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# Generalized Darcy–Oseen resolvent problem

Dagmar Medková<sup>a</sup>, Mariya Ptashnyk<sup>b\*†</sup> and Werner Varnhorn<sup>c</sup>

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**In this paper, we study the well-posedness of a coupled Darcy–Oseen resolvent problem, describing the fluid flow between free-fluid domains and porous media separated by a semipermeable membrane. The influence of osmotic effects, induced by the presence of a semipermeable membrane, on the flow velocity is reflected in the transmission conditions on the surface between the free-fluid domain and the porous medium. To prove the existence of a weak solution of the generalized Darcy–Oseen resolvent system, we consider two auxiliary problems: a mixed Navier–Dirichlet problem for the generalized Oseen resolvent system and Robin problem for an elliptic equation related to the general Darcy equations. © 2016 The Authors Mathematical Methods in the Applied Sciences Published by John Wiley & Sons Ltd.**

**Keywords:** Darcy–Oseen resolvent problem; semipermeable membrane; Brinkman–Darcy equations; fluid flow between free-fluid domains and porous media; compact perturbations

## 1. Introduction

The transport of macromolecules from a free-fluid domain to a porous medium separated by a semipermeable membrane is an important biological question as well as an interesting and challenging problem for the mathematical modelling and analysis. As examples for selective transport of macromolecules between free-fluid and porous medium, we can consider transport of macromolecules by water in plant tissues or by blood in arteries. Water flow in plant tissues takes place in two different physical domains separated by semipermeable membranes, denoted as symplast and apoplast [1]. The apoplast is composed of cell walls and intercellular spaces, while the symplast is constituted by cell insides, which can be connected by plasmodesmata. The complex microstructure of the apoplast, composed of polymers and microfibrils, can be represented as a porous medium, and the water flow in the cell walls and intercellular space is modelled by Darcy's law. The Stokes and/or Brinkmann equations can be used to define the flow velocity in the cell inside (cytoplasm) and plasmodesmata. Similar situation is found in models of early atherosclerotic lesions [2] or low-density lipoproteins transfer through arterial walls, for example, [3–5]. Here, the blood velocity in artery lumen is described by Stokes or Navier–Stokes equations, while arterial walls are modelled as porous media and Darcy equations are considered.

The aim of this article is to study the well-posedness of a general coupled free-fluid and porous-medium model for transport processes in biological tissues. The main difference of our problem to coupled Stokes–Darcy, Brinkman–Darcy and Navier–Stokes–Darcy problems studied before (see, for example, [6–9] and references therein) is that the free-fluid and the porous media domains do not interact directly. A semipermeable membrane separates there two domains and controls actively and passively the fluxes of the water (or blood) and the solutes [1, 2]. Thus, the appropriate transmission conditions on the boundary between free-fluid domain and porous media need to be considered to represent the regulation of the water (or blood) flow from the cell inside (or artery lumen) into the intercellular space (or arterial walls), [1, 2].

In contrast to previous work on transport of macromolecules from artery lumen to arterial walls [2], where the dependence of fluid flow across the membrane on the oncotic pressure difference was neglected (to simplify the analysis), we consider the influence of the osmotic effect on the flow velocity. This is reflected in the transmission conditions on the boundary between cell insides and cell walls, comprising the normal component of the Darcy velocity  $\mathbf{v}^D \cdot \mathbf{n}$  and a given function  $\mathbf{g} \cdot \mathbf{n}$ . The function  $\mathbf{g} \cdot \mathbf{n}$  corresponds to the difference between the solute concentrations in the cell insides and the intercellular space (or in the artery lumen and arterial walls), respectively (for example [1, 2]). Our model includes also the situation when only a part of the boundary between the cell insides and the intercellular space (cell walls) is semipermeable.

The analysis presented here is a generalization of the results obtained in [10], where a coupled Stokes–Darcy system with a constant permeability in the Darcy equations was analysed. For the flow velocity in the domain with free fluid, we consider generalized Oseen

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resolvent equations, while the fluid flow in the porous medium is modelled by the Darcy equations with a general space-dependent permeability tensor. Although, there are many results on existence of solutions of Oseen system [11–16], coupled generalized Darcy–Oseen problem was not considered before.

To prove the existence of a weak solution of the generalized Darcy–Oseen resolvent problem, we consider two auxiliary problems: a mixed Navier–Dirichlet problem for the generalized Oseen resolvent system and a Robin problem for an elliptic equation related to the general Darcy equations. The Riesz representation theorem for a continuous linear functional on the Hilbert space  $H^1(\Omega)$  and the fact that  $H^{-1/2}(\partial\Omega)$  is a closed subspace of the dual space  $[H^1(\Omega)]'$  are used to prove the existence of a weak solution of the Robin-type problem. To show the existence of a solution of a mixed Navier–Dirichlet problem for the generalized Oseen resolvent system, we combine the existence results for a mixed Navier–Dirichlet problem for Stokes system, obtained in [10], and a compact perturbation argument.

Notice that considering some of the parameters in the generalized Oseen resolvent equations to be 0, we recover Stokes, Oseen or Brinkmann system. Brinkman equations describe flows through some types of porous media (i.e. fibrous porous media for swarms of particles of low concentration) [17–19]. The Brinkman system is also an extension of Darcy’s law when boundary layer regions near the soil phase of a porous media cannot be neglected [20] and can be derived from the Stokes or Navier–Stokes system with slip boundary conditions at the surfaces of solid structures [21]. Brinkman system can also be viewed as an approximation of the Navier–Stokes equations at low Reynolds numbers (for example [8]). The Oseen problem is a popular linearization of the Navier–Stokes equations [11, 22]. Thus, modelling the fluid flow in the symplast of plant tissues by generalized Oseen resolved equations allows us to consider the situation when the water flow inside plant cells is modelled by Stokes equations and Brinkmann equations describe the water velocity in plasmodesmata.

The paper is organized as follows. In Section 2, we formulate mathematical model for transport processes in a domain composed of free-fluid domain and a porous medium separated by a semipermeable membrane. In Sections 3 and 4, two auxiliary problems: the mixed Navier–Dirichlet problem for an Oseen resolvent system and the Robin-type problem are analysed. The main result of the paper, that is, the existence and uniqueness (up to a constant for pressures) of a weak solution of the generalized Darcy–Oseen resolvent problem, is proved in Section 5.

## 2. Formulation of the mathematical model

Let  $\Omega \subset \mathbb{R}^3$  be a bounded domain, that is, a bounded open connected set, with Lipschitz boundary  $\partial\Omega$ , and suppose that  $\Omega_S$  is a nonempty subdomain of  $\Omega$  with Lipschitz boundary  $\partial\Omega_S$  such that  $\Omega_S \neq \Omega$ . Then  $\Omega_D := \Omega \setminus \overline{\Omega_S}$  is a bounded open set, not necessarily connected. We suppose that  $\Omega_D$  has a Lipschitz boundary, too. Note that  $\partial\Omega_S \cap \partial\Omega_D \cap \Omega$  is always nonempty, and it is locally the graph of a Lipschitz function. Let  $\Gamma$  denote a nonempty closed subset of  $\partial\Omega_S \cap \partial\Omega_D$ . Then  $\Gamma$  might reach the boundary  $\partial\Omega$  or not. If  $G$  is a component of  $\Omega_D$ , we suppose that  $\Gamma \cap \partial G$  has positive surface measure.

In  $\Omega$  we consider the following coupled boundary value transmission problem

$$-\eta \Delta \mathbf{v}^S + \lambda \mathbf{v}^S + \mathbf{k} \cdot \nabla \mathbf{v}^S + \nabla p^S = 0, \quad \operatorname{div} \mathbf{v}^S = 0 \quad \text{in } \Omega_S, \tag{1}$$

$$\mathbf{v}^D + K \nabla p^D = 0, \quad \operatorname{div} \mathbf{v}^D = 0 \quad \text{in } \Omega_D, \tag{2}$$

$$\mathbf{v}^S = \mathbf{f} \quad \text{on } \partial\Omega_S \setminus \Gamma, \tag{3}$$

$$\mathbf{v}^D \cdot \mathbf{n} = h \quad \text{on } \partial\Omega_D \setminus \Gamma, \tag{4}$$

$$\mathbf{v}^D \cdot \mathbf{n} - \mathbf{v}^S \cdot \mathbf{n} = h, \quad \mathbf{v}_\tau^S = \mathbf{f}_\tau \quad \text{on } \Gamma, \tag{5}$$

$$\left[ -T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \alpha \mathbf{v}^S + \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right] \cdot \mathbf{n} = \beta p^D + \gamma \mathbf{v}^D \cdot \mathbf{n} - \mathbf{g} \cdot \mathbf{n} \quad \text{on } \Gamma. \tag{6}$$

Here,  $\mathbf{k} \in \mathbb{R}^3$ , and  $\eta, \beta > 0, \alpha, \gamma \geq 0$  are constants,  $K = (K_{ij})$  is a symmetric  $3 \times 3$ -matrix function with entries  $K_{ij} \in L^\infty(\Omega_D)$  and  $\lambda = (\lambda_{ij})$  is a measurable  $3 \times 3$ -matrix function with entries  $\lambda_{ij} \in L^\infty(\Omega_D)$ . Notice that  $\mathbf{k} \cdot \nabla \mathbf{v} = (\mathbf{k} \cdot \nabla v_1, \mathbf{k} \cdot \nabla v_2, \mathbf{k} \cdot \nabla v_3)$ .

The vector  $\mathbf{v}^D = (v_1^D, v_2^D, v_3^D)$  denotes the generalized Darcy velocity vector in  $\Omega_D$ , and  $\mathbf{v}^S = (v_1^S, v_2^S, v_3^S)$  represents the generalized Oseen resolvent velocity in  $\Omega_S$ . In the transmission condition (6),

$$T_\eta(\mathbf{v}^S, p^S) = 2\eta \mathbf{Dv}^S - p^S I$$

means the stress tensor, where

$$\mathbf{Dv} = \frac{1}{2} [\nabla \mathbf{v} + (\nabla \mathbf{v})^T]$$

is the symmetric  $3 \times 3$ -gradient of  $\mathbf{v}$  and  $I$  the  $3 \times 3$ -unity matrix. We denote by  $\mathbf{n}^S$  and by  $\mathbf{n}^D$  the exterior unit normal vectors of  $\Omega_S$  and  $\Omega_D$ , respectively. By  $\mathbf{n}$ , we mean  $\mathbf{n} = \mathbf{n}^S$  on  $\partial\Omega_S$  and  $\mathbf{n} = -\mathbf{n}^D$  on  $\partial\Omega_D$ . Moreover, we use  $\mathbf{v}_n = (\mathbf{v} \cdot \mathbf{n}) \mathbf{n}$  for the normal part of  $\mathbf{v}$ , and  $\mathbf{v}_\tau = \mathbf{v} - \mathbf{v}_n$  for the tangential part of  $\mathbf{v}$ . Finally, we suppose that the matrix function  $K$  satisfies a uniform ellipticity condition, that is, there exists a constant  $a > 0$  such that for a.a.  $\mathbf{x} \in \Omega_D$  we have

$$\sum_{i,j=1}^3 K_{ij}(\mathbf{x}) \xi_i \xi_j \geq a |\boldsymbol{\xi}|^2 \quad \forall \boldsymbol{\xi} = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3, \tag{7}$$

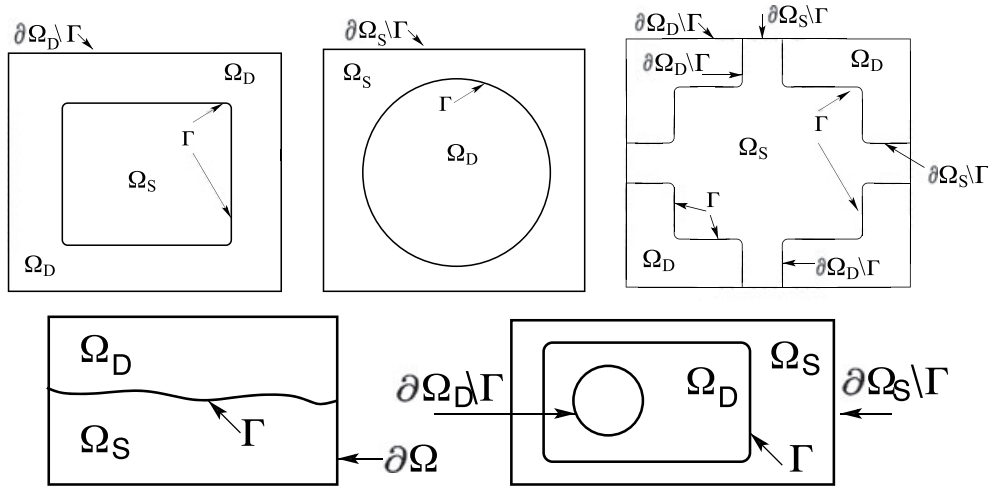


Figure 1. Representative geometry of  $\Omega$ .

and that the matrix  $\lambda$  is nonnegative, that is, for a.a.  $\mathbf{x} \in \Omega_S$  we have

$$\sum_{ij=1}^3 \lambda_{ij}(\mathbf{x}) \xi_i \xi_j \geq 0 \quad \forall \xi = (\xi_1, \xi_2, \xi_3) \in \mathbb{R}^3. \quad (8)$$

The Equation (2) represents a generalization of the standard Darcy equations where  $K = I$ . For  $\lambda = 0$  and  $k = 0$  the system (1) coincides with the Stokes system, for  $\lambda = 0$  and  $k \neq 0$  with the Oseen system, and for  $\lambda = cI$  with  $c > 0$  and  $k \neq 0$  with the Oseen resolvent system. Finally, if  $k = 0$  and  $\xi \cdot (\lambda(\mathbf{x}) \xi) > 0$  for a.a.  $\mathbf{x} \in \Omega_S$  and  $\xi \in \mathbb{R}^3$ , with  $\xi \neq \mathbf{0}$ , then (1) is called the Brinkman system.

In the next, we suppose that there exists some  $\Theta \in [H^1(\Omega_S)]^3$  with  $\Theta = 0$  on  $\partial\Omega \setminus \Gamma$ , with  $\Theta_\tau = 0$  on  $\Gamma$ , and with

$$\int_{\Gamma} \Theta \cdot \mathbf{n} \, d\sigma_{\mathbf{y}} = 1. \quad (9)$$

Note that the latter condition is certainly satisfied if the surface  $\Gamma$  contains a nontrivial sufficiently smooth part (of class  $C^2$ ).

Notice that the situations when  $\partial\Omega_S \setminus \Gamma = \emptyset$  or  $\partial\Omega_D \setminus \Gamma = \emptyset$  are included in the analysis presented here. If  $\overline{\Omega_S} \subset \Omega$  and  $\partial\Omega_S = \Gamma$ , then the condition (3) disappears. As examples, we can consider a domain  $\Omega = \{\mathbf{x} \in \mathbb{R}^3; |x_j| < 2, j = 1, 2, 3\}$ ,  $\Omega_S = \{\mathbf{x} \in \mathbb{R}^3; |x| < 1\}$ ,  $\Omega_D = \Omega \setminus \Omega_S$ , and  $\Gamma = \{\mathbf{x} \in \mathbb{R}^3; |x| = 1\}$ . In the situation when  $\overline{\Omega_D} \subset \Omega$  and  $\partial\Omega_D = \Gamma$ , the boundary condition (4) disappears. For example, if  $\Omega = \{\mathbf{x} \in \mathbb{R}^3; |x_j| < 2, j = 1, 2, 3\}$ ,  $\Omega_D = \{\mathbf{x} \in \mathbb{R}^3; |x| < 1\}$ , and  $\Gamma = \{\mathbf{x} \in \mathbb{R}^3; |x| = 1\}$ . The situation when  $\partial\Omega_S \setminus \Gamma \neq \emptyset$  and  $\partial\Omega_D \setminus \Gamma \neq \emptyset$  is depicted in Figure 1. Such structure is characteristic for plant cells that are connected with each other via plasmodesmata. Then the domain  $\Omega_S$  represents cell inside and plasmodesmata, and the domain  $\Omega_D$  describes a cell wall and intercellular space.

The interface  $\Gamma$  might reach the boundary, for example,  $\Omega = \{\mathbf{x} \in \mathbb{R}^3; |x_j| < 3\}$ ,  $\Omega_S = \{\mathbf{x} \in \Omega; x_1 < 0\}$ ,  $\Omega_D = \{\mathbf{x} \in \Omega; 0 < x_1\}$ , and  $\Gamma = \{\mathbf{x} \in \mathbb{R}^3; x_1 = 0, |x_2| \leq 3, |x_3| \leq 3\}$ , or might not reach the boundary, for example,  $\Omega = \{\mathbf{x} \in \mathbb{R}^3; 1 < |x| < 3\}$ ,  $\Omega_S = \{\mathbf{x} \in \mathbb{R}^3; 2 < |x| < 3\}$ ,  $\Omega_D = \{\mathbf{x} \in \mathbb{R}^3; 1 < |x| < 2\}$ ,  $\Gamma = \{\mathbf{x} \in \mathbb{R}^3; |x| = 2\}$ ,  $\partial\Omega_D \setminus \Gamma = \{\mathbf{x} \in \mathbb{R}^3; |x| = 1\}$ ,  $\partial\Omega_S \setminus \Gamma = \{\mathbf{x} \in \mathbb{R}^3; |x| = 3\}$ .

### 3. A mixed Navier–Dirichlet problem for the Oseen resolvent system

To prove the main result of the paper, that is, the existence of a unique (up to a constant for pressures) weak solution of the model (1)–(6), we shall use two auxiliary problems.

As a first auxiliary problem, in this section, we consider a mixed Navier–Dirichlet problem for the generalized Oseen resolvent system: For a given  $\mathbf{f} \in [H^{1/2}(\partial\Omega_S)]^3$  and  $\mathbf{g} \in [H^{-1/2}(\partial\Omega_S)]^3$ , find a weak solution  $\mathbf{v} \in [H^1(\Omega_S)]^3$  and  $p \in L^2(\Omega_S)$  of the problem

$$\begin{aligned} -\eta \Delta \mathbf{v} + \lambda \mathbf{v} + \mathbf{k} \cdot \nabla \mathbf{v} + \nabla p &= 0, & \operatorname{div} \mathbf{v} &= 0 & \text{in } \Omega_S, \\ \mathbf{v} &= \mathbf{f} & & & \text{on } \partial\Omega_S \setminus \Gamma, \\ \mathbf{v}_\tau &= \mathbf{f}_\tau & & & \text{on } \Gamma, \\ \left[ T_\eta(\mathbf{v}, p) \mathbf{n} + b \mathbf{v} - \frac{1}{2} (\mathbf{k} \cdot \mathbf{n}) \mathbf{v} \right]_n &= \mathbf{g}_n & & & \text{on } \Gamma, \end{aligned} \quad (10)$$

where  $\mathbf{k} \in \mathbb{R}^3$  and  $\eta, b > 0$  are constants.

Definition 3.1

We say that  $(\mathbf{v}, p) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)$  is a weak solution of the problem (10) if the boundary conditions  $\mathbf{v} = \mathbf{f}$  on  $\partial\Omega_S \setminus \Gamma$  and  $\mathbf{v}_\tau = \mathbf{f}_\tau$  on  $\Gamma$  are fulfilled in the sense of traces, and if it holds

$$\int_{\Omega_S} \left\{ 2\eta \mathbf{D}\mathbf{v} : \mathbf{D}\Phi + \lambda \mathbf{v} \cdot \Phi - p \operatorname{div} \Phi + \frac{1}{2} [\Phi \cdot (\mathbf{k} \cdot \nabla \mathbf{v}) - \mathbf{v} \cdot (\mathbf{k} \cdot \nabla \Phi)] \right\} dy + \int_{\partial\Omega_S} b \mathbf{v} \cdot \Phi d\sigma_y = \langle \mathbf{g}, \Phi \rangle_{H^{-1/2}, H^{1/2}} \tag{11}$$

for all  $\Phi \in \mathcal{V}_\Gamma$ , where

$$\mathcal{V}_\Gamma = \{ \mathbf{v} \in [H^1(\Omega_S)]^3; \mathbf{v} = 0 \text{ on } \partial\Omega_S \setminus \Gamma, \mathbf{v}_\tau = 0 \text{ on } \Gamma \}.$$

The integral relation (11) follows from (10) with the help of Green's formula, which also implies that a weak solution of the problem (10) is contained in the space

$$W_{\eta, \lambda, \mathbf{k}}(\Omega_S) = \{ (\mathbf{v}, p) \in H^1(\Omega_S)^3 \times L^2(\Omega_S); -\eta \Delta \mathbf{v} + \lambda \mathbf{v} + \mathbf{k} \cdot \nabla \mathbf{v} + \nabla p = 0, \operatorname{div} \mathbf{v} = 0 \}.$$

Moreover, if  $\mathbf{v} \in H^2(\Omega_S)^3$  and  $p \in H^1(\Omega_S)$ , then all boundary conditions are satisfied in the sense of traces.

For the following, we need some special Sobolev trace spaces on the boundary  $\partial\Omega_S$ , that is, the space

$$V_\Gamma = \{ \mathbf{v} \in [H^{1/2}(\partial\Omega_S)]^3; \mathbf{v} = 0 \text{ on } \partial\Omega_S \setminus \Gamma, \mathbf{v}_\tau = 0 \text{ on } \Gamma \}$$

of traces of the space  $\mathcal{V}_\Gamma$  and the space of restrictions

$$W_\Gamma = \{ (\mathbf{v}|_{\partial\Omega_S \setminus \Gamma}, \mathbf{v}_\tau|_\Gamma); \mathbf{v} \in [H^{1/2}(\partial\Omega_S)]^3 \},$$

the latter equipped with the norm

$$\| \mathbf{v} \|_{W_\Gamma} = \inf \{ \| \mathbf{u} \|_{H^{1/2}(\partial\Omega_S)}; \mathbf{u} \in [H^{1/2}(\partial\Omega_S)]^3, \mathbf{u} = \mathbf{v} \text{ on } \partial\Omega_S \setminus \Gamma, \mathbf{u}_\tau = \mathbf{v}_\tau \text{ on } \Gamma \}.$$

Because  $W_\Gamma$  coincides with the factor-space  $[H^{1/2}(\partial\Omega_S)]^3 / V_\Gamma$ , it is a Banach space. By  $V'_\Gamma$  we denote the dual space of  $V_\Gamma$ . According to the theorem of Hahn–Banach this dual space can be represented by

$$V'_\Gamma = \{ \mathbf{g}_n|_\Gamma; \mathbf{g} \in [H^{-1/2}(\partial\Omega_S)]^3 \}.$$

According to [23, Theorem 6.9.2], there exists a bounded linear extension operator  $E : H^{1/2}(\partial\Omega_S) \rightarrow H^1(\Omega_S)$  such that  $\varphi$  is the trace of  $E\varphi$ .

For  $(\mathbf{v}, p) \in W_{\eta, \lambda, \mathbf{k}}(\Omega_S)$ , we define  $[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v}]_n|_\Gamma \in V'_\Gamma$  by

$$\left\langle \left[ T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v} \right]_n|_\Gamma, \Phi \right\rangle_{H^{-1/2}, H^{1/2}} = \int_{\Omega_S} \left\{ 2\eta \mathbf{D}\mathbf{v} : \mathbf{D}(E\Phi) + (\lambda \mathbf{v}) \cdot (E\Phi) + \frac{1}{2} [(E\Phi) \cdot (\mathbf{k} \cdot \nabla \mathbf{v}) - \mathbf{v} \cdot (\mathbf{k} \cdot \nabla E\Phi)] - p \operatorname{div}(E\Phi) \right\} dy$$

for all  $\Phi \in V_\Gamma$ .

In order for  $[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v}]_n|_\Gamma$  to be well defined, we shall show that its definition does not depend on the choice of the extension operator  $E$ .

Proposition 3.2

If  $(\mathbf{v}, p) \in W_{\eta, \lambda, \mathbf{k}}(\Omega_S)$ , then  $[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v}]_n|_\Gamma$  does not depend on the choice of the extension operator  $E$ .

Proof

Consider  $\Phi \in V_\Gamma$  and  $\tilde{\Phi} \in [H^1(\Omega_S)]^3$  with  $\tilde{\Phi} = \Phi$  on  $\partial\Omega_S$ . Then  $E\Phi - \tilde{\Phi} \in [H_0^1(\Omega_S)]^3$ . If  $(\mathbf{v}, p) \in W_{\eta, \lambda, \mathbf{k}}(\Omega_S)$ , then Green's formula implies

$$\int_{\Omega_S} \left\{ 2\eta \mathbf{D}\mathbf{v} : \mathbf{D}(E\Phi - \tilde{\Phi}) + (\lambda \mathbf{v}) \cdot (E\Phi - \tilde{\Phi}) + \frac{1}{2} [(E\Phi - \tilde{\Phi}) \cdot (\mathbf{k} \cdot \nabla \mathbf{v}) - \mathbf{v} \cdot (\mathbf{k} \cdot \nabla (E\Phi - \tilde{\Phi}))] - p \operatorname{div}(E\Phi - \tilde{\Phi}) \right\} dy = 0.$$

Hence,  $[T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v}]_n|_\Gamma$  does not depend on the choice of the extension operator  $E$ . □

The following proposition ensures the unique solvability of the mixed Navier–Dirichlet problem (10) for the Oseen resolvent system.

**Proposition 3.3**

For a constant  $b > 0$  define

$$U^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}^S, p^S) = \left( \mathbf{v}^S|_{\partial\Omega_S \setminus \Gamma}, \mathbf{v}_\tau^S|_\Gamma, \left[ T_\eta(\mathbf{v}, p)\mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v} \right]_\mathbf{n}|_\Gamma + b\mathbf{v}_\mathbf{n}|_\Gamma \right).$$

Then  $U^{b,\eta,\lambda,\mathbf{k}} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \rightarrow W_\Gamma \times V'_\Gamma$  is an isomorphism.

Hence, for  $\mathbf{f} \in [H^{1/2}(\partial\Omega_S)]^3$ ,  $\mathbf{g} \in [H^{-1/2}(\partial\Omega_S)]^3$ , there exists a unique solution of the problem (10), and  $(\mathbf{v}^S, p^S)$  is a solution of the problem (10) if and only if  $(\mathbf{v}^S, p^S) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$  and

$$U^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}^S, p^S) = (\mathbf{f}|_{\partial\Omega_S \setminus \Gamma}, \mathbf{f}_\tau|_\Gamma, \mathbf{g}_\mathbf{n}|_\Gamma).$$

**Proof**

Using the Green formula, we obtain that  $(\mathbf{v}^S, p^S)$  is a solution of problem (10) if and only if

$$(\mathbf{v}^S, p^S) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \quad \text{and} \quad U^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}^S, p^S) = (\mathbf{f}|_{\partial\Omega_S \setminus \Gamma}, \mathbf{f}_\tau|_\Gamma, \mathbf{g}_\mathbf{n}|_\Gamma).$$

Moreover, the definition of  $U^{b,\eta,\lambda,\mathbf{k}}$  implies that

$$U^{b,\eta,\lambda,\mathbf{k}} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \rightarrow W_\Gamma \times V'_\Gamma$$

is a bounded linear operator.

As next, we have to show that  $U^{b,\eta,\lambda,\mathbf{k}} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \rightarrow W_\Gamma \times V'_\Gamma$  is an isomorphism.

We suppose first that  $\lambda \equiv 0$  and  $\mathbf{k} = 0$ . In this case, it is shown in [10] that the problem (10) is uniquely solvable and

$$U^{b,\eta,0,0} : W_{\eta,0,0}(\Omega_S) \rightarrow W_\Gamma \times V'_\Gamma$$

is an isomorphism.

We prove the proposition for arbitrary  $\lambda$  and  $\mathbf{k}$  using a compact perturbation argument. One of the problem in the analysis of the system (10) is that the operator  $U^{b,\eta,\lambda,\mathbf{k}}$  is defined on the space  $W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$  that depends on  $\eta, \lambda$  and  $\mathbf{k}$ . To overcome this problem, we extend these operators on some common space. Denote

$$H(\Omega_S, \text{div}) = \{ \mathbf{v} \in [H^1(\Omega_S)]^3; \text{div } \mathbf{v} = 0 \}.$$

Define the operator  $T_1^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \text{div}) \times L^2(\Omega_S) \rightarrow V'_\Gamma$  by

$$\langle T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p), \Phi \rangle = \int_{\Omega_S} \left[ 2\eta \mathbf{D}\mathbf{v} : \mathbf{D}\Phi + \lambda \mathbf{v} \cdot \Phi - p \text{div } \Phi + \frac{1}{2} [\Phi \cdot (\mathbf{k} \cdot \nabla \mathbf{v}) - \mathbf{v} \cdot (\mathbf{k} \cdot \nabla \Phi)] \right] \mathbf{d}\mathbf{y} + \int_{\partial\Omega_S} b \mathbf{v} \cdot \Phi \, d\sigma_{\mathbf{y}}$$

for  $\Phi \in V_\Gamma$ , and the operator  $T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \text{div}) \times L^2(\Omega_S) \rightarrow W_\Gamma \times V'_\Gamma$  by

$$T_2^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p) = \left( \mathbf{v}|_{\partial\Omega_S \setminus \Gamma}, \mathbf{v}_\tau|_\Gamma, T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p) \right).$$

Notice that

$$T_2^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p) = (\mathbf{f}, \mathbf{g})$$

for  $\mathbf{f} \in W_\Gamma$  and  $\mathbf{g} \in V'_\Gamma$  means that

$$\begin{aligned} -\eta \Delta \mathbf{v} + \lambda \mathbf{v} + \mathbf{k} \cdot \nabla \mathbf{v} + \nabla p &= \mathbf{g}, & \text{div } \mathbf{v} &= 0 & \text{in } \Omega_S, \\ \mathbf{v} &= \mathbf{f} & & & \text{on } \partial\Omega_S \setminus \Gamma, \\ \mathbf{v}_\tau &= \mathbf{f}_\tau, & \left[ T_\eta(\mathbf{v}, p)\mathbf{n} + b\mathbf{v} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n})\mathbf{v} \right]_\mathbf{n} &= \mathbf{g}_\mathbf{n} & \text{on } \Gamma. \end{aligned}$$

In the following, we show that

$$T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \text{div}) \times L^2(\Omega_S) \rightarrow W_\Gamma \times V'_\Gamma$$

is an isomorphism.

To do so, first suppose that  $\lambda \equiv 0$ ,  $\mathbf{k} = 0$ . If  $T_2^{b,\eta,0,0}(\mathbf{v}, p) = 0$  then  $(\mathbf{v}, p)$  is a solution of the problem (10) with  $\lambda \equiv 0$ ,  $\mathbf{k} = 0$  and with homogeneous boundary conditions. For this case, we have proved in [10] that  $(\mathbf{v}, p) = 0$ .

Let now  $(\mathbf{f}, \mathbf{g}) \in W_\Gamma \times \mathcal{V}'_\Gamma$  be given. Then  $\mathbf{g} \in [H^1(\Omega_S)^3]'$  and according to [22, Chapter IV, Theorem 1.1] there exists  $(\tilde{\mathbf{v}}, \tilde{p}) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)$  such that

$$\begin{aligned} -\eta \Delta \tilde{\mathbf{v}} + \nabla \tilde{p} &= \mathbf{g}, & \operatorname{div} \tilde{\mathbf{v}} &= 0 & \text{in } \Omega_S, \\ \tilde{\mathbf{v}} &= 0 & & & \text{on } \partial\Omega_S. \end{aligned}$$

This means that  $T_1^{b,\eta,0,0}(\tilde{\mathbf{v}}, \tilde{p}) - \mathbf{g}$  is supported on  $\partial\Omega_S$ , and hence  $\mathbf{g} - T_1^{b,\eta,0,0}(\tilde{\mathbf{v}}, \tilde{p}) \in V'_\Gamma$ . Moreover, in [10] we have proved that there exists  $(\mathbf{u}, \pi) \in W_{\eta,0,0}(\Omega_S)$  such that

$$\left(\mathbf{u}|_{\partial\Omega_S \setminus \Gamma}, \mathbf{u}_\tau|_\Gamma\right) = \left(\mathbf{f}|_{\partial\Omega_S \setminus \Gamma}, \mathbf{f}_\tau|_\Gamma\right) \quad \text{and} \quad T_1^{b,\eta,0,0}(\mathbf{u}, \pi) = \mathbf{g} - T_1^{b,\eta,0,0}(\tilde{\mathbf{v}}, \tilde{p}).$$

Consider  $\mathbf{v} = \tilde{\mathbf{v}} + \mathbf{u}$  and  $p = \tilde{p} + \pi$ . Then from the definition of  $\tilde{\mathbf{v}}$  and  $\mathbf{u}$  we obtain  $\operatorname{div} \mathbf{v} = 0$  and

$$T_2^{b,\eta,0,0}(\mathbf{v}, p) = \left(\mathbf{f}|_{\partial\Omega_S \setminus \Gamma}, \mathbf{f}_\tau|_\Gamma, \mathbf{g}\right).$$

This implies that

$$T_2^{b,\eta,0,0} : H(\Omega_S, \operatorname{div}) \times L^2(\Omega_S) \rightarrow W_\Gamma \times \mathcal{V}'_\Gamma$$

is an isomorphism.

Next, we consider a general matrix function  $\lambda$  with bounded entries and  $\mathbf{k} \in \mathbb{R}^3$ , such that  $\lambda$  satisfies the assumption (8). Then, Sobolev's imbedding theorem ensures that

$$T_2^{b,\eta,\lambda,\mathbf{k}} - T_2^{b,\eta,0,0} : H(\Omega_S, \operatorname{div}) \times L^2(\Omega_S) \rightarrow W_\Gamma \times \mathcal{V}'_\Gamma$$

is a compact operator. Because  $T_2^{b,\eta,0,0} : H(\Omega_S, \operatorname{div}) \times L^2(\Omega_S) \rightarrow W_\Gamma \times \mathcal{V}'_\Gamma$  is an isomorphism, the operator  $T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \operatorname{div}) \times L^2(\Omega_S) \rightarrow W_\Gamma \times \mathcal{V}'_\Gamma$  is Fredholm with index 0.

Consider now  $(\mathbf{v}, p) \in H(\Omega_S, \operatorname{div}) \times L^2(\Omega_S)$  such that  $T_2^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p) = 0$ . Then  $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$ ,  $\mathbf{v} = 0$  on  $\partial\Omega_S \setminus \Gamma$ , and  $\mathbf{v}_\tau = 0$  on  $\Gamma$ . Because  $\operatorname{div} \mathbf{v} = 0$ , we have

$$0 = \langle T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p), \mathbf{v} \rangle = \int_{\Omega_S} [2\eta |\mathbf{D}\mathbf{v}|^2 + (\lambda \mathbf{v}) \cdot \mathbf{v}] \, dy + \int_{\partial\Omega_S} b |\mathbf{v}|^2 \, d\sigma_y.$$

Thus,  $\mathbf{D}\mathbf{v} \equiv 0$  and  $\mathbf{v} = 0$  on  $\partial\Omega_S$ . Because  $\mathbf{D}\mathbf{v} \equiv 0$ , we have that  $\mathbf{v}$  is linear by [24, Lemma 3.1]. Because  $\Delta \mathbf{v} = 0$  in  $\Omega$  and  $\mathbf{v} = 0$  on  $\partial\Omega$ , we infer that  $\mathbf{v} \equiv 0$ . So,  $(\mathbf{v}, p)$  is a solution of the problem (10) with  $\lambda \equiv 0$ ,  $\mathbf{k} = 0$ ,  $\mathbf{f} \equiv 0$ ,  $\mathbf{g} \equiv 0$ . Therefore,  $p \equiv 0$ . Hence, the operator  $T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \operatorname{div}) \times L^2(\Omega_S) \rightarrow W_\Gamma \times \mathcal{V}'_\Gamma$  is a Fredholm operator with index 0 and with trivial kernel, so it is an isomorphism.

From the definition of  $T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p)$ , we have that  $T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p)$  is supported on  $\partial\Omega_S$  if and only if  $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$ . Hence,  $T_1^{b,\eta,\lambda,\mathbf{k}}(\mathbf{v}, p) \in V'_\Gamma$  if and only if  $(\mathbf{v}, p) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$ . Then, because  $T_2^{b,\eta,\lambda,\mathbf{k}} : H(\Omega_S, \operatorname{div}) \times L^2(\Omega_S) \rightarrow W_\Gamma \times \mathcal{V}'_\Gamma$  is an isomorphism, we have that

$$U^{b,\eta,\lambda,\mathbf{k}} = T_2^{b,\eta,\lambda,\mathbf{k}} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \rightarrow W_\Gamma \times \mathcal{V}'_\Gamma$$

is also an isomorphism. □

#### 4. Robin-type problem

The second auxiliary problem is the following Robin problem, related to the generalized Darcy equations,

$$\operatorname{div}(K \nabla p) = 0 \quad \text{in } \Omega_D, \tag{12}$$

$$\mathbf{n}^D \cdot K \nabla p + p = f \quad \text{on } \partial\Omega_D. \tag{13}$$

Here,  $\mathbf{n}^D$  is the unit exterior normal of  $\Omega_D$  and  $f \in H^{-1/2}(\partial\Omega_D)$ . For  $p \in H^1(\Omega_D)$ , we define

$$\langle \partial_K^D p, \varphi \rangle \equiv \int_{\Omega_D} (K \nabla p) \cdot \nabla \varphi \, dy, \quad \varphi \in H^1(\Omega_D). \tag{14}$$

We remark that  $\partial_K^D p$  is supported on  $\partial\Omega_D$ , that is,  $\partial_K^D p \in H^{-1/2}(\partial\Omega_D)$ , if and only if  $p$  is a solution of (12) in the sense of distributions (a weak solution). If this is true then  $\partial_K^D p$  has a meaning of the conormal derivative  $\mathbf{n}^D \cdot K(x) \nabla p$  from (13), that is,  $\partial_K^D p = \mathbf{n}^D \cdot K \nabla p$ .

*Definition 4.1*

We say that  $p \in H^1(\Omega_D)$  is a solution of the Robin problem (12)–(13) if

$$\langle U_K p, \varphi \rangle := \langle \partial_K^D p, \varphi \rangle + \int_{\partial\Omega_D} p \varphi \, d\sigma_y = \langle f, \varphi \rangle \quad \forall \varphi \in H^1(\Omega_D). \tag{15}$$

Proposition 4.2

The operator

$$U_K : H^1(\Omega_D; K) \rightarrow H^{-1/2}(\partial\Omega_D)$$

is an isomorphism, where

$$H^1(\Omega_D; K) = \{p \in H^1(\Omega_D); \operatorname{div}(K\nabla p) = 0\}.$$

Hence, if  $f \in H^{-1/2}(\partial\Omega_D)$ , then there exists a unique solution of the Robin problem (12)–(13), and  $u \in H^1(\Omega_D)$  is a solution of this problem if and only if  $u \in H^1(\Omega_D; K)$  and  $U_K u = f$ .

Proof

From (14) and (15), and the assumption on  $K$ , we have that  $\langle U_K p, \varphi \rangle$  defines an inner product on  $H^1(\Omega_D)$ , and the corresponding norm is equivalent to the original norm in  $H^1(\Omega_D)$ . According to the Riesz representation theorem,  $U_K : H^1(\Omega_D) \rightarrow [H^1(\Omega_D)]'$  is an isomorphism. The space  $H^{-1/2}(\partial\Omega_D)$  is a closed subspace of the dual space  $[H^1(\Omega_D)]'$  formed by distributions from  $[H^1(\Omega_D)]'$  supported on  $\partial\Omega_D$ . Because  $H^1(\Omega_D; K) = U_K^{-1}([H^1(\Omega_D)]')$ , we obtain the assertion stated in the proposition.  $\square$

### 5. Generalized Oseen–Darcy problem

In this section, we shall study the original coupled problem (1)–(6). Because  $\mathbf{v}^D \cdot \mathbf{n} = \mathbf{v}^S \cdot \mathbf{n} + h$  on  $\Gamma$ , we can rewrite (6) as

$$\left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right] \cdot \mathbf{n} + \beta p^D + (\alpha + \gamma) \mathbf{v}^S \cdot \mathbf{n} = \tilde{\mathbf{g}} \cdot \mathbf{n}$$

or in the vector form

$$\left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right]_{\mathbf{n}} + \beta p^D \mathbf{n} + (\alpha + \gamma) \mathbf{v}^S_{\mathbf{n}} = \tilde{\mathbf{g}}_{\mathbf{n}},$$

where  $\tilde{\mathbf{g}}_{\mathbf{n}} = \mathbf{g}_{\mathbf{n}} - h \mathbf{n}$ . Instead of this transmission condition, we can consider a bit more general condition

$$\left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right]_{\mathbf{n}} + \beta p^D \mathbf{n} + (A \mathbf{v}^S)_{\mathbf{n}} = \tilde{\mathbf{g}}_{\mathbf{n}} \quad \text{on } \Gamma, \tag{16}$$

where  $A$  is a matrix function of the type  $3 \times 3$  with  $A_{ij} \in L^\infty(\partial\Omega_S)$ , which is nonnegative on  $\Gamma$ , that is,

$$(A(\mathbf{x})\boldsymbol{\xi}) \cdot \boldsymbol{\xi} \geq 0 \quad \forall \mathbf{x} \in \Gamma, \quad \forall \boldsymbol{\xi} \in \mathbb{R}^3. \tag{17}$$

We shall study the transmission problem (1)–(5), (16) under assumption  $\tilde{\mathbf{g}} \in [H^{-1/2}(\partial\Omega_S)]^3$ ,  $h \in H^{-1/2}(\partial\Omega_D)$ , and  $\mathbf{f} \in [H^{1/2}(\Omega_S)]^3$ .

Definition 5.1

We say that  $(\mathbf{v}^S, p^S) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)$  and  $(\mathbf{v}^D, p^D) \in [L^2(\Omega_D)]^3 \times H^1(\Omega_D)$  is a solution of the problem (1)–(5), (16) if

- (i)  $(\mathbf{v}^S, p^S)$  is a solution of the system (1) in the sense of distributions, that is,  $(\mathbf{v}^S, p^S) \in W_{\eta, \lambda, \mathbf{k}}(\Omega_S)$ ,
- (ii)  $(\mathbf{v}^D, p^D)$  is a solution of the system (2) in the sense of distributions, that is,  $p^D \in H^1(\Omega_D; K)$  and  $\mathbf{v}^D = -K\nabla p^D$  in  $\Omega_D$ ,
- (iii) the boundary conditions  $\mathbf{v}^S = \mathbf{f}$  on  $\partial\Omega_S \setminus \Gamma$  and  $\mathbf{v}^S_{\boldsymbol{\tau}} = \mathbf{f}_{\boldsymbol{\tau}}$  on  $\Gamma$  are satisfied in the sense of traces,
- (iv)  $\left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right]_{\mathbf{n}} \Big|_{\Gamma} + \beta p^D \mathbf{n} \chi_{\Gamma} \Big|_{\Gamma} + [A \mathbf{v}^S]_{\mathbf{n}} \Big|_{\Gamma} = \tilde{\mathbf{g}}_{\mathbf{n}} \Big|_{\Gamma}$  and  $\partial_K^D p^D = h + \mathbf{v}^S \cdot \mathbf{n} \chi_{\Gamma}$ .

Here,  $\chi_{\Gamma}$  is the characteristic function of  $\Gamma$ .

First, we show the uniqueness of a solution

$$(\mathbf{v}^S, p^S) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)/\mathbb{R} \quad \text{and} \quad (\mathbf{v}^D, p^D) \in [L^2(\Omega_D)]^3 \times H^1(\Omega_D)/\mathbb{R}$$

of the problem (1)–(5), (16).

Proposition 5.2

Let  $(\mathbf{v}^S, p^S) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)$  and  $(\mathbf{v}^D, p^D) \in [L^2(\Omega_D)]^3 \times H^1(\Omega_D)$  be a solution of the problem (1)–(5), (16) with  $\mathbf{f} \equiv 0$ ,  $h \equiv 0$  and  $\tilde{\mathbf{g}} \equiv 0$ . Then, there exists a constant  $c$  such that  $p^S = \beta c$ ,  $\mathbf{v}^S \equiv 0$ ,  $\mathbf{v}^D \equiv 0$  and  $p^D = c$ .

On the other hand, if  $p^S = \beta c$ ,  $\mathbf{v}^S \equiv 0$ ,  $\mathbf{v}^D \equiv 0$ ,  $p^D = c$  for some constant  $c$ , then  $(\mathbf{v}^S, p^S, \mathbf{v}^D, p^D)$  is a solution of the problem (1)–(5), (16) with  $\mathbf{f} \equiv 0$ ,  $h \equiv 0$  and  $\tilde{\mathbf{g}} \equiv 0$ .



*Proof*

Because  $\mathbf{v}^S = 0$  on  $\partial\Omega_S \setminus \Gamma$  and  $\mathbf{v}^S_\tau = 0$  on  $\Gamma$ , we have  $\mathbf{v}^S \in \mathcal{V}_\Gamma$ . Because  $\mathbf{v}^S_\tau = 0$  on  $\Gamma$ , we have  $\mathbf{v}^S \cdot (A\mathbf{v}^S)_n = \mathbf{v}^S \cdot (A\mathbf{v}^S)$  on  $\Gamma$ . Using the fact that

$$\partial_K^D p^D = \mathbf{v}^S \cdot \mathbf{n} \chi_\Gamma$$

and

$$\left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right]_n \Big|_\Gamma + \beta p^D \mathbf{n} \chi_\Gamma \Big|_\Gamma + (A\mathbf{v}^S)_n \Big|_\Gamma = 0,$$

we obtain

$$\begin{aligned} 0 &= \left\langle \left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right]_n \Big|_\Gamma + \beta p^D \mathbf{n} \chi_\Gamma \Big|_\Gamma + (A\mathbf{v}^S)_n \Big|_\Gamma, \mathbf{v}^S \right\rangle \\ &= \int_{\Omega_S} [2\eta |\mathbf{D} \mathbf{v}^S|^2 + (\lambda \mathbf{v}^S) \cdot \mathbf{v}^S] \, d\mathbf{y} + \beta \langle \partial_K^D p^D, p^D \rangle + \int_\Gamma \mathbf{v}^S \cdot (A\mathbf{v}^S)_n \, d\sigma_{\mathbf{y}} \\ &= \int_{\Omega_S} [2\eta |\mathbf{D} \mathbf{v}^S|^2 + (\lambda \mathbf{v}^S) \cdot \mathbf{v}^S] \, d\mathbf{y} + \beta \int_{\Omega_D} (K \nabla p^D) \cdot \nabla p^D \, d\mathbf{y} + \int_\Gamma \mathbf{v}^S \cdot (A\mathbf{v}^S) \, d\sigma_{\mathbf{y}} \\ &\geq \int_{\Omega_S} 2\eta |\mathbf{D} \mathbf{v}^S|^2 \, d\mathbf{y} + a\beta \int_{\Omega_D} |\nabla p^D|^2 \, d\mathbf{y}. \end{aligned}$$

From the last inequality, we deduce that  $\nabla p^D = 0$  in  $\Omega_D$  and  $\mathbf{D} \mathbf{v}^S = 0$  in  $\Omega_S$ . Hence,  $p^D$  is constant on each component of  $\Omega$  and  $\mathbf{v}^D = -K \nabla p^D = 0$ .

According to (5), we have  $\mathbf{v}^S \cdot \mathbf{n} = \mathbf{v}^D \cdot \mathbf{n} = 0$  on  $\Gamma$ . Because  $\mathbf{v}^S = 0$  on  $\partial\Omega_S \setminus \Gamma$  and  $\mathbf{v}^S_\tau = 0$  on  $\Gamma$ , we infer that  $\mathbf{v}^S = 0$  on  $\partial\Omega_S$ . Using the fact that  $\mathbf{D} \mathbf{v}^S \equiv 0$ , we obtain that the functions  $\mathbf{v}^S_j$ , for  $j = 1, 2, 3$ , are affine ([25, Lemma 6]) and therefore harmonic. The maximum principle for harmonic functions yields that  $\mathbf{v}^S_j \equiv 0$ , for  $j = 1, 2, 3$ . Using the Equation (1) and the fact that  $\mathbf{v}^S \equiv 0$ , we obtain  $\nabla p^S = \eta \Delta \mathbf{v}^S - \lambda \mathbf{v}^S - \mathbf{k} \cdot \nabla \mathbf{v} = 0$ . Thus, there exists a constant  $\tilde{c}$  such that  $p^S = \tilde{c}$ .

We have proved that  $p^D$  is constant on each component of  $\Omega_D$ ,  $\mathbf{v}^D \equiv 0$ ,  $\mathbf{v}^S \equiv 0$  and  $p^S \equiv \tilde{c}$ . Then, using the boundary conditions  $0 = [T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S] \cdot \mathbf{n} + \beta p^D + (A\mathbf{v}^D) \cdot \mathbf{n} = -\tilde{c} + \beta p^D$  on  $\Gamma$ , we conclude that  $p^D = \tilde{c}/\beta$ . Considering  $c = \beta \tilde{c}$ , we obtain the first statement of the proposition.

Substituting  $p^S = \beta c$ ,  $\mathbf{v}^S \equiv 0$ ,  $\mathbf{v}^D \equiv 0$ ,  $p^D = c$ , with some constant  $c$ , into (1)–(5), (16), we obtain that the equations are satisfied for  $\mathbf{f} \equiv 0$ ,  $h \equiv 0$  and  $\tilde{\mathbf{g}} \equiv 0$ .  $\square$

*Lemma 5.3*

Let  $\tilde{\mathbf{g}} \in [H^{-1/2}(\partial\Omega_S)]^3$ ,  $\mathbf{f} \in [H^{1/2}(\partial\Omega_S)]^3$ ,  $h \in H^{-1/2}(\partial\Omega_D)$ . If there exists a solution of the problem (1)–(5), (16), then

$$\langle h, 1 \rangle = \int_{\partial\Omega_S \setminus \Gamma} \mathbf{f} \cdot \mathbf{n}^S \, d\sigma_{\mathbf{y}}. \tag{18}$$

*Proof*

Let  $(\mathbf{v}^S, p^S) \in [H^1(\Omega_S)]^3 \times L^2(\Omega_S)$  and  $(\mathbf{v}^D, p^D) \in [L^2(\Omega_D)]^3 \times H^1(\Omega_D)$  be a solution of the problem (1)–(5), (16). For  $\varphi \equiv 1$ , we have

$$\langle \partial_K^D p^D, 1 \rangle = \int_{\Omega_D} (K \nabla p^D) \cdot \nabla \varphi \, d\mathbf{y} = 0.$$

Because  $\operatorname{div} \mathbf{v}^S = 0$ , Green's theorem yields

$$\int_{\partial\Omega_S} \mathbf{v}^S \cdot \mathbf{n}^S \, d\sigma_{\mathbf{y}} = 0, \tag{19}$$

(for example [22, Chapter 4]). Using the fact that  $\partial_K^D p^D = \mathbf{v}^D \cdot \mathbf{n}$  and the equality (19), we obtain

$$\begin{aligned} 0 &= \langle \partial_K^D p^D, 1 \rangle = \langle h, 1 \rangle + \int_\Gamma \mathbf{v}^S \cdot \mathbf{n}^S \, d\sigma_{\mathbf{y}} - \int_{\partial\Omega_S} \mathbf{v}^S \cdot \mathbf{n}^S \, d\sigma_{\mathbf{y}} \\ &= \langle h, 1 \rangle - \int_{\partial\Omega_S \setminus \Gamma} \mathbf{f} \cdot \mathbf{n}^S \, d\sigma_{\mathbf{y}}. \end{aligned} \tag{20}$$

$\square$

*Theorem 5.4*

Consider  $\tilde{\mathbf{g}} \in [H^{-1/2}(\partial\Omega_S)]^3$ ,  $\mathbf{f} \in [H^{1/2}(\partial\Omega_S)]^3$ ,  $h \in H^{-1/2}(\partial\Omega_D)$ . Then there exists a solution of the problem (1)–(5), (16) if and only if the compatibility condition (18) is satisfied.

*Proof*

We notice that  $(\mathbf{v}^S, p^S)$  and  $(\mathbf{v}^D, p^D)$  is a solution of the problem (1)–(5), (16) if and only if  $(\mathbf{v}^S, p^S) \in W_{\eta, \lambda, \mathbf{k}}(\Omega_S)$ ,  $p^D \in H^1(\Omega_D; K)$  and  $\mathbf{v}^D = -K \nabla p^D$  together with

$$U(\mathbf{v}^S, p^S, p^D) = \left( \mathbf{f} \Big|_{\partial\Omega_S \setminus \Gamma}, \mathbf{f}_\tau \Big|_\Gamma, \tilde{\mathbf{g}}_n \Big|_\Gamma, h \right),$$

where the operator  $U : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \times H^1(\Omega_D; K) \rightarrow W_\Gamma \times V'_\Gamma \times H^{-1/2}(\partial\Omega_D)$  is given by

$$U(\mathbf{v}^S, p^S, p^D) = \left( \mathbf{v}^S|_{\partial\Omega_S \setminus \Gamma}, \mathbf{v}^S_\tau|_{\Gamma}, \left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right]_{\mathbf{n}} \Big|_{\Gamma} + \beta p^D \mathbf{n} \Big|_{\Gamma} + [A\mathbf{v}^S]_{\mathbf{n}} \Big|_{\Gamma}, \partial_k^D p^D - \mathbf{v}^S \cdot \mathbf{n} \chi_\Gamma \right).$$

We also define the operator  $\tilde{U} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \times H^1(\Omega_D; K) \rightarrow W_\Gamma \times V'_\Gamma \times H^{-1/2}(\partial\Omega_D)$  as

$$\tilde{U}(\mathbf{v}^S, p^S, p^D) = \left( \mathbf{v}^S|_{\partial\Omega_S \setminus \Gamma}, \mathbf{v}^S_\tau|_{\Gamma}, \left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right]_{\mathbf{n}} \Big|_{\Gamma} + [\mathbf{v}^S_{\mathbf{n}} + \beta p^D \mathbf{n}] \Big|_{\Gamma}, \partial_k^D p^D + p^D \right).$$

Notice that

$$(\tilde{U} - U)(\mathbf{v}^S, p^S, p^D) = (0, 0, \mathbf{v}^S_{\mathbf{n}}|_{\Gamma} - [A\mathbf{v}^S]_{\mathbf{n}}|_{\Gamma}, \mathbf{v}^S \cdot \mathbf{n} \chi_\Gamma + p^D).$$

Consider fixed  $[\mathbf{F}, \mathbf{G}] \in W_\Gamma \times V'_\Gamma$  and  $h \in H^{-1/2}(\partial\Omega_D)$ . According to Proposition 4.2, there exists a unique  $p^D \in H^1(\Omega_D; K)$  such that  $\partial_k^D p^D + p^D = h$ . Proposition 3.3 ensures the existence of a unique  $(\mathbf{v}^S, p^S) \in W_{\eta,\lambda,\mathbf{k}}(\Omega_S)$  such that

$$\left( \mathbf{v}^S|_{\partial\Omega_S \setminus \Gamma}, \mathbf{v}^S_\tau|_{\Gamma}, \left[ T_\eta(\mathbf{v}^S, p^S) \mathbf{n} - \frac{1}{2}(\mathbf{k} \cdot \mathbf{n}) \mathbf{v}^S \right]_{\mathbf{n}} \Big|_{\Gamma} + \mathbf{v}^S_{\mathbf{n}} \Big|_{\Gamma} \right) = (\mathbf{F}, [\mathbf{G}_{\mathbf{n}} - \beta p^D \mathbf{n}] \Big|_{\Gamma}).$$

Hence,  $\tilde{U} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \times H^1(\Omega_D; K) \rightarrow W_\Gamma \times V'_\Gamma \times H^{-1/2}(\partial\Omega_D)$  is an isomorphism.

The operator  $U - \tilde{U} : W_{\eta,\lambda,\mathbf{k}}(\Omega_S) \times H^1(\Omega_D; K) \rightarrow W_\Gamma \times V'_\Gamma \times H^{-1/2}(\partial\Omega_D)$  is compact by the Sobolev imbedding theorem. Thus, the operator  $U$  is a Fredholm operator with index 0. By Proposition 5.2, we have that the dimension of the kernel of  $U$  is equal to 1. Therefore, the range of  $U$  is a subspace of  $W_\Gamma \times V'_\Gamma \times H^{-1/2}(\partial\Omega_D)$  of the codimension 1. Hence, using Lemma 5.3, we deduce that there exists a solution of the problem (1)–(5), (16) if and only if (18) holds true.  $\square$

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