(a)}^t \phi h \Delta + \phi^\nabla(t) \int_t^b \psi h \Delta, \quad t \in \mathbb{T}, \\ W(u^\nabla)^{\Delta g}(t) &= -Wh(t) + (\psi^\nabla)^{\Delta g}(t) \int_{\sigma(a)}^{\sigma(t)} \phi h \Delta + (\phi^\nabla)^{\Delta g}(t) \int_{\sigma(t)}^b \psi h \Delta \\ &= -Wh(t) + q(t) \left\{ \psi(t) \int_{\sigma(a)}^{\sigma(t)} \phi h \Delta + \phi(t) \int_{\sigma(t)}^b \psi h \Delta \right\} \\ &= -Wh(t) + q(t) \left\{ \psi(t) \int_{\sigma(a)}^t \phi h \Delta + \phi(t) \int_t^b \psi h \Delta \right\} \\ &= -Wh(t) + q(t) Wu(t), \quad \mu_{\mathbb{T}\text{-a.e.}} t \in \mathbb{T}. \end{aligned}$$

It follows directly from these formulae and (3.10) that  $u = Gh \in \mathcal{D}$ , with

$$\|Gh\|_{\mathcal{D}} = \|u\|_{\mathcal{D}} := |u|_{\nabla, \mathbb{T}} + \|u^\nabla\|_{\mathbb{T}_\kappa, \Delta} \leq C \|h\|_{\mathbb{T}_\kappa}, \quad h \in C^0(\mathbb{T}_\kappa), \quad (3.13)$$

for some constant  $C$ . On the other hand, if  $a$  is right-scattered the calculation of  $Wu(a)$  and  $Wu^\nabla(\sigma(a))$  needs to be amended slightly to avoid reference to the (undefined) value  $h(a)$ , in the following manner. Directly from (3.12),

$$\begin{aligned} Wu(a) &= \phi(a) \int_{\sigma(a)}^b \psi h \Delta, \quad Wu(\sigma(a)) = \int_{\sigma(a)}^b \psi h \Delta, \\ Wu^\nabla(\sigma(a)) &= W \frac{u(\sigma(a)) - u(a)}{\sigma(a) - a} = \phi^\nabla(\sigma(a)) \int_{\sigma(a)}^b \psi h \Delta = W \gamma_a u(\sigma(a)). \end{aligned}$$

Thus the boundary condition at  $\sigma(a)$  also holds in this case, and it follows from the preceding formulae that if  $\sigma(a)$  is right-dense then  $\lim_{t \rightarrow \sigma(a)^+} u(t) = u(\sigma(a))$ ,  $\lim_{t \rightarrow \sigma(a)^+} u^\nabla(t) = u^\nabla(\sigma(a))$ , (here,  $t \rightarrow \sigma(a)^+$  through points in  $\mathbb{T}_\kappa$ ), so that  $u \in C^1(\mathbb{T}, \nabla)$ , that is we again have  $u \in \mathcal{D}$ . Clearly, the inequality (3.13) also holds in this case.

Now, since  $C^0(\mathbb{T}_\kappa)$  is dense in  $L^2(\mathbb{T}_\kappa)$  (see Lemma 3.5 in [11]), and  $\mathcal{D}$  is a Banach space (with respect to the above norm  $\|\cdot\|_{\mathcal{D}}$ ), the inequality (3.13) extends from the set  $C^0(\mathbb{T}_\kappa)$  to the whole of  $L^2(\mathbb{T}_\kappa)$  by continuity. Hence, in particular,  $G$  is bounded as an operator from  $L^2(\mathbb{T}_\kappa)$  into  $\mathcal{D}$ . The assertions about the operators  $L$  and  $G$

now follow immediately (the compactness of  $G$  follows from (3.13), Lemma 2.1 and the compactness of the embedding  $C_{rd}^1(\mathbb{T}, \Delta) \rightarrow C^0(\mathbb{T})$ , see [6, Lemma 2.2]).  $\square$

We can now prove that  $L$  is self-adjoint (irrespective of injectivity of  $L$ ).

**Theorem 3.12.** *The domain  $D(L)$  is dense in  $L^2(\mathbb{T}_\kappa)$ , and the operator  $L$  is self-adjoint with respect to the inner product  $\langle \cdot, \cdot \rangle_{\mathbb{T}_\kappa}$  on  $L^2(\mathbb{T}_\kappa)$ .*

*Proof.* It suffices to prove the result for  $L_c$ , for arbitrary  $c \in \mathbb{R}$ , so without loss of generality we suppose that  $c = 0$  and  $L$  is injective. If  $D(L)$  is not dense in  $L^2(\mathbb{T}_\kappa)$  then there exists  $0 \neq w \in L^2(\mathbb{T}_\kappa)$  such that

$$\langle u, w \rangle_{\mathbb{T}_\kappa} = 0, \quad \forall u \in D(L).$$

Since  $R(L) = L^2(\mathbb{T}_\kappa)$ , we have  $w = Lz$  for some  $z \in D(L)$ , so by Lemma 3.2,

$$0 = \langle u, Lz \rangle_{\mathbb{T}_\kappa} = \langle Lu, z \rangle_{\mathbb{T}_\kappa}, \quad \forall u \in D(L),$$

and hence  $z = 0$  (again, since  $R(L) = L^2(\mathbb{T}_\kappa)$ ). However, this implies that  $w = 0$ , which contradicts the choice of  $w$ , and so proves that  $D(L)$  is dense in  $L^2(\mathbb{T}_\kappa)$ .

Now suppose that

$$\langle Lu, v \rangle_{\mathbb{T}_\kappa} = \langle u, w \rangle_{\mathbb{T}_\kappa}, \quad \forall u \in D(L), \quad (3.14)$$

for some  $v, w \in L^2(\mathbb{T}_\kappa)$ . We again have  $w = Lz$ , for some  $z \in D(L)$ , and so from (3.14),

$$\langle Lu, v - z \rangle_{\mathbb{T}_\kappa} = 0, \quad \forall u \in D(L),$$

and hence  $v = z$ . That is,  $v \in D(L)$  and  $w = Lv$ , which proves that  $L$  is self-adjoint.  $\square$

**Corollary 3.13.** *Suppose that  $L$  is injective. Then the operator  $G$  is self-adjoint with respect to the inner product  $\langle \cdot, \cdot \rangle_{\mathbb{T}_\kappa}$  on  $L^2(\mathbb{T}_\kappa)$ .*

*Proof.* Let  $u, v \in L^2(\mathbb{T}_\kappa)$  be arbitrary. Then by Theorem 3.10,  $u = Lx$ ,  $v = Ly$ , for some  $x, y \in D(L)$ . Hence, by Lemma 3.2 and Theorem 3.10,

$$\langle Gu, v \rangle_{\mathbb{T}_\kappa} = \langle x, Ly \rangle_{\mathbb{T}_\kappa} = \langle Lx, y \rangle_{\mathbb{T}_\kappa} = \langle u, Gv \rangle_{\mathbb{T}_\kappa},$$

which proves that  $G$  is self-adjoint.  $\square$

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FORDYCE A. DAVIDSON  
DIVISION OF MATHEMATICS, DUNDEE UNIVERSITY, DUNDEE DD1 4HN, SCOTLAND  
*E-mail address:* `fdavidso@maths.dundee.ac.uk`

BRYAN P. RYNNE  
DEPARTMENT OF MATHEMATICS AND THE MAXWELL INSTITUTE FOR MATHEMATICAL SCIENCES,  
HERIOT-WATT UNIVERSITY, EDINBURGH EH14 4AS, SCOTLAND  
*E-mail address:* `bryan@ma.hw.ac.uk`