Statistical mechanics of ocean waves: Landau damping, Rogue Waves & Hilbert transforms

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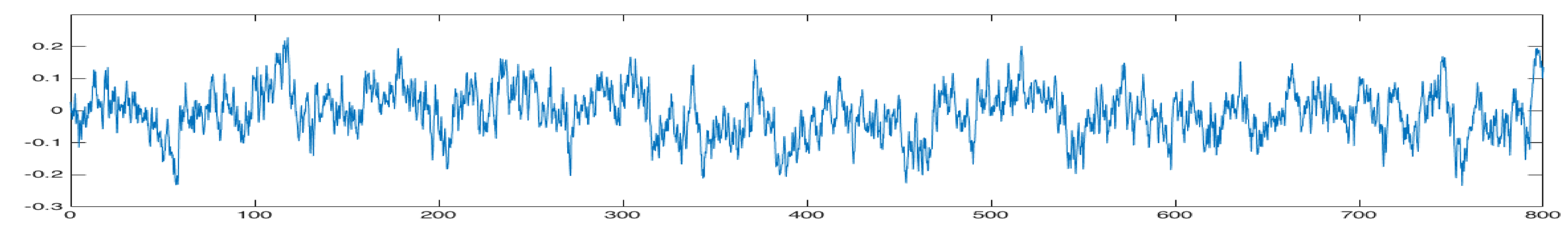
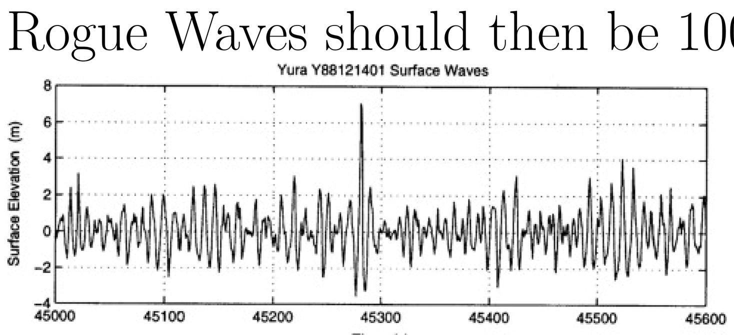
1. The physical problem: Rogue Waves

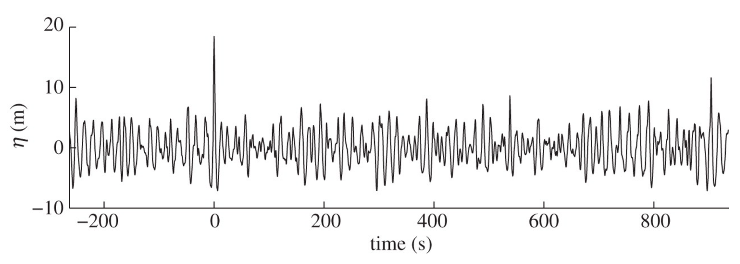
> 99% of the time, ocean waves are "nice" like this:

weakly nonlinear, quasi-gaussian, quasi-homogeneous+stationary, ...



Rogue Waves should then be 1000 year events – but they're not.





Is there some effect that switches on < 1% of the time?

2. The mathematical problem: the Alber equation

Consider a sea-state with autocorrelation

$$R(\alpha,\beta,t) = \underbrace{\Gamma(\alpha-\beta)}_{\text{homogeneous background}} + \varepsilon \underbrace{\rho(\alpha,\beta,t)}_{\text{inhomogeneity}}$$

Under the "nice" assumptions, the inhomogeneity ρ evolves with

$$i\partial_t\rho + \frac{p}{2}\left(\Delta_\alpha - \Delta_\beta\right)\rho + $$
$$+ q[\Gamma(\alpha-\beta) + \varepsilon\rho(\alpha,\beta)][\rho(\alpha,\alpha) - \rho(\beta,\beta)] = 0.$$

- Autocorrelation Γ is where real-world data comes into the problem.
- If ρ grows, "nice" regime fails under its own assumptions.
- Derivation and first formal results by Alber [1].

3. Linear LD for non-mean-zero inhomogeneity

Denote $P(k) = \mathcal{F}_{y\to k}[\Gamma(y)]$ the power spectrum, and

$$f(X,k) = \mathcal{F}^{-1}_{\alpha\to X}\mathcal{F}_{\beta\to k}\Big[\rho(\alpha+\frac{\beta}{2},\alpha-\frac{\beta}{2},t)\Big], \qquad \breve{n}(X,t) = \int f(X,\xi,t)d\xi.$$

Then the linearised equation for f is

$$\partial_t f - 4\pi^2 ipk\cdot Xf + qi\Big[P\big(k-\tfrac{X}{2}\big) - P\big(k+\tfrac{X}{2}\big)\Big]\breve{n}(X,t) = 0.$$

leading to a closed equation for the position density

$$\underbrace{\breve{n}(X,t)}_{\text{position density}} = \underbrace{\breve{n}_f(X,t)}_{\text{free-space pos. dens.}} + \int_0^t h(X,t-\tau)\breve{n}(X,\tau)d\tau,$$

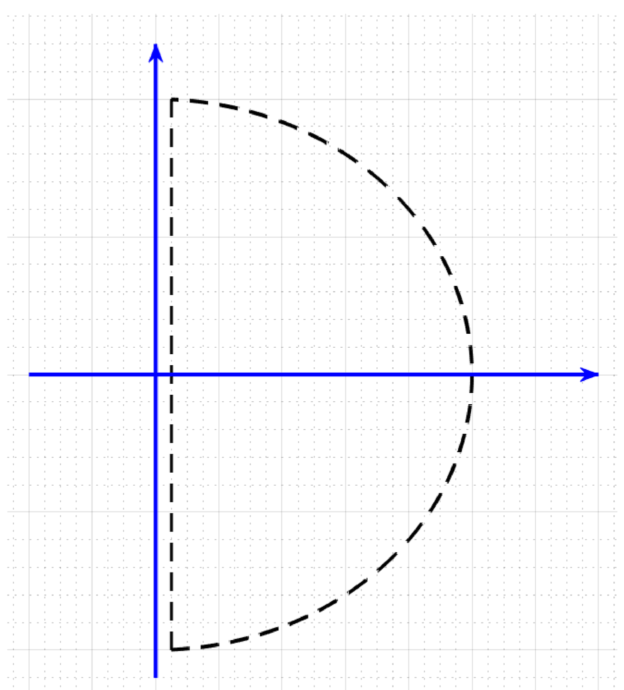
In the Laplace domain

$$\widetilde{n}(X,\omega) = \frac{1}{1-\widetilde{h}(X,\omega)}\widetilde{n}_f(X,\omega) \qquad \text{or} \qquad X\widetilde{n} - X\widetilde{n}_f = X\frac{\widetilde{h}}{1-\widetilde{h}}\widetilde{n}_f.$$

To Laplace invert note:

- **Jump across the imaginary axis** (by construction)

$$\widetilde{h}(X,\omega) = qi\int_{\mathbb{R}} \frac{\frac{P(k+\frac{X}{2})-P(k-\frac{X}{2})}{X}}{\frac{\omega}{X} - 4\pi^2 ipk}dk = \frac{q}{4\pi p}\mathbb{H}\left[D_X P\right]\left(\frac{\omega}{4\pi^2 ipX}\right)$$



- **Decay at $|\omega| \to \infty$** (by construction)
- **Analytic on right half-plane** (stability condition)

Linear Landau Damping for the Alber equation [2]

As long as **the stability condition holds**

$$\exists\kappa > 0 \qquad \inf_{X,\mathrm{Re}\,\omega>0}\left|1-\widetilde{h}(X,\omega)\right| \geqslant \kappa,$$

there exists a $C > 0$ such that

$$\|\nabla_X n\|_{L^2_{X,t}} \leqslant C\frac{\kappa+1}{\kappa^2}\|f_0\|_{\Sigma^r}$$

while allowing $\breve{n}(0,0) = \int n(x,0)dx = \int R(x,x,0)dx \neq 0$.

4. Differences with the state of the art

Most similar in the literature: free-space "screened Vlasov" [5].

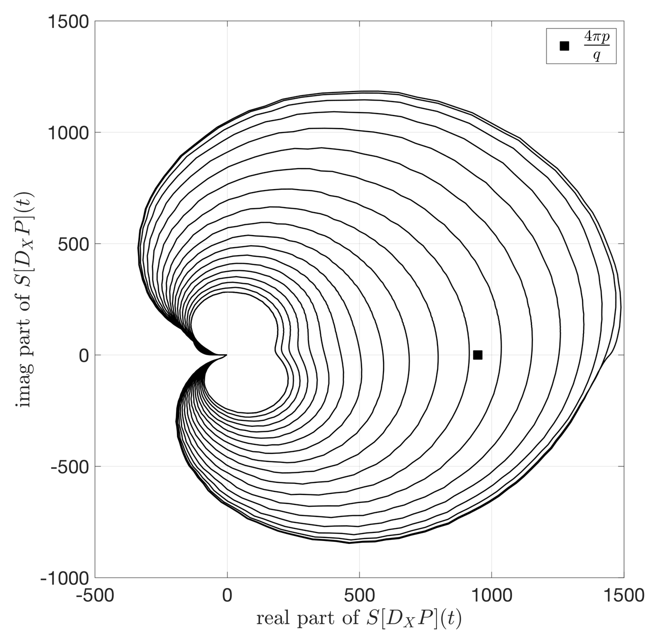
$$\text{Vlasov}: \quad h(X,t) \approx X^2\widetilde{W}(X)t\breve{P}(Xt)$$
$$\text{Alber}: \quad h(X,t) \approx \sin(X^2 t)\breve{P}(Xt)$$

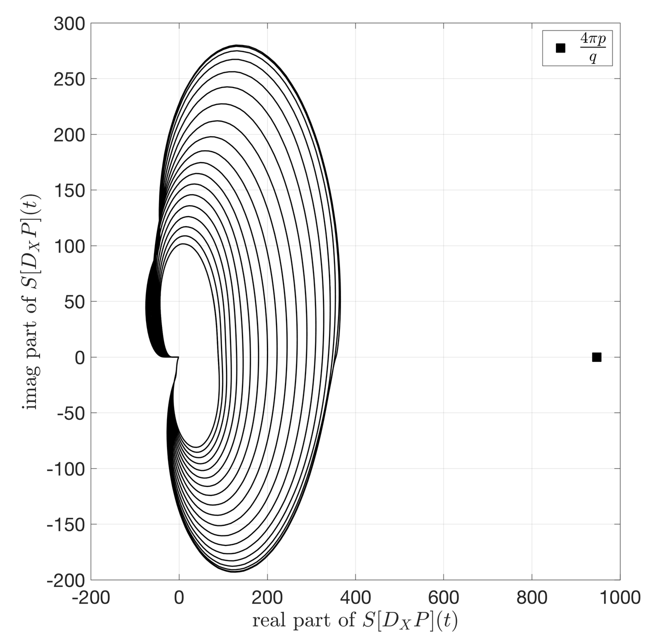
- Fine properties of the Hilbert transform, $\int_{s\in\mathbb{R}}|\mathbb{H}\left[D_X P\right](s)|\,ds < \infty$
- the extra X allows control without assuming $\breve{n}(0,0) = 0$.
- Fine $L^2_{x,t}$ estimates for the free-space position density

A high-dimensional version of the argument is possible.

5. Resolving the Penrose-Alber condition

Checking the stability condition [2]:

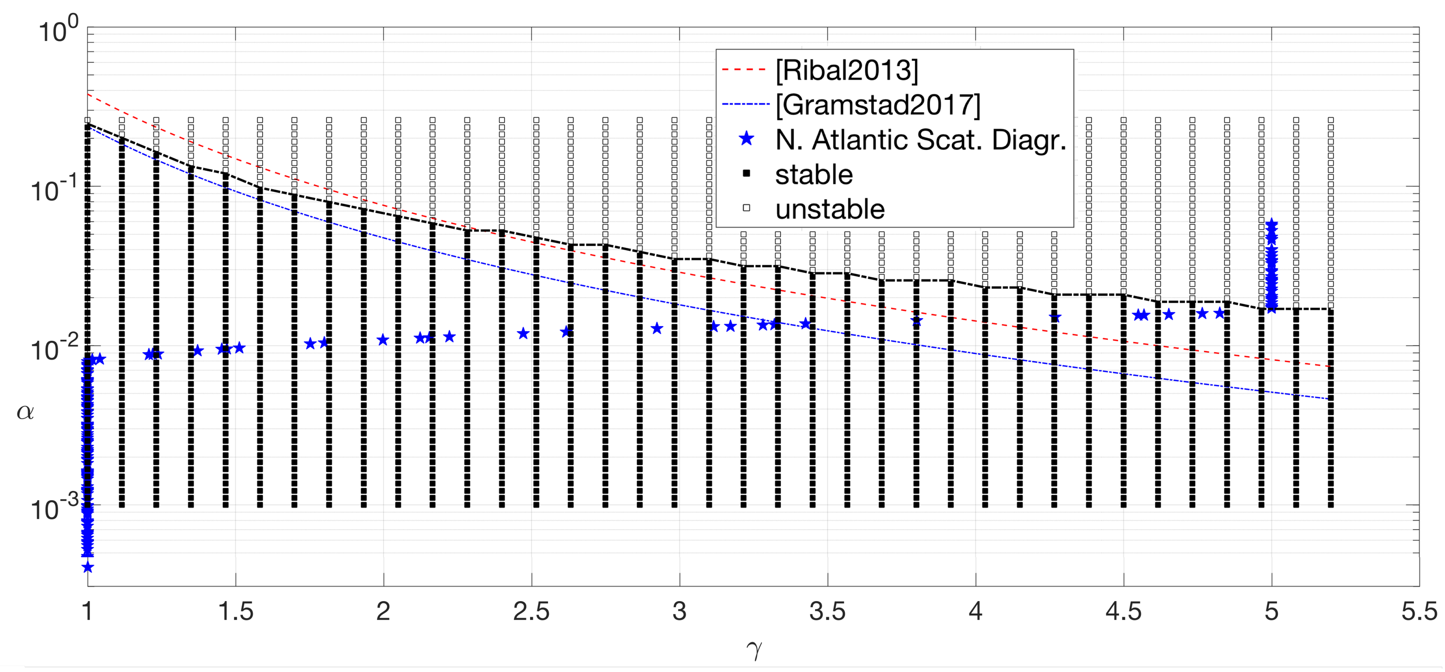




Using measurements fitted to JONSWAP spectra

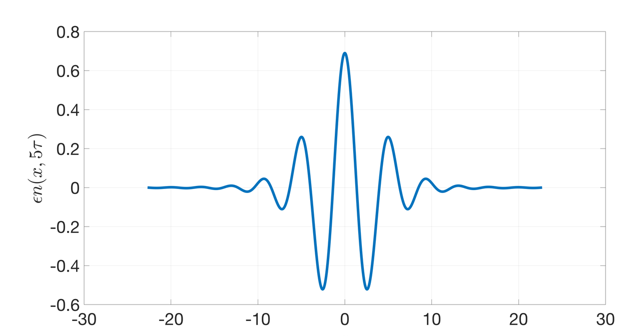
$$P(k) = \frac{\alpha}{2k^3}e^{-\frac{5}{4}(\frac{k_0}{k})^2}\gamma^{exp[-(1-\sqrt{k/k_0})^2/2\delta^2]},$$

we have instability $O(2/1000)$ of the time in the North Atlantic [2]

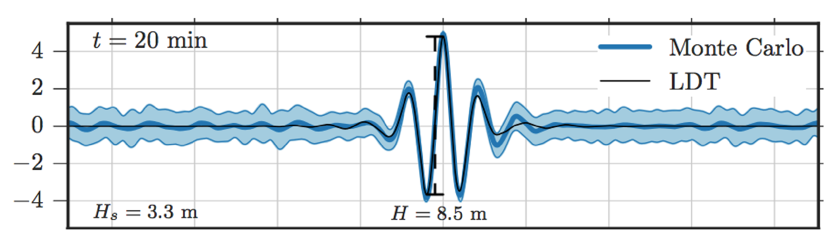


6. Beyond Landau damping

Universal profiles of Rogue Waves [3], see also [6]:

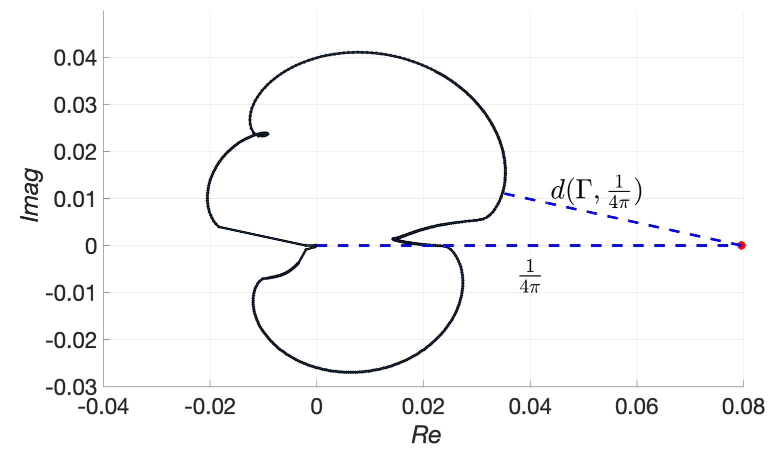


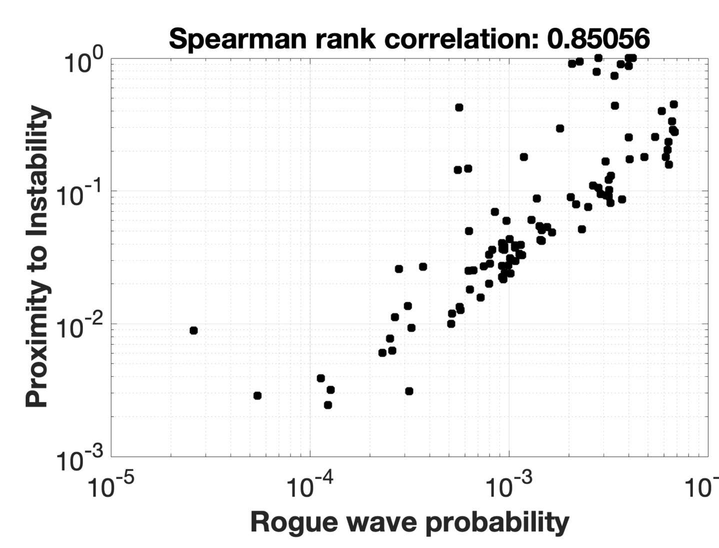
A., Athanassoulis, Sapsis [3]



Dematteis, Grafke, Vanden-Eijden [6]

Proximity to instability [4]:





References

[1] I. E. Alber, The Effects of Randomness on the Stability of Two-Dimensional Surface Wavetrains, Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences, 363 (1978), pp. 525–546.

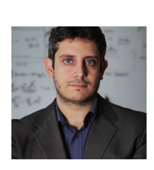
[2] A. Athanassoulis, G. Athanassoulis, M. Ptashnyk, and T. Sapsis, Strong solutions for the Alber equation and stability of unidirectional wave spectra, Kinetic & Related Models, 13 (2020), pp. 703–737.

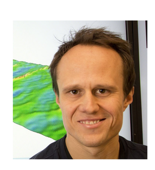
[3] A. Athanassoulis, G. Athanassoulis, and T. Sapsis, Localized instabilities of the Wigner equation as a model for the emergence of Rogue Waves, Journal of Ocean Engineering and Marine Energy, 3 (2017), pp. 353–372.

[4] A. Athanassoulis and O. Gramstad, Modelling of Ocean Waves with the Alber Equation: Application to Non-Parametric Spectra and Generalisation to Crossing Seas, Fluids, 6 (2021), p. 291.

[5] J. Bedrossian, N. Masmoudi, and C. Mouhot, Landau Damping in Finite Regularity for Unconfined Systems with Screened Interactions, Communications on Pure and Applied Mathematics, 71 (2018), pp. 537–576.

[6] G. Dematteis, T. Grafke, and E. Vanden-Eijnden, Rogue Waves and Large Deviations in Deep Sea, Proceedings of the National Academy of Sciences, (2018), p. 201710670.





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